# Canonical formulation of the conformal $p$-brane

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Abstract: We introduce a new Weyl invariant action for bosonic p-branes. The key ingredient is the introduction of an auxiliary field that transform the non-linear conformal action into a quadratic action. For these actions we construct, following the standard Dirac's method, the canonical formalism. Based in this construction we obtain the physical degrees of freedom of the system.

## 1. p-branes Actions

The study of the dynamics of extended systems was introduced by Dirac in 1962 [1]. In this paper Dirac tried to describe elementary particles as vibrational modes of a membrane. After this initial step a general principle to describe the dynamics of extended systems in the context of strings was developed by Nambu and Goto [2, 3] that is known as the Nambu-Goto action principle. This principle states that the action of a extended system of $p$-dimensions ($p$-brane) embedded in a $D$-dimensional space-time is proportional to the world volume,

$$S[X^\mu] = -T\int d^{p+1}\xi\sqrt{-\det\partial_i X^\mu\partial_j X^\nu\eta_{\mu\nu}}, \tag{1.1}$$

where $\xi^i$, $i = 0, 1, \ldots, p$ are the $p+1$ dimensional world volume coordinates and $X^\mu = X^\mu(\xi^i)$, $\mu = 0, 1, \ldots, D-1$ are the $D$ space-time coordinates. We assume a flat space-time with Minkowski metric $\eta_{\mu\nu}$ and signature $(-, +, \ldots, +)$. The tension of the object is given by the constant $T$ which has the necessary units for a dimensionless action ($[T] = [mass]^{(p+1)}$). This action has $p+1$ dimensional diffeomorphism invariance given by

$$\delta X^\mu = \varepsilon^i\partial_i X^\mu, \tag{1.2}$$

where $\varepsilon^i(\xi)$ are the parameters of the transformation.

In the case of string theory (1-brane), Schild [4] proposed the introduction of an auxiliary field $e$ to eliminate the square root in (1.1). The generalization of this idea for the $p$-brane action (1.1) has the form [5]:

$$S_s[X^\mu] = \int d^{p+1}\xi\ \frac{1}{2}\left(\frac{h}{e} - T^2 e\right), \tag{1.3}$$

here $h$ is the determinant of the induced metric

$$h_{ij} = \partial_i X^\mu\partial_j X^\nu\eta_{\mu\nu}. \tag{1.4}$$

The action (1.3) is invariant under the diffeomorphisms (1.2), provided $e$ transform as

$$\delta e = \partial_i(\varepsilon^i e) \tag{1.5}$$

i.e., $e$ has the role of an einbein. Another interesting property of the Schild action is that we can get the limit of null tension that correspond to the ultra relativistic limit of the $p$-brane. In a more geometrical context Brink, et al [6] wrote a different extension of the Nambu-Goto action, now called the Polyakov action. For this action instead of the auxiliary field $e$ we introduce a world sheet metric $\Gamma_{ij}$. In the case of a $p$-brane the Polyakov form of the action is

$$S[X^\mu, \Gamma_{ij}] = -\frac{T}{2}\int d^{p+1}\xi\sqrt{-\Gamma}\left(\Gamma^{ij}\partial_i X^\mu\partial_j X^\nu\eta_{\mu\nu} - (p-1)\right). \tag{1.6}$$

Here $\Gamma^{ij}$ is the inverse of $\Gamma_{ij}$ and $\Gamma$ denotes the determinant. For any $p$-brane the action (1.6) is invariant under diffeomorphisms,

$$\delta X^\mu = \varepsilon^i\partial_i X^\mu, \tag{1.7}$$

$$\delta\Gamma_{ij} = \varepsilon^k\partial_k\Gamma_{ij} + \partial_i\varepsilon^k\Gamma_{kj} + \partial_j\varepsilon^k\Gamma_{ik}. \tag{1.8}$$

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In the case of the 1-brane this action is also invariant under Weyl transformations $\delta\Gamma_{ij} = 2\omega\Gamma_{ij}$. Taking into account that the Weyl symmetry is a very important factor in the quantization of the string, several people tried to develop a Weyl invariant extension of the action for the $p$-brane [7]. This action reads as follows

$$S[X^\mu, \Gamma_{ij}] = -T \int d^{p+1}\xi\sqrt{-\Gamma} \left(\frac{1}{p+1}\Gamma^{ij}\partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}\right)^{(p+1)/2} \quad (1.9)$$

