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On the Casimir Energy of κ-Deformed Quantum Fields

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A simple and effective method for summing up over zero-point non-trivial energy spectra stemming from the imposition of boundary conditions quantum fields is discussed and then applied to obtain the Casimir energy of κ-deformed quantum scalar and electromagnetic fields.

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1. Introduction

The macroscopically observable vacuum energy shift associated with a quantum field is the regularised difference between the vacuum expectation value of the corresponding Hamiltonian with and without the external conditions demanded by the particular physical situation at hand. At the one-loop level, when the external conditions are represented by boundary conditions, this leads to the usual Casimir effect [1]. In the evaluation of the zero point energies associated with confined quantum fields some configurations, which depend on the nature of the quantum field, the type of spacetime manifold, its dimensionality, and the specific boundary condition imposed on the quantum field on certain surfaces, lead to relatively simple spectra, but others lead to rather complex ones. The main obstacle is the evaluation of the spectral sum that results at the one-loop level from the definition of the Casimir energy. In order to be physically meaningful this evaluation demands the employment of regularization and renormalization techniques. These techniques range from the relatively simple cutoff method, employed by Casimir himself [1], to a number of powerful and elegant generalised zeta function techniques [2]. Contour integral representations of spectral sums are a great improvement in the techniques of evaluating zero-point energies; they are especially useful when the spectra are not simple, and have been employed before see for instance [3, 4] and references therein. Here, a simple and effective method for summing up over zero-point non-trivial energy spectra stemming from the imposition of boundary conditions quantum fields is discussed and then applied to obtain the Casimir energy of κ-deformed quantum scalar and electromagnetic fields. We employ natural units $\hbar = c = 1$.

2. A useful sum formula

For completeness we sketch the derivation of a simple sum formula. For a detailed derivation see [3, 4]. Consider a field theory in $3+1$ ultrastatic flat space-time under boundary conditions imposed on two parallel planes of area L^2 kept at a fixed distance ℓ from each other. Suppose also that the condition $L \gg \ell$ is in force. The one-loop Casimir energy reads

$$E_0 = \alpha \frac{L^2}{2} \int \sum_n \frac{d^2 p_\perp}{(2\pi)^2}\, \Omega_n\,, \tag{2.1}$$

where α is a dimensionless factor that takes into account the number of internal degrees of freedom of the quantum field and

$$\Omega_n := \sqrt{p_\perp^2 + \frac{\lambda_n^2}{\ell^2} + m^2}\,, \tag{2.2}$$

where λ_n is the n-th real root of the transcendental equation defined by the boundary conditions, $p_\perp = \sqrt{p_x^2 + p_y^2}$, and m is the mass of an elementary excitation of the quantum field. Cauchy's integral formula allow us to write

$$\Omega_n = -\oint_\Gamma \frac{dq}{2\pi} \frac{2q^2}{q^2 + \Omega_n^2}\,, \tag{2.3}$$

where in principle Γ is a Jordan curve on the complex q-plane with $\operatorname{Im} q > 0$, that can be chosen to be a semicircle of infinitely large radius whose diameter is the entire real axis. Taking Eq. (2.2) into Eq. (2.1) we have

$$E_0 = -\alpha \frac{L^2}{2} \int \frac{d^2 p_\perp}{(2\pi)^2} \oint_\Gamma \frac{dq}{2\pi} \sum_n \frac{2q^2}{q^2+\Omega_n^2} . \tag{2.4}$$

The summation over the discrete index n can be performed as follows. Consider a complex function $G(z)$ of a single complex variable z symmetrical on the real axis and such that its roots are simple, nonzero and symmetrical with respect to the origen. The assumption that $z=0$ is not a root of $G(z)$ is not an obstacle because if this happens to be so we can divide $G(z)$ by some convenient power of z eliminating thus the zero from the set of roots without introducing new singularities. Due to the symmetry of the roots we can also order and count the roots of $G(z)$ in such a way that

$$\lambda_n = -\lambda_{-n}, \quad n = \pm 1, \pm 2, \ldots . \tag{2.5}$$

Consider now the meromorphic function

$$J(z) := \sum_{n=-\infty}^{\infty}{}' \frac{1}{z - i\lambda_n} , \tag{2.6}$$

where the prime indicates that the term corresponding to $n=0$ is omitted from this sum. The function $J(z)$ has the following easily verifiable properties: (i) it has first order poles determined by the roots of $G(iz)$; (ii) the corresponding residua are all equal to one. Taking into account the symmetry of the roots expressed by Eq. (2.5) we can rewrite $J(z)$ as

$$\begin{aligned} J(z) &:= \frac{1}{2}\left(\sum_{n=-\infty}^{\infty}{}' \frac{1}{z-i\lambda_n} - \sum_{n=-\infty}^{\infty}{}' \frac{1}{z+i\lambda_n} \right) \\ &= \sum_{n=1}^{\infty} \frac{2z}{z^2+\lambda_n^2} . \end{aligned} \tag{2.7}$$

