PROCEEDINGS

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Yangians and Kac-Moody Loop Algebras in Superconformal Gauge Theory

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Abstract: We discuss the Yangian symmetry recently found for superconformal Yang Mills theory, and contrast it with the Kac-Moody loop algebra that can be defined in various integrable models.

1. Yangian Symmetry and Kac-Moody Algebras

We describe an infinite-dimensional non-abelian symmetry algebra that is found in planar superconformal Yang-Mills theory. The generators have a basis $\mathcal{J}_n^A$ where $\mathcal{J}_0^A = J^A$, $\mathcal{J}_1^A = Q^A$, and $\mathcal{J}_n^A$, $n = 0, 1, 2, \ldots$. We have given the first two generators special names, since only those occur in the defining relations for the algebra. The higher charges will arise from commutators of the Q's. The algebra, called a Yangian $Y(G)$, is an associative Hopf algebra [1]-[5] that satisfies:

$$[J^A, J^B] = f_C^{AB} J^C \,, \qquad [J^A, Q^B] = f_C^{AB} Q^C \,, \tag{1.1}$$

and the Serre relations

$$\begin{aligned}
&[Q^A, [Q^B, J^C]] + [Q^B, [Q^C, J^A]] + [Q^C, [Q^A, J^B]] \\
&\quad = \frac{1}{24} f^{ADK} f^{BEL} f^{CFM} f_{KLM} \{J_D, J_E, J_F\} \,,
\end{aligned}\tag{1.2}$$

$$\begin{aligned}
&[[Q^A, Q^B], [J^C, Q^D]] + [[Q^C, Q^D], [J^A, Q^B]] \,, \\
&= \frac{1}{24} (f^{AGL} f^{BEM} f^{KFN} f_{LMN} f_K^{CD} \\
&\qquad + f^{CGL} f^{DEM} f^{KFN} f_{LMN} f_K^{AB}) \{J_G, J_E, J_F\} \,,
\end{aligned}\tag{1.3}$$

for J^A taking values in the Lie algebra of an arbitrary semi-simple Lie group G.

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Here $\{J^D, J^E, J^F\}$ is the totally symmetrized product,

$$\begin{aligned}\{J^D, J^E, J^F\} &= J^D J^E J^F + J^E J^D J^F + J^F J^E J^D \\ &+ J^E J^F J^D + J^D J^F J^E + J^F J^D J^E \,. \end{aligned} \tag{1.4}$$

For $SU(2)$ the relation (1.2) is trivial. For other cases such as $SU(N)$ with $N \geq 3$, the relation (1.2) implies the following one (1.3).

In [6] we give the construction of the generators for the free superconformal Yang-Mills theory, and conjecture that they will exist to all orders of perturbation theory. We then check that the free generators commute with the planar one-loop anomalous dimension operator, *i.e.* the spin chain Hamiltonian [7]-[9]. Our construction is motivated by the existence of non-local symmetries in the AdS/CFT dual worldsheet theory of the type IIB superstring on $AdS_5 \times S^5$, as conjectured [10] and found in [11].

Non-local symmetries in two-dimensional models and their possible role in four-dimensional gauge theory has a long history. In 1980, Polykov discussed "rings of glue" [12],[13], where the untraced Wilson loop of ordinary Yang-Mills theory, $\psi[\xi_\mu(s)] = Pe^{\oint A^a_\mu T^a d\xi^\mu}$, satisfies the on-shell functional equation

$$\frac{\delta}{\delta \xi^\mu}(\psi^{-1}\frac{\delta}{\delta \xi_\mu}\psi) \sim D^\mu F^a_{\mu\nu} T^a \dot{\xi}^\nu \sim 0 \,, \tag{1.5}$$

with $0 \leq \mu \leq 3$. He used the similarity of (1.5) to the equations of motions of the two-dimensional principal chiral model (PCM)

$$\partial_\mu (g^{-1}\partial^\mu g) = 0 \,, \tag{1.6}$$

where g is a matrix valued field talking values in a group G, $g_{ij}(x,t) \in G$, and now $0 \leq \mu \leq 1$. In 1978, Lusher and Pohlmeyer [14] had shown that the conserved current of a PCM Lagrangian $\mathcal{L} = \frac{1}{2} Tr \partial_\mu g \partial^\mu g$, given by $j_\mu(x,t) = g^{-1}\partial_\mu g = j^A_\mu(x,t) T^A$, which takes values in the Lie algebra of G, was also a flat connection

$$\partial_\mu j_\nu - \partial_\nu j_\mu + [j_\mu, j_\nu] = 0 \,. \tag{1.7}$$

From this they could construct an infinite number of non-local symmetries iteratively, since in two dimensions the conservation of the ordinary global current, $\partial^\mu j_\mu = 0$, implies $j_\mu = \epsilon_{\mu\nu}\partial^\nu \chi$, and this in turn leads to the conservation of the non-local current

$$j^{(1)}_\mu \equiv \partial_\mu \chi + g^{-1}\partial_\mu g \, \chi \,. \tag{1.8}$$