The action in (1.9) is invariant under the following transformations

$$\begin{aligned} \delta X^\mu &= \varepsilon^i \partial_i X^\mu, & (1.10)\\ \delta\Gamma_{ij} &= \varepsilon^k\partial_k\Gamma_{ij} + \partial_i\varepsilon^k\Gamma_{kj} + \partial_j\varepsilon^k\Gamma_{ik} \\ &\quad +2\omega\Gamma_{ij}, & (1.11)\end{aligned}$$

then, we obtain Weyl and diffeomorphism invariance, but we have again, as in Nambu-Goto, a non quadratic action. As a consequence of this fact, the canonical formalism is not well defined, since is not possible to get a Hamiltonian function of the coordinates and momenta but not of the velocities. An attempt to develop the canonical analysis of (1.9) was proposed in [8].

In this paper we present a different solution to this problem. We construct an action equivalent to (1.9) for which the standard rules of the canonical approach can be applied. Then we completely work out the algebra of constraints of this theory. Following Schild [4] we introduce an auxiliary field $e$ to eliminate the power in the integrand of the action (1.9) to get the following expression

$$S[X^\mu, \Gamma^{ij}, e] = -\frac{T}{2}\int d^{p+1}\xi \,\sqrt{-\Gamma}\,\Big(e^{1-\frac{2}{p+1}}\Gamma^{ij}h_{ij} - e(p-1)\Big). \quad (1.12)$$

This action is invariant under Weyl symmetry if the auxiliary field $e$ changes as $e \to e\exp(-\omega d)$. Thus the infinitesimal transformations of the fields that leave the action invariant up to a boundary term are

$$\begin{aligned} \delta X^\mu &= \varepsilon^i \partial_i X^\mu, & (1.13)\\ \delta\Gamma_{ij} &= \varepsilon^k\partial_k\Gamma_{ij} + \partial_i\varepsilon^k\Gamma_{kj} + \partial_j\varepsilon^k\Gamma_{ik} \\ &\quad +2\omega\Gamma_{ij}, & (1.14)\\ \delta e &= \varepsilon^i\partial_i e - \omega(p+1)e. & (1.15)\end{aligned}$$

An interesting point is that the transformation law for the auxiliary field $e$ is quite different in the actions (1.3) and (1.12). In the case of Schild's action $e$ is an einbein, whereas in (1.12) $e$ is a scalar under diffeomorphisms –like a space-time coordinate $X^\mu(\xi)$– and transform under conformal transformations. The classical equivalence between (1.9) and (1.12) is obtained from the solution of the equation of motion for $e$ and its substitution in (1.12). Meanwhile, the equivalence between (1.6) and (1.9) with (1.1) is obtained from the solution to the equation of motion for the auxiliary field $\Gamma_{ij}$ and its substitution in the respective action.

## 2. Canonical analysis of the conformal action

In this section we develop the canonical formulation of the action (1.12) and compute the algebra of constraints.

To construct the associated canonical analysis for the action (1.12) we assume that the topology of the world-volume $\mathcal{M}^{p+1}$ is of the form $\Sigma^p \times \Re$, where $\Sigma^p$ is a $p$ dimensional compact manifold. Following the ADM construction we introduce a shift vector $N^a$ $(a = 1, ..., p)$ and a lapse function $N$. Using these variables the metric of the world-volume $\Gamma_{ij}$ can be rewritten as

$$\begin{aligned} \Gamma_{00} &= -N^2\gamma + \gamma_{ab}N^aN^b, \\ \Gamma_{0a} &= \gamma_{ab}N^b, \\ \Gamma_{ab} &= \gamma_{ab}. \end{aligned} \quad (2.1)$$

With this metric the Lagrangian action (1.12) is given by

$$S = -\frac{T}{2}\int d^{p+1}d\xi\bigg(\frac{e^{\frac{p-1}{p+1}}}{N}\Big[-\dot{X}^2 + 2N^a\dot{X}^\mu\partial_a X_\mu + \left(N^2\gamma\gamma^{ab} - N^aN^b\right)h_{ab}\Big] - (p-1)eN\gamma\bigg) \quad (2.2)$$