Consider now the function $K(z) := G(iz)$. We can say that

$$J(z) = \frac{1}{K(z)} \frac{dK(z)}{dz} . \tag{2.8}$$

In fact, the rhs of Eq. (2.8) has the same pole structure as the original $J(z)$, and also the same residue at each pole. By invoking the Mittag-Leffler theorem on the pole expansion of meromorphic functions [5] we conclude that Eq. (2.8) is true up to an entire function that does not contribute to the evaluation of the Casimir energy. It follows that we can write

$$\frac{1}{K(z)} \frac{dK(z)}{dz} = \frac{d}{dz} \log[K(z)] = \sum_{n=1}^{\infty} \frac{2z}{z^2+\lambda_n^2} . \tag{2.9}$$

In order to make use of Eq. (2.9) we relate the complex variable z and the complex momemtum variable q by writing

$$q^2 + \Omega_n^2 = \frac{z^2+\lambda_n^2}{\ell^2} , \tag{2.10}$$

hence

$$z = z(q, p_\perp) = \ell \sqrt{q^2 + p_\perp^2 + m^2} , \tag{2.11}$$

and (notice the omission of $\lambda_0 = 0$)

$$\sum_{n=1}^{\infty} = \frac{2q^2}{q^2+\omega_n^2} = \frac{\ell^2 q^2}{z} \sum_{n=1}^{\infty} \frac{2z}{z^2+\lambda_n^2} \, . \tag{2.12}$$

Upon changing variables ($d/dz = \left(z/\ell^2 q\right) d/dq$) we obtain the unregularised Casimir energy

$$E_0 = -\alpha \frac{L^2}{2} \int \frac{d^2 p_\perp}{(2\pi)^2} \oint_\Gamma \frac{dq}{2\pi} q \frac{d}{dq} \log\left[K(z)\right] . \tag{2.13}$$

Equation (2.13) can be integrated by parts on an open curve that lies on the Riemann surface of the integrand the projection of which on the original q-complex plane is the curve Γ, and after discarding a phase term we obtain for the unregularised Casimir energy the expression

$$E_0 = \alpha \frac{L^2}{2} \int \frac{d^2 p_\perp}{(2\pi)^2} \oint_\Gamma \frac{dq}{2\pi} \log\left[K(z)\right] . \tag{2.14}$$

In order to regularise (2.14) we split the function $K(z)$ into two separate terms

$$K(z) = K_1(z) + K_2(z) \, , \tag{2.15}$$

with follwing properties: (i) all terms whose integrals diverge when Re $z > 0$ are expressed by K_1; (ii) all terms whose integrals diverge when Re $z < 0$ are expressed by K_2, (iii) the condition $K_1(z) = K_2(-z)$ holds. From this it follows that we can write for the *regularised* Casimir energy the expression

$$E_0 = \alpha \frac{L^2}{2} \int \frac{d^2 p_\perp}{(2\pi)^2} \int \frac{dq}{2\pi} \log\left[1 + \frac{K_1(z)}{K_2(z)}\right] . \tag{2.16}$$

All along the real axis z does not change sign and is function of q and $p_\perp$. This procedure can be extended and applied to the case where the integrand leading to the Casimir energy is a more complicated function of Ω_n. It suffices to write

$$f(\Omega_n) = -\oint \frac{dq}{2\pi} \frac{2 f(q)}{q^2+\Omega_n^2} \, , \tag{2.17}$$

that holds if $i f(\Omega_n) = f(i\Omega_n)$. This will be the case of the kappa-deformed dispersion relation given by Eq. (3.1) below since $\operatorname{arcsinh}(ix) = i \arcsin(x)$.

3. The Casimir energy of κ-deformed theories

A quantum field theory is said to be κ-deformed when its spacetime symmetry is described by the κ-deformed Poincaré algebra. As is the case with non-deformed theories, κ-deformed ones lead to mass shell conditions and dispersion relations. For an introduction to these type of theories see [6] and the references therein. The important fact for us here is that the complexity of the dispersion relation makes it an ideal test for our technique. The dispersion relation for a kappa-deformed scalar field, the prototype quantum field theory, reads [6]

$$f(\Omega_n) = \frac{1}{\eta} \arcsin(\eta \Omega_n) \, , \tag{3.1}$$

where $\eta = 1/\kappa$ and

$$\Omega_n = \sqrt{p_\perp^2 + \frac{\lambda_n^2}{\ell^2} + m^2}\,. \tag{3.2}$$

The Casimir energy is given by

$$E_0\left(\ell,\eta,m\right) = L^2 \int \sum_n \frac{d^2 p_\perp}{\left(2\pi\right)^2} f\left(\Omega_n\right)\,. \tag{3.3}$$

Cauchy's integral formula allow us to write the integral representation

$$\text{arcsinh}\left(\eta\Omega_n\right) = -\oint \frac{dq}{2\pi} \frac{2q\,\arcsin\left(\eta q\right)}{q^2+\Omega_n^2}\,. \tag{3.4}$$