The conserved charges associated with these currents are

$$\begin{gathered} J^A = \int_{-\infty}^{\infty} dx j^{0A}(x,t) \\ Q^A = f^A_{BC} \int_{-\infty}^{\infty} dx \int_x^{\infty} dy j^{0B}(x,t) j^{0c}(y,t) - 2\int_{-\infty}^{\infty} dx j^A_1(x,t) \,. \end{gathered} \tag{1.9}$$

A set of infinitesimal field transformations that leave (1.6) invariant

$$g(x,t) \rightarrow g(x,t) + \delta^A_n g(x,t), \tag{1.10}$$

for $m, n \geq 0$, can be used to define generators

$$M_n^A \equiv \int d^2x \delta_n^A g(x,t) \frac{\delta}{\delta g(x)} \tag{1.11}$$

which were shown to close a partial Kac-Moody loop algebra [15]

$$[M_n^A, M_m^B] = f_C^{AB} M_{n+m}^c \,. \tag{1.12}$$

These transformations are related to those generated via Poisson brackets by the non-local charges, *eg.*

$$\{Q^A, g(x,t)\} = -\delta_1^A g(x,t) + \frac{1}{2} f_{ABC} J^B \, \delta_0^C(g(x,t)). \tag{1.13}$$

So in these models, along with the Yangian which the charges satisfy, it is possible to find non-local transformations that obey the algebra of the partial Kac-Moody algebra. The infinitesimal transformations generated by the charges are simply linear combinations of the Kac-Moody transformations (but with field dependent coefficients). The Yangian and Kac-Moody algebras generate the same equivalence relation even though they are different algebras. It doesn't matter which symmetry algebra we use since transformations of one can be recovered from those of the other by changing the coefficients.

The occurrence of these symmetries in two dimensions led to speculation that four-dimensional ordinary Yang-Mills theory might carry representations of an infinite-dimensional algebra [15]-[17]. At the time, this was conjectured to be the Kac-Moody algebra $\hat{g}$, for g given by the gauge group $g = SU(N)$, and later extended to include the classical conformal group $g = SO(2,4)$. Restriction to self-dual gauge fields fascilitated the computations.

With the more recent precise AdS/CFT correspondence between four-dimensional gauge theories and string theories (whose worldsheets are two-dimensional), we have been able to derive the Yangian generators $Y(G)$, for $G = PSU(2,2|4)$, in the planar limit of the superconformal Yang-Mills theory [6].

2. Yangian Symmetry in Superconformal Yang-Mills Theory

Neglecting for the moment the closed string boundary conditions, we will describe how the non-local charges form an infinite-dimensional extension of $PSU(2,2|4)$. The AdS/CFT correspondence implies a realization of the Yangian on a chain of partons. In the four-dimensional $\mathcal{N} = 4$ superconformal Yang-Mills theory (SYM) with gauge group $SU(N)$, we will consider the 't Hooft limit, *i.e.* large N (which restricts us to string tree graphs), and a range of values for the 't Hooft coupling $g_{YM}^2 N$ (which is inversely proportional to $(\alpha')^2$). Bena, Polchinski and Roiban displayed worldsheet non-local currents in the supergravity limit, $\alpha' \to 0$, using a classical Green-Schwarz formalism. This was generalized to the pure spinor formulation in ([18]). The symmetries are expected to exist for all values of the coupling, so we will construct the charges in the AdS/CFT dual gauge theory at the other end of the parameter space, $g_{YM}^2 N \sim (\alpha')^{-2} \to 0$, where we can treat the gauge theory perturbatively.

We consider radial quantization on $\mathbf{R} \times S^3$ in the superconformal gauge theory. The Hamiltonian is the dilatation operator D of $PSU(2,2|4)$. It can be shown to be conjugate to the linear combination of momentum and special conformal generators $D \sim \frac{1}{2}(P^0+K^0)$. In this quantization, the states of the conformal gauge theory are in one to one correspondence with local operators $\mathcal{O}(x)$ via $\lim_{|x|\to 0}\mathcal{O}(x)|0> \sim |\mathcal{O}>$.

