PROCEEDINGS OF SCIENCE

On the Integrability and Chaos of an N=2 Maxwell-Chern-Simons-Higgs Mechanical Model

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We apply different procedures to analyse integrability in a reduced (spatially homogeneous) mechanical system derived from an off-shell non-minimally coupled N=2-D=3 Maxwell-Chern-Simons-Higgs model. The possibility of chaotic dynamics in the evolution of the spatially homogeneous reduced system is considered, as one conjectures its counterpart rôle with respect to static topological excitations present in the spectrum of the full N=2 model. The analysis is performed by means of global integrability Noether point symmetries and Painlevé criteria, chosen to probe both the general and the critical coupling regimes. These analytical methods point to a non-integrable behavior, which is further explored as we seek for chaotic patterns through numerical simulations.

Fifth International Conference on Mathematical Methods in Physics — IC2006
April 24-28 2006
Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro, Brazil

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1. Introduction and Motivation

The dynamical properties of gauge systems appear as a topic of renewed interest, as one can realize from the remarkable efforts recently devoted to the analysis of the stability of gauge field configurations [1]. Being a cornerstone of theoretical physics, gauge theories have been intensively investigated; we may say that we presently have a comprehensive, but non-exhaustive, picture of such a successful theoretical framework. One (promising) aspect of such systems is the room for coherent soliton solutions, that may play a crucial rôle in the understanding of physical phenomena like, for instance, quark confinement. On the other hand, the search for chaotic regime windows in gauge theories seems to be as much important as the former analysis; it sets up a stimulating approach that may lead to answers to key-problems, like for instance (and again) the confinement phenomenon [1, 2, 3]. In the eighties, a method for investigating chaos in field theories has been developed and applied to Yang-Mills systems [2, 4]. The main idea is to reduce the model to its mechanical limit, by considering *spatially homogeneous* field configurations. The verification of chaotic evolution in this restricted regime is conjectured to be sufficient to ensure chaotic behavior for the full field theory [2, 5].

In this context, a relevant matter is the possibility of establishing a systematic procedure to clarify the relationship between gauge symmetries and the control of chaotic dynamics. In the framework of gauge field theories, one should eventually care about *supersymmetric systems*, considered either as manifestations of a more fundamental symmetry or enriched models conceived to be a tool to better describe physical situations. It is advisable to stress that our work has been motivated by a question stated some years ago [6], which, for the time being, has not been answered; namely, whether or not supersymmetry would have a stabilizing rôle for those field theories that, in their non-supersymmetric version, display chaotic behavior.

Planar (2+1) Maxwell-Chern-Simons-Higgs (MCSH) theories, as candidates for an effective description of high-T_c superconducting phenomena, have recently been chosen to be elegible models in considering order-to-chaos transition studies. In this scenario, Bambah *et al.* [7] have considered both the (proven to be) integrable minimally coupled Chern-Simons-Higgs (CSH) model and its higher momenta natural extension, namely, the minimally coupled Maxwell-Chern-Simons-Higgs system. The latter failed when submitted to an integrability criterion, the Painlevé test, leaving room for a chaotic regime that happened to be confirmed by numerical Lyapunov exponents and phase plots analysis. Recently, Escalona *et al.* [8] have performed a similar work in a MCSH system endowed with both minimal *and non-minimal* couplings in the interaction sector. The non-minimal coupling stands for a Pauli-type term describing a field-strength/matter-current interaction, admitted in (2+1)-D regardless the spin of the matter field [9]. Moreover, if quantum extension is a claim, such a non-minimal coupling should be considered from the start [10]. In the work of Ref.[8], the extended CSH system is argued to be still integrable, while the non-minimal MCSH exhibits "alternating windows of order and chaos", as the non-minimal coupling constant g is varied, while the other parameters define a set of constant inputs. The model they adopt is the bosonic projection of an already established N=1-*supersymmetric* system. As a matter of fact, a non-minimally interacting MCSH system had formerly deserved an extension endowed with *on-shell* N=2-*susy*[11]. As far as soliton solutions are a subject of interest, the N=2 extension defines the proper framework, allowing for the self-dual regime [12, 13]. On this token, Antillón, Escalona *et al.* had found, in a previous paper [14], by working with such an $N = 2$ extendable model, a self-dual *static* non-topological vortex solution, motivating their later search for *spatially homogeneous* chaotic dynamics as an interesting counterpart of their previous result. Nevertheless, even if one assumes the validity of the conjecture that relates the mechanical limit to the full theory, a problem is immediately raised: the vortex has been found in an N=2-*susy* framework, *while the procedure of varying g, adopted in [8], necessarily drives the system out of the N=2-susy-bosonic projection situation*. As clearly assumed in [11], a *critical coupling*, namely, $g = -e/\kappa$, has to be verified to ensure on-shell N=2-*susy*, where e is the minimal coupling constant and κ is the Chern-Simons mass parameter. Moreover, the scalar potential is forbidden to be anything but the non-topological mass-like ϕ^2 term. So, varying g, while keeping e and κ constant, and adopting $V = \lambda(\phi^2 - v^2)^2$ render their model a sector of, at most, an N=1 system.