The fields content of this action is $(X^\mu, N, N^a, \gamma^{ab}, e)$. The associated canonical momenta are

$$P_\mu = T\frac{e^{\frac{p-1}{p+1}}}{N}\left(\dot{X}_\mu - N^a\partial_a X_\mu\right), \quad (2.3)$$

$$\pi \approx 0, \qquad \pi_a \approx 0, \qquad \pi_{ab} \approx 0, \qquad \pi_e \approx 0. \quad (2.4)$$

with the basic Poisson brackets

$$\begin{aligned}
\{X^\mu(\xi), P_\nu(\xi')\} &= \delta^\mu_\nu \delta^p(\xi-\xi'),\\
\{N(\xi), \pi(\xi')\} &= \delta^p(\xi-\xi'),\\
\{N^a(\xi), \pi_b(\xi')\} &= \delta^a_b \delta^p(\xi-\xi'),\\
\{e(\xi), \pi_e(\xi')\} &= \delta^p(\xi-\xi'),\\
\{\gamma^{ab}(\xi), \pi_{cd}(\xi')\} &= \frac{1}{2}(\delta^a_c\delta^b_d + \delta^b_c\delta^a_d)\delta^p(\xi-\xi').
\end{aligned} \tag{2.5}$$

From the definition of the momenta we obtain $\frac{(p+1)(p+2)}{2}+1$ primary constraints. The total Hamiltonian associated to (2.2) is

$$\begin{aligned}
H_T = & \int d^p\xi \Bigg( \frac{N}{2}\Big(\frac{1}{T} e^{\frac{1-p}{p+1}} P_\mu P^\mu + T e^{\frac{p-1}{p+1}} \gamma\gamma^{ab} h_{ab} \\
& -Te\gamma(p-1)\Big) + N^a P_\mu \partial_a X^\mu \\
& + \lambda_e \pi_e + \lambda\pi + \lambda^a \pi_a + \lambda^{ab}\pi_{ab}\Bigg),
\end{aligned} \tag{2.6}$$

here the $\lambda$'s are the Lagrangian multipliers associated to the primary constraints. By using the Dirac method the evolution in time of these constraints generate the following $(p+1)(p+2)/2$ secondary constraints

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2}\Big(\frac{1}{T} P_\mu P^\mu + T e^{2-\frac{4}{p+1}} \gamma\gamma^{ab} h_{ab} \\
& \quad - T e^{2-\frac{2}{p+1}} \gamma(p-1)\Big) \approx 0,\\
\mathcal{H}_a &= P_\mu \partial_a X^\mu \approx 0,\\
\Omega_{ab} &= e^{-\frac{2}{p+1}} h_{ab} - \gamma_{ab} \approx 0.
\end{aligned} \tag{2.7}$$

The evolution in time of these secondary constraints does not produce new constraints.

To split the constraints according to its first or second class character we observe that the constraints $\Omega_{ab}$ and $\pi_{ab}$ are second class. This are $p(p+1)$ constraints. Furthermore, from the range of the matrix defined by the Poisson brackets between all the constraints, we conclude that there are no more second class constraints.

By a redefinition of the constraints $\mathcal{H}$, $\mathcal{H}_a$ and $\mathcal{T}_{\pi_e}$ on the constraint surface, the algebra of the complete set of $2p+3$ first class constraints can be closed up to quadratic pieces in second class constraints. To that end we propose the following complete set of constraints

*First class:*

$$\pi \approx 0, \quad \pi_a \approx 0, \tag{2.8}$$

$$\mathcal{T}_{\pi_e} \equiv \pi_e + \frac{2}{p+1} e^{-\frac{2}{p+1}-1} \pi^{ab} h_{ab} \approx 0, \tag{2.9}$$

$$\mathcal{T} \equiv \mathcal{H} + \frac{2}{T} P_\mu \partial_a (e^{-\frac{2}{p+1}} \pi^{ab} \partial_b X^\mu) \approx 0, \tag{2.10}$$

$$\mathcal{T}_a \equiv \mathcal{H}_a + 2\partial_a X^\mu \partial_b (e^{-\frac{2}{p+1}} \pi^{bc} \partial_c X_\mu) \approx 0. \tag{2.11}$$

*Second class:*

$$\pi_{ab} \approx 0, \quad \Omega_{ab} \approx 0. \tag{2.12}$$

Denoting the first class constraints as $G_r$ ($r = 1, ..., 2p+3$) and the second class as $\chi_\alpha$ ($\alpha = 1, ..., p(p+1)$) the algebra has the general form

$$\begin{aligned}
\{G_r, G_s\} &= C^t_{rs} G_t + B^{\alpha\beta}_{rs} \chi_\alpha \chi_\beta\\
\{G_r, \chi_\alpha\} &= A^s_{r\alpha} G_s + D^\beta_{r\alpha} \chi_\beta,\\
\{\chi_\alpha, \chi_\beta\} &= C_{\alpha\beta},
\end{aligned}$$

where the explicit expressions of the structure functions are given in [10].