The local operators in the planar limit are given by a single trace of a product of "letters", which are the elementary fields of the gluon supermultiplet and their derivatives, $\Phi_{ij}(x) \sim \phi^I = \phi^{I\mathcal{A}}(x)\mathcal{T}^{\mathcal{A}}_{ij}, \psi^a_\alpha = \psi^{a\mathcal{A}}_\alpha(x)\mathcal{T}^{\mathcal{A}}_{ij}, F_{\mu\nu} = F^{\mathcal{A}}_{\mu\nu}(x)\mathcal{T}^{\mathcal{A}}_{ij}$. (The indices $1 \le I \le 6$, $1 \le a \le 4$ label the vector and spinor $SU(4)$ R-symmetry representations, and $1 \le i,j \le N$ denote the N-dimensional representation of $SU(N)$.) A single-trace operator $\mathcal{O}(x)$ is said to be of length L if it is the trace of a product of L letters, $\mathcal{O}(x) = \Phi^{(1)}_{ij}(x)\Phi^{(2)}_{jk}(x)\ldots\Phi^{(L)}_{li}(x) = Tr\Phi^{(1)}\Phi^{(2)}\ldots\Phi^{(L)}$. This local operator represents a state of a *chain* of L *spins*, or *partons*. From this description, we can see how a four-dimensional planar theory might be integrable, since the words are like spin chains. In the correspondence between operators and states, the letters form a basis for the one-particle states of the free $\mathcal{N}=4$ vector multiplet on $\mathbf{R}\times S^3$.

At $g^2_{YM}N=0$, the Yangian generators are [6]

$$\begin{aligned} J^A &= \sum_i J^A_i\,, \\ Q^A &= f^A_{BC}\sum_{i<j} J^B_i J^C_j\,, \end{aligned} \tag{2.1}$$

where at each site $[J^A_i, J^B_j] = f^{AB}_C J^C_j \delta_{ij}$, with f^{AB}_C given by the structure constants of the supergroup $PSU(2,2|4)$. The brackets denote either commmutators or anticommutators. These generators satisfy the defining relations (1.1). When J^A_i is the in the $(4|4)$ fundamental representation of $SU(2,2|4)$, the generators in (2.1) also satisfy the nesting relation (1.2) as we have discussed in [19].

Therefore, from (2.1), we can construct the entire set of Yangian generators $\mathcal{J}^A_n$ where $\mathcal{J}^A_0 = J^A$, $\mathcal{J}^A_1 = Q^A$, and $\mathcal{J}^A_n$, $n = 0,1,2,\ldots$, for the free superconformal Yang-Mills theory, $g^2_{YM}N=0$. We consider the exact generators as an expansion in $g^2_{YM}N$,

$$\begin{aligned} \tilde{J}^A &= J^A + (g^2N)\delta J^A + \mathcal{O}((g^2N)^2) \qquad \text{for } n=0\,, \\ \tilde{\mathcal{J}}^A_n &= \mathcal{J}^A_n + (g^2N)\delta\mathcal{J}^A_n + \mathcal{O}((g^2N)^2)\,, \end{aligned} \tag{2.2}$$

where $\delta\mathcal{J}^A_n$ are corrections to the Yangian generators to one-loop in the 't Hooft coupling.

We will assume that $\mathcal{N}=4$ SYM has a Yangian symmetry in the planar limit for all $g^2_{YM}N$:

$$[\tilde{J}^A, \tilde{J}^B] = f^{AB}_C \tilde{J}^C\,, \qquad [\tilde{J}^A, \tilde{Q}^B] = f^{AB}_C \tilde{Q}^C\,, \tag{2.3}$$

and likewise for the Serre relations. Then the right-hand equation expanded to one-loop is

$$[\delta J^A, Q^B] + [J^A, \delta Q^B] = f^{ABC}\delta Q^C\,. \tag{2.4}$$

For $J^A = D$, (2.4) becomes

$$[\delta D, Q^B] + [D, \delta Q^B] = \lambda^B \delta Q^B\,, \tag{2.5}$$

where λ^B is the bare conformal dimension, and δD is the one-loop planar (spin chain) Hamiltonian. Note that the structure constants are given by the algebra, *i.e.* they do not receive quantum corrections. Since $[D, \delta Q^B] = \lambda^B \delta Q^B$, we must have

$$[\delta D, Q^B] = 0\,. \tag{2.6}$$

Our requirement parallels one verified by Beisert [9], that

$$[\delta D, J^B] + [D, \delta J^B] = \lambda^B \delta J^B\,, \tag{2.7}$$

implies

$$[\delta D, J^B] = 0\,. \tag{2.8}$$

Therefore, δD must commute with the $g^2_{YM}N = 0$ limit of all the Yangian generators. The operator δD is a sum of operators local along the chain. Its eigenvalues are the (one-loop) anomalous dimensions [7]-[9], [20]-[22]. Operators like this that commute with the ($g^2_{YM}N = 0$) Yangian are called the Hamiltonians of the integrable spin chain.