Alternatively, another planar N=2 non-minimal MCSH model has been recently proposed [15], defining a richer spectrum that presents both non-topological *and topological* self-dual static vortex solutions [16]. These solutions are numerically obtained after the adoption of the critical coupling relation. Such a system exhibits *off-shell-realized N=2-susy*, and it is derived from an N=1-D=4 ansatz, after dimensional reduction is carried out and a suitable N=2-covariant superfield identification is implemented. Two important differences arise, if one compares both versions of the non-minimal MCSH models: in the N=2-off-shell case, there appears an “additional" [1] neutral scalar field ; also, in the N=2-off-shell case, *no relation between the coupling constants and parameters* is required to ensure N=2-*susy* (though the vortex excitations have so far been shown to prevail in the particular $g = -e/\kappa$ regime). In other words, if the model of Ref.[15] is considered, the strategy of varying g freely and the presence of a topologically non-trivial scalar potential happen to be compatible with N=2-*susy*.

Motivated by the features reported above, we carry out the analysis of the reduced (mechanical) version for the Lagrangian extracted from the Ref.[16] for the bosonic sector. In Section 2, we present the theory and its spatially homogeneous version, the one-dimensional effective Lagrangian. The associated conjugate momenta and a constant of motion are then taken into account, leading to a suitable reparametrization and, ultimately, to the proper Hamiltonian and canonical equations of motion. In Section 3, we move back to a (general regime) second-order formulation, and we start up our analysis of integrability; we adopt two alternative analytical criteria - the Noether point symmetries approach [17] and the Painlevé test procedure [19, 20]. Both present negative results as one seeks for a global integrability pattern. In Section 4, we adopt the critical coupling relation regime, and we arrive at *first-order* equations for the gauge degrees of freedom. We re-consider the Painlevé test, which indicates that the critical coupling regime shows up an even worse feature concerning the presumed strict negativeness of the dominant exponents. Revisiting the Noether point symmetries approach also gives no clues on possible integrable setups. In Section 5, an analysis of chaos is performed with physically acceptable values of the parameters. Finally, we present our Concluding comments.

[1] A common, improper, terminology. The “minimum" content forbids interesting excitations, like topological vortices.

2. Describing the Model

We start off [16] with:

$$\mathscr{L}_{\text{boson}} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}\partial_\mu M \partial^\mu M + \frac{1}{2}(\nabla_\mu\phi)(\nabla^\mu\phi)^* + \\ -\frac{g}{2}(\partial_\mu M)(\partial^\mu|\phi|^2) + \frac{\kappa}{2}A_\mu\tilde{F}^\mu - U, \tag{2.1}$$

where

$$U = \frac{e^2}{8G}\left(|\phi|^2 - v^2 + \frac{2m}{e}M + 2g|\phi|^2 M\right)^2 + \frac{e^2}{2}M^2|\phi|^2 ,$$

and $\nabla_\mu\phi \equiv (\partial_\mu - ieA_\mu - ig\tilde{F}_\mu)\phi$. G is defined as $G \equiv 1 - g^2|\phi|^2$.