To compute the algebra *on* the second class constraint surface we introduce the Dirac bracket [9],

$$\begin{aligned}
\{F \;\; , \;\; G\}^* = & \{F, G\} \\
& + \int d^p\xi \{F, \Omega_{ab}(\xi)\} \gamma^{ac}\gamma^{bd}(\xi) \{\pi_{cd}(\xi), G\} \\
& - \int d^p\xi \{F, \pi_{ab}(\xi)\} \gamma^{ac}\gamma^{bd}(\xi) \{\Omega_{cd}(\xi), G\}.
\end{aligned} \tag{2.13}$$

The relevant Dirac brackets between the canonical variables are

$$\begin{aligned}
\{X^\mu(\xi), P_\nu(\xi')\}^* &= \delta^\mu_\nu \delta^p(\xi-\xi'),\\
\{N(\xi), \pi(\xi')\}^* &= \delta^p(\xi-\xi'),\\
\{N^a(\xi), \pi_b(\xi')\}^* &= \delta^a_b \delta^p(\xi-\xi'),\\
\{e(\xi), \pi_e(\xi')\}^* &= \delta^p(\xi-\xi'),\\
\{\gamma^{ab}(\xi), \pi_{cd}(\xi')\}^* &= 0,\\
\{\gamma^{ab}(\xi), P_\mu(\xi')\}^* &= e^{-\frac{2}{p+1}} \gamma^{ac}\gamma^{bd} \\
& \quad \times \Big[\partial_d X_\mu \partial_c \delta^p(\xi-\xi') \\
& \quad + \partial_c X_\mu \partial_d \delta^p(\xi-\xi')\Big],\\
\{\gamma^{ab}(\xi), \pi_e(\xi')\}^* &= \frac{2}{e(p+1)} \gamma^{ab} \delta^p(\xi-\xi').
\end{aligned} \tag{2.14}$$

The expressions for the Dirac brackets allow to show that the constraint (2.9) generates the Weyl transformation. Using these relevant Dirac brackets the full Dirac algebra of the densitized constraints is

$$
\begin{aligned}
\{F[f],\Omega_{ab}[g]\}^* &= \{F[f],\pi_{ab}[g]\}^* = 0,\\
\{\mathcal{T}_{\pi_e}[f],\mathcal{T}[g]\}^* &= \{\mathcal{T}_{\pi_e}[f],\mathcal{T}_a[g]\}^* = 0,\\
\{\mathcal{T}[f],\mathcal{T}[g]\}^* &= \mathcal{T}_a \\
&\qquad \left[(f\partial_b g - g\partial_b f)e^{(2-\frac{4}{p+1})}\gamma\gamma^{ab}\right],\\
\{\mathcal{T}[f],\mathcal{T}_a[g]\}^* &= \mathcal{T}[f\partial_a g - g\partial_a f], \qquad (2.15)\\
\{\mathcal{T}[f]_a,\mathcal{T}_b[g]\}^* &= \mathcal{T}_b[f\partial_a g] - \mathcal{T}_a[g\partial_b f].
\end{aligned}
$$

We observe that the standard diffeomorphism algebra for the $p$-brane is reproduced with different structure functions that are now modified to preserve the Weyl symmetry. The constraints analysis is consistent with the number of physical degrees of freedom. We have $D+1+(p+1)(p+2)/2$ variables, $2p+3$ first class constraints and $p(p+1)$ second class constraints. Then the number of physical degrees of freedom per space-time point is, $D-(p+1)$.

## 3. Conclusions

From the above construction we see that the action (1.12) have exactly the same physical degrees of freedom that the usual $p$-brane action (1.6). However, the action (1.12) have an additional symmetry given by the Weyl invariance. We hope that this new action can be useful in the study of the quantization of the $p$-branes in similar form to the case of the Polyakov action in strings where the Weyl anomaly is the more easy way to compute the critical dimension. The idea of a Weyl invariant action for p-branes introducing an auxiliary field can be extendable to the supersymmetric case [10] where the $\kappa$-symmetry is now Weyl invariant. Furthermore, for the case of $d$-branes it is also possible to consider this construction using the action for $d$-branes proposed in [11]. For $d$-branes it is also possible introduce a dynamical field instead of an auxiliary field, taking into account the Weyl symmetry.

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