So, from our construction of the free field limit ($g^2_{YM}N = 0$) of the Yangian generators (which was motivated by the non-local symmetries found by Bena, Polchinski, and Roiban) and their commutation with δD, we have deduced the conclusion of Beisert and Staudacher [8], shown earlier in a special case by Minahan and Zarembo, that δD is a Hamiltonian of an integrable spin chain.

For the remainder of the talk, we verify this picture by using formulas from [8],[9] to argue that $[\delta D, Q^A] = 0$. We compute the commutation of Q^A with the planar one-loop Hamiltonian as follows. First, define

$$\begin{aligned} H &\equiv \delta D \\ H &= \sum_{i=1}^{L-1} H_{i,i+1}\,, \end{aligned} \tag{2.9}$$

which is a sum of operators acting on nearest neighbors. We introduce the lattice version of a total derivative,

$$q^A = \sum_{i=1}^{L-1} \left(J_i^A - J_{i+1}^A\right) = J_1^A - J_{\mathrm{L}}^A\,. \tag{2.10}$$

Then, using the specific form of H determined in [9], we will be able to show

$$[H, Q^A] = q^A\,. \tag{2.11}$$

So, for an infinite chain, assuming we can drop terms at infinity, then $[H, Q^A] = 0$. For a finite chain with periodic boundary conditions, where a total derivative will sum to zero, then we have $[H, \mathcal{C}(Q^A)] = 0$, when $\mathcal{C}(Q^A)$ are the Yangian Casimir opertators which can be defined for periodic boundary conditions.

To show $[H, Q^A] = q^A$, we consider the commutator $[H_{12}, Q^A_{12}]$ of quantities acting on a two-particle system,

$$Q^A_{12} = f^A_{BC} J^B_1 J^C_2\,,$$

$$\begin{aligned} J_{12}^2 &= \sum_A (J_1^A + J_2^A)(J_1^A + J_2^A)\,, \\ q_{12}^A &= J_1^A - J_2^A\,. \end{aligned} \tag{2.12}$$

H_{12} acts on the two-particle system $V_F \otimes V_F = \oplus_{j=o}^{\infty} V_j$, where V_F is the space of one-particle states in the free superconformal gauge theory on $\mathbf{R} \times S^3$. The decomposition of the two-site system in terms of single-site states is discussed in [9],[6], where it is also shown how $J_{12}^2 V_j = j(j+1)V_j$. The direct sum of modules $\oplus_{j=o}^{\infty} V_j$ can be thought of as towers of $PSU(2,2|4)$ representations, with the towers labelled by $j = 0, 1, 2, \ldots$. The highest weight of each tower, ν_j is annihilated by the supergroup charges $K_\mu \nu_j = S_\alpha \nu_j = 0$. In a given tower, states are raised and lowered by the ordinary charges J_{12}^A. States can be moved from tower to tower by the non-local charges Q_{12}^A. In [6] we also prove that $q_{12}^A V_j \in V_{j-1} \oplus V_{j+1}$. We use this, together with the identity

$$Q_{ij}^A = \frac{1}{4}[J_{ij}^2, q_{ij}^A] \tag{2.13}$$

and the expression for H_{12} derived from Feynman graphs [9],

$$H_{12} = \sum_{j=0}^{\infty} 2h(j) P_{12,j}\,, \tag{2.14}$$

where $h(j) = \sum_{n=1}^{j} \frac{1}{n}$ are the harmonic numbers. Then we can show

$$\begin{aligned} &[H_{12}, Q_{12}^A]|\lambda(j)\rangle \\ &= \frac{1}{4}[H_{12}, [J_{12}^2, q_{12}^A]]|\lambda(j)\rangle \\ &= \frac{1}{4}(H_{12}J_{12}^2 - j(j+1)H_{12} - 2h(j)J_{12}^2 + 2h(j)j(j+1))q_{12}^A|\lambda(j)\rangle \\ &= j\,(h(j) - h(j-1))\;|\chi^A(j-1)\rangle \;+\; (j+1)(h(j+1) - h(j))\;|\rho^A(j+1)\rangle \\ &= |\chi^A(j-1)\rangle + |\rho^A(j+1)\rangle \\ &= q_{12}^A|\lambda(j)\rangle\,. \end{aligned} \tag{2.15}$$

We can extend this proof [6] to deduce $[H, Q^A] = q^A$ from $[H_{12}, Q_{12}^A] = q_{12}^A$.