Adopting the gauge choice $A_0 = 0$ and imposing the spatial homogeneity, namely, $\partial_i(\forall\, field) = 0$, the phase of the scalar field becomes a variable with vanishing time-derivative, and one can eliminate it without loss of generality. So, one ends up with a real scalar field. Also, as we start from the field equations obtained by means of the extremization of (2.1), an effective Lagrangian for the mechanical system can be derived:

$$L = \frac{G}{2}\left[(\dot{A}_1)^2 + (\dot{A}_2)^2\right] - \frac{Q}{2}\left(A_1\dot{A}_2 - A_2\dot{A}_1\right) - \frac{e^2\phi^2}{2}\left({A_1}^2 + {A_2}^2\right) + \\ + \frac{(\dot{M})^2}{2} - g\phi\dot{\phi}\dot{M} + \frac{(\dot{\phi})^2}{2} - \frac{e^2(\phi^2 - v^2 + (2\kappa/e)M + 2gM\phi^2)^2}{8G} - \frac{e^2\phi^2 M^2}{2} \quad ,$$

where $G \equiv 1 - g^2\phi^2$ and $Q \equiv \kappa + 2eg\phi^2$. The corresponding equations of motion coincide with the reduced version of those obtained from (2.1).

The canonically conjugate momenta, defined, as usually, by $p = \frac{\partial L}{\partial \dot{q}}$, have the expressions:

$$\pi_1 = G\dot{A}_1 + \frac{Q}{2}A_2 \; ; \; \pi_2 = G\dot{A}_2 - \frac{Q}{2}A_1 \; ; \; p_\phi = \dot{\phi} - g\phi\dot{M} \; ; \; P_M = \dot{M} - g\phi\dot{\phi} \; .$$

Before we proceed to the canonical Hamilton equations, let us notice that the quantity

$$I \equiv A_2\pi_1 - A_1\pi_2 \tag{2.2}$$

is a constant of motion. Motivated by this fact we reparametrize the gauge sector adopting polar coordinates, instead of Cartesian ones. We have: $A_1 = Acos\zeta$, $A_2 = Asen\zeta$, and the “new" set of variables is (A,ζ,ϕ,M). The Lagrangian now reads:

$$L = \frac{G}{2}\left[(\dot{A})^2 + A^2(\dot{\zeta})^2\right] - \frac{Q}{2}\left(A^2\dot{\zeta}\right) + \frac{(\dot{M})^2}{2} - g\phi\dot{\phi}\dot{M} + \frac{(\dot{\phi})^2}{2} \\ -\frac{e^2\phi^2A^2}{2} - \frac{e^2(\phi^2 - v^2 + (2\kappa/e)M + 2gM\phi^2)^2}{8G} - \frac{e^2\phi^2M^2}{2} \quad , \tag{2.3}$$

yielding the same expressions for p_ϕ and P_M, and bringing about $p_A \equiv \frac{\partial L}{\partial \dot{A}} = G\dot{A}$, $p_\zeta \equiv \frac{\partial L}{\partial \dot{\zeta}} = GA^2\dot{\zeta} - \frac{Q}{2}A^2$. One can easily check that $p_\zeta = -I$, resulting $\dot{p}_\zeta = 0$. The corresponding Hamiltonian follows:

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$$H_{CAN.} = \frac{1}{2G}\left[{p_A}^2 + \frac{{p_\zeta}^2}{A^2} + Qp_\zeta + {p_\phi}^2 + {P_M}^2 + 2g\phi p_\phi P_M\right] +$$
$$+\frac{1}{2G}\left[(Q/2)^2 + e^2G\phi^2\right]A^2 + \frac{e^2\phi^2M^2}{2} + \frac{e^2}{8G}\left(\phi^2 - v^2 + (2\kappa/e)M + 2g\phi^2M\right)^2 \quad ,$$

and the canonical equations of motion may be written down:

$$\dot{A} = \frac{p_A}{G} \; ; \; \dot{p}_A = \frac{1}{G}\left[\frac{{p_\zeta}^2}{A^3} - \left((Q/2)^2 + e^2G\phi^2\right)A\right] \; ; \; \dot{\zeta} = \frac{1}{G}\left[\frac{p_\zeta}{A^2} + \frac{Q}{2}\right] \; ; \; \dot{p}_\zeta = 0 \; ;$$