It follows that

$$E_0\left(\ell,\eta,m\right) = -\frac{L^2}{\eta}\int \frac{d^2 p_\perp}{\left(2\pi\right)^2}\oint dq \sum_n \frac{2q\,\arcsin\left(\eta q\right)}{q^2+\Omega_n^2}\,. \tag{3.5}$$

Introducing the variable $z := \ell\sqrt{p\perp^2 + q^2 + m^2}$, we can write

$$\sum_{n=1}^{\infty} \frac{2q^2}{q^2+\Omega_n^2} = \left(\frac{\ell q}{z}\right)^2 \sum_{n=1} \frac{2z^2}{z^2+\lambda_n^2}\,, \tag{3.6}$$

and $d/dz = \left(z/\ell^2 q\right) d/dz$. Therefore the Casimir energy can be recasted into the form

$$E_0\left(\ell,\eta,m\right) = -\frac{L^2}{\eta}\int \frac{d^2 p_\perp}{\left(2\pi\right)^2}\oint \frac{dq}{2\pi}\arcsin\left(\eta\, q\right)\frac{d}{dq}\log K\left(z\right)\,. \tag{3.7}$$

Integrating by parts on the q-complex plane and discarding an unimportant phase term we arrive at

$$E_0\left(\ell,\eta,m\right) = L^2 \int \frac{d^2 p_\perp}{\left(2\pi\right)^2}\oint \frac{dq}{2\pi}\frac{\log K\left(z\right)}{\sqrt{1-\eta^2 q^2}}\,. \tag{3.8}$$

Keep in mind that now $z = z\left(p_\perp, q, m\right)$. The closed integral on the complex q-plane can be evaluated as follows: we divide the contour in such way that we have to evaluate an integral over the real axis plus the contribution of large semicircle whose radius $\to\infty$. It can be shown then that this last integral does not contribute to the Casimir energy. The real axis must be partitioned according to its branch points and cuts. A careful analysis leads then to the following expression for the Casimir energy

$$E_0\left(\ell,\eta,m\right) = L^2 \int \frac{d^2 p_\perp}{\left(2\pi\right)^3}\int_0^{1/\eta} dq \frac{1}{\sqrt{1-\eta^2 q^2}} \log\left(1+\frac{K_1\left(z\right)}{K_2\left(z\right)}\right)\,. \tag{3.9}$$

If the boundary conditions are given by those of Dirichlet, we know that $K\left(z\right) = G\left(iz\right) = \sin\left(z\right)$. Then making use of the relation $\sin\left(iz\right) = i\sinh\left(z\right)$, we can easily identify the divergent and convergent parts of $K = K_1 + K_2$, and we write

$$E_0\left(\ell,\eta,m\right) = L^2 \int_0^{1/\eta} \frac{dq}{\sqrt{1-\eta^2 q^2}} \int_0^{\infty} \frac{dp_\perp}{\left(2\pi\right)^2} p_\perp \log\left(1-\mathrm{e}^{-2\ell\sqrt{p_\perp^2+q^2+m^2}}\right)\,. \tag{3.10}$$

These result can be easily adapted for the case of κ-deformed Maxwellian electromagnetism. Equation (3.10) can be recast into a more illuminating form by writing

$$E_0(\ell,\eta,m) = \frac{L^2}{(2\pi)^2}\int_0^{1/\eta}\frac{dq}{\sqrt{1-\eta^2 q^2}} I\left(q^2\right)\,, \tag{3.11}$$

where

$$\begin{aligned} 4\ell^2\sqrt{q^2+m^2}I\left(q^2\right) &= -2\mathrm{Li}_2\left(\mathrm{e}^{-2\ell\sqrt{q^2+m^2}}\right)\ell q^2 - \mathrm{Li}_2\left(\mathrm{e}^{-2\ell\sqrt{q^2+m^2}}\right)\ell m^2 \\ &- \sqrt{q^2+m^2}\mathrm{Li}_2\left(\mathrm{e}^{-2\ell\sqrt{q^2+m^2}}\right)\,. \end{aligned} \tag{3.12}$$

As a check we can set $\eta \to 0$ and also $m \to 0$. The integrals are easily manipulated and final result agrees with the standard Casimir energy for a massless scalar field. In the massless case, equation (3.10) can be also manipulated to yield

$$E_0(\ell,\eta,m=0) = -\frac{L^2}{4\pi^2\ell^3}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^{1/\eta} dy\left(1+\frac{1}{2n}\right)\frac{\mathrm{e}^{-2ny}}{\sqrt{1-\left(\frac{ny}{\ell}\right)^2}}\,, \tag{3.13}$$

in agreement with [6].

4. Conclusions

Here we have sketched the employment of a simple but effective technique of evaluating the Casimir energy associated with non-trivial dispersion relations. This technique can be also employed to evaluate photon emission rates associated with κ-deformed scalar fields. Extensions to more than one constrained dimension and finite temperature are presently being considered.

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