3. Non-local Currents as Noether Currents

To interpret our construction of the non-local symmetry charges in the gauge theory from a field theory point of view [6], we consider the super Yang-Mills Lagrangian

$$\mathcal{L} = \frac{1}{g_{YM}^2} Tr\left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi^I D^\mu\phi^I - \frac{1}{2}[\phi^I,\phi^J][\phi^I,\phi^J] + \text{fermions}\right)\,. \tag{3.1}$$

For simplicity, we will discuss the charges for $A \in so(2,4)$. Classically, we have

$$j^{A\mu}(x) = \kappa_\nu^A \theta^{\mu\nu}(x) \tag{3.2}$$

where κ_μ^A are the conformal Killing vectors, and

$$\theta^{\mu\nu} = 2TrF^{\mu\rho}F_\rho^\nu + 2TrD^\mu\phi^I D^\nu\phi^I - g^{\mu\nu}\mathcal{L} - \frac{1}{3}Tr(D^\mu D^\nu - g^{\mu\nu}D_\rho D^\rho)\phi^I\phi^I + \text{fermions}\,. \tag{3.3}$$

The currents (3.2) are conserved at any $g^2_{YM}N$ using the classical interacting equations of motion. If we set $g^2_{YM}N = 0$, we note that the *untraced* matrix

$$\begin{aligned}(\theta^{\mu\nu})_i^{\ j} &= F^{\mu\rho}F^\nu_{\ \rho} + F^{\nu\rho}F^\mu_{\ \rho} + \partial^\mu\phi^I\partial^\nu\phi^I + \partial^\nu\phi^I\partial^\mu\phi^I - g^{\mu\nu}(\frac{1}{2}F_{\rho\sigma}F^{\rho\sigma} + \partial_\mu\phi^I\partial^\mu\phi^I) \\ &-\frac{1}{3}(\partial^\mu\partial^\nu - g^{\mu\nu}\partial_\rho\partial^\rho)\phi^I\phi^I + \text{fermions}\,,\end{aligned} \tag{3.4}$$

is also conserved, as is $\kappa^A_\nu(\theta^{\mu\nu})_i^{\ j}$. Here i,j are the matrix labels of the gauge group generators $(T^{\mathcal{A}})_i^{\ j}$. Thus we can construct non-local conserved charges by

$$Q_0^{AB\ldots} = \int_M \kappa^A_\nu(\theta^{0\nu})_i^{\ j}\int_M \kappa^B_\rho(\theta^{0\rho})_j^{\ k}\ldots\,, \tag{3.5}$$

where M is an initial value surface in spacetime. In the free gauge field theory, this acts on a chain of partons in a similar fashion as (2.1) does, but it is not apparent how to extend the definition to $g^2_{YM}N \neq 0$.

4. Conclusions

We have described a non-local set of symmetry charges in four-dimensional planar superconformal Yang-Mills theory. They form a Yangian algebra, which generates the same equivalence relation as a partial Kac-Moody loop algebra [6]. The existence of a Hamiltonian that commutes with this Yangian depends on expanding the dilation operator to first order near $g^2N = 0$. In the exact theory (at nonzero $g^2_{YM}N$), the exact dilatation operator $\mathcal{D}$ (which depends on $g^2_{YM}N$) is one of the Yangian generators. In the exact theory it not the case that we have a Yangian algebra *and* a dilatation operator that commutes with it. Furthermore, in string theory, or SYM, we want to compactify the string (or the spin chain) with periodic boundary conditions, since the string is closed. Periodic boundary conditions make it impossible to define the Yangian, because the restriction to the integration region $x < y$ in (1.9) does not make sense. The global $PSU(2,2|4)$ generators J^A still make sense, as do the traces of holonomies (which are like Casimir operators of the Yangian). These Casimirs, which commute with $PSU(2,2|4)$ may be useful in computing the spectrum of the superconformal Yang-Mills theory in the planar limit. Some of them are odd under charge conjugation (the symmetry that reverses the order of the spin chain), so their commutation with $PSU(2,2|4)$ leads to denegeracies among states of opposite charge conjugation properties, as found by [21].

In this talk, we have made contact between the integrability of perturbative planar SYM with its interpretation of the anomalous dimensions as a spin chain Hamiltonian, and the existence of non-local worldsheet symmetry currents closing an infinite-dimensional symmetry algebra generic to non-linear sigma models with a target coset space. We did this by giving a construction of the non-local charges to lowest order in the gauge coupling constant, and then showing that they commuted with the planar one-loop ordinary dilatation charge.

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