$$\dot{p}_\phi = -\frac{1}{G^2}\Bigg\{g^2\phi\left[{p_A}^2 + \frac{{p_\zeta}^2}{A^2} + {p_\phi}^2 + {P_M}^2\right] + g\phi(\kappa g + 2e)p_\zeta + g(1 + g^2\phi^2)p_\phi P_M$$
$$+(\kappa g + 2e)^2\frac{\phi A^2}{4} + e^2G^2\phi M^2 + \frac{e^2g^2\phi}{4}\left(\phi^2 - v^2 + (2\kappa/e)M + 2gM\phi^2\right)^2$$
$$+ \frac{e^2G\phi}{2}\left(\phi^2 - v^2 + (2\kappa/e)M + 2gM\phi^2\right)(1 + 2gM)\Bigg\} \quad ;$$

$$\dot{\phi} = \frac{1}{G}\left[p_\phi + g\phi P_M\right] \; ; \; \dot{M} = \frac{1}{G}\left[P_M + g\phi p_\phi\right] \; ;$$

$$\dot{P}_M = -\frac{1}{G}\left[e^2G\phi^2M + \frac{e^2}{4}\left(\phi^2 - v^2 + (2\kappa/e)M + 2gM\phi^2\right)\left(\frac{2\kappa}{e} + 2g\phi^2\right)\right] \quad . \tag{2.4}$$

3. Integrability Analysis: General Case

We present two Lagrangian analytical criteria to address the issue of integrability: Noether point symmetries, better suited for establishing the constants of motion, and the Painlevé test, taylored to check for an overall property (the dependent variables being meromorphic for movable singularities on the complex time plane) that indicates integrability.

3.1 Noether point symmetries

An important issue regarding a Lagrangian system concerns its Noether point symmetries, linking symmetries of the action to conserved quantities. Here we address the question of Noether point symmetries in our system following the method shown in reference [17].

We seek for infinitesimal point transformations of the form

$$\bar{A} = A + \varepsilon\eta_A \; ; \; \bar{\zeta} = \zeta + \varepsilon\eta_\zeta \; ; \; \bar{\phi} = \phi + \varepsilon\eta_\phi \; ; \; \bar{M} = M + \varepsilon\eta_M \; ; \; \bar{t} = t + \varepsilon\tau \, ,$$

for $\eta_A, \eta_\zeta, \eta_M, \eta_\phi$ and τ functions of the fields and the time, and ε an infinitesimal parameter. These infinitesimal transformations leave the action functional invariant up to the addition of an irrelevant numerical constant if and only if the following Noether symmetry condition [17] is satisfied,

$$\begin{aligned}\tau\frac{\partial L}{\partial t} &+ \eta_A\frac{\partial L}{\partial A}+\eta_\zeta\frac{\partial L}{\partial \zeta}+\eta_\phi\frac{\partial L}{\partial \phi}+\eta_M\frac{\partial L}{\partial M}+(\dot{\eta}_A-\dot{\tau}\dot{A})\frac{\partial L}{\partial \dot{A}}+\\ &+ (\dot{\eta}_\zeta-\dot{\tau}\dot{\zeta})\frac{\partial L}{\partial \dot{\zeta}}+(\dot{\eta}_\phi-\dot{\tau}\dot{\phi})\frac{\partial L}{\partial \dot{\phi}}+(\dot{\eta}_M-\dot{\tau}\dot{M})\frac{\partial L}{\partial \dot{M}}+\\ &+ \dot{\tau}L=\dot{F}\,, \end{aligned}\tag{3.1}$$

F being a function of the fields and the time. If such a function can be found, there is a Noether point symmetry and an associated Noether invariant I given by

$$I=\eta_A\frac{\partial L}{\partial \dot{A}}+\eta_\zeta\frac{\partial L}{\partial \dot{\zeta}}+\eta_\phi\frac{\partial L}{\partial \dot{\phi}}+\eta_M\frac{\partial L}{\partial \dot{M}}-\tau(\dot{A}\frac{\partial L}{\partial \dot{A}}+\dot{\zeta}\frac{\partial L}{\partial \dot{\zeta}}+\dot{\phi}\frac{\partial L}{\partial \dot{\phi}}+\dot{M}\frac{\partial L}{\partial \dot{M}}-L)-F\,. \tag{3.2}$$

In the Noether symmetry condition, the time derivatives are to be understood as total derivatives, e.g.,

$$\dot{\tau}=\frac{\partial \tau}{\partial A}\dot{A}+\frac{\partial \tau}{\partial \zeta}\dot{\zeta}+\frac{\partial \tau}{\partial \phi}\dot{\phi}+\frac{\partial \tau}{\partial M}\dot{M}+\frac{\partial \tau}{\partial t}\,. \tag{3.3}$$

Inserting the Lagrangian given by (2.3) in (3.1), we obtain that a cubic polynomial in the velocities must vanish. The coefficients of equal powers of velocities vanishing, we obtain a coupled set of linear partial differential equations determining both the symmetries and the Noether invariants. The cubic terms yield simply

$$\tau=\tau(t)\,, \tag{3.4}$$

that is, the transformed independent variable is a function of time only. The equations associated to quadratic terms, however, are a set of ten coupled partial differential equations which, apparently, do possess a closed form solution only in the minimal coupling case $g=0$. Restricting the treatment to this almost trivial case, and proceeding to the first and zeroth order terms, we just found time-translation and ζ translation symmetries. These symmetries are linked, according to (3.2), to the energy and p_ζ conservation laws. These are almost obvious results, showing that the nonlinearity and coupling in the potential gives no much space for the existence of conservation laws of the system, even in the $g=0$ case. This is a signature of non-integrability. However, other methods for investigating conservation laws of the system like Lie point symmetries [18] were not used.

3.2 Painlevé test

We now go back to a second-order configuration space formalism in order to settle the framework for the application of Painlevé test [19, 20]. The iterated equations stemming from the set (2.4) are cast below:

$$G^2\ddot{A} = 2g^2G\phi\dot{\phi}\dot{A}+\frac{C^2}{A^3}-(\frac{\kappa}{2})^2A-e(\kappa g+e)\phi^2A\text{ , where }C\equiv p_\zeta\,, \tag{3.5}$$

$$\begin{aligned}G^3\ddot{\phi} &= g^2G^2\phi\dot{\phi}^2-g^2G^2\phi\dot{A}^2-\frac{g^2C^2\phi}{A^2}-gC(\kappa g+2e)\phi-\frac{1}{4}(\kappa g+2e)^2\phi A^2+\\ &+ \frac{e\phi}{4}(\phi^2-v^2)\left[eg^2v^2-2(\kappa g+e)+g^2(2\kappa g-e)\phi^2+2eg^4\phi^4\right]\\ &- (\kappa g+e)\left[\kappa-ev^2g+(3eg-\kappa g^2)\phi^2-2eg^3\phi^4+(\kappa g+e)M\right]\phi M\end{aligned}\tag{3.6}$$

$$
\begin{aligned}
\text{and}\quad G^3\ddot{M} &= gG^2\dot{\phi}^2 - g^3G^2\phi^2\dot{A}^2 + \frac{g^3C^2\phi^2}{A^2} - \frac{g}{4}(\kappa g+2e)^2A^2\phi^2 - g\,(\kappa g+e)^2M^2\phi^2 + \\
&+ \frac{e}{4}(\phi^2-v^2)[-2\kappa+g\,(-4e+eg^2v^2+2\kappa g)\phi^2+3eg^3\phi^4] + \\
&+ [-\kappa^2+(\kappa^2g^2+e\kappa g^3v^2-e^2-3e\kappa g+e^2g^2v^2)\phi^2+eg^2(2\kappa g-e)\phi^4+e^2g^4\phi^6]M\,. \qquad (3.7)
\end{aligned}
$$

Assuming time to be a complex variable, the first step of the Painlevé test is concerned with the leading singularity behavior. One supposes the leading terms to be of the general form $A \sim a(t-t_0)^{\alpha}, \phi \sim b(t-t_0)^{\beta}, M \sim c(t-t_0)^{\gamma}$, where $\alpha,\beta,\gamma<0$. Such an assumption turns the last three equations into the following asymptotic ($t \to t_0$) relations (with $\tau = t-t_0$):

$$g^4ab^4\alpha\,[\alpha-1+2\beta]\,\tau^{\alpha+4\beta-2} \sim 0 \;;$$

$$g^6b^7\beta\,[2\beta-1]\,\tau^{7\beta-2} \;\sim\; g^6b^5a^2\alpha^2\tau^{2\alpha+5\beta-2} \;-\; 2eg^3(\kappa g+e)b^5c\tau^{5\beta+\gamma} \;+\; (\kappa g+e)^2bc^2\tau^{\beta+2\gamma} \;;$$

$$c\gamma(\gamma-1)\tau^{4\beta+\gamma-2} \sim gb^2\beta(2\beta-1)\tau^{6\beta-2} \;,$$

Starting from the last equation, one gets $\gamma = 2\beta$ and $c = gb^2/2$. Inserting $\gamma = 2\beta$ in the second equation reduces the balancing to the first two terms, so leading to $\alpha = \beta$ and $a^2 = (2\beta-1)b^2/\beta$. But the first equation shows the impossibility of having $\alpha,\beta<0$, as $\alpha+2\beta-1=0$ is required, spoiling the Painlevé test procedure. Another possibility would be to set $\alpha = 0$ in the first equation, leaving it behind as an identity. One could then drop the second term (first on right-hand side) of the second equation, and the balancing of the remaining three terms would lead to an interesting set of negative values for γ and β: $\gamma = -4$, $\beta = -1$, provided that the following relation holds:

$$(\kappa g+e)^2c^2 - 2(\kappa g+e)eg^3b^4c + 3g^6b^6 = 0\,.$$

Still, one has to deal with a zero “dominant" exponent, which spoils the Painlevé test.

As one faces a problem with the gauge sector dynamics, the adoption of the critical coupling relation (vortex excitations have been established for such a coupling) may serve as a valuable tool of investigation. In fact, $g = -e/\kappa$ leads to first-order equations for the gauge field [16].

4. Critical coupling regime

If $g = -e/\kappa$, one gets $\kappa\tilde{F}_\nu = -\mathscr{J}_\nu$, and the reduction to spatially homogeneous configurations yields

$$\kappa G\mathscr{E}_{ij}\dot{A}_j = -e^2A_i\phi^2 \;, \qquad (4.1)$$

where $\mathscr{E}_{12} = +1 = -\mathscr{E}_{21}$. From this set of equations one can arrive at $G\frac{d}{dt}(A_1^2+A_2^2) = 0$, and, as far as $G>0$ (a condition inherited from the original N=2-*susy* framework), this implies that $A_1^2+A_2^2$ is a constant of motion (thus reproducing the “pure" minimally coupled Chern-Simons-Higgs situation). Adopting polar coordinates, $A_1 = C cos\,\zeta$, $A_2 = C sen\,\zeta$ (so $A_1^2+A_2^2 = C^2$), and manipulating the set (4.1), one finds $\dot{\zeta} = -e^2\phi^2/\kappa G$. Following the same route chosen in the general (non-critical) case, we seek for the effective Lagrangian and Hamiltonian, settle the canonical equations of motion and, as we aim at the Painlevé test for integrability, iterate them to get second-order coupled differential equations. One can easily verify that the following Hamiltonian leads to the proper set of field equations:

$$H_{CAN.} = \frac{1}{2G}\left[\frac{{p_\zeta}^2}{A^2} + Qp_\zeta + {p_\phi}^2 + {P_M}^2 + 2g\phi p_\phi P_M\right] + \\ + \frac{1}{2G}\left[(Q/2)^2 + e^2G\phi^2\right]A^2 + \frac{e^2\phi^2M^2}{2} + \frac{e^2}{8G}\left(\phi^2 - v^2 + (2\kappa/e)M + 2g\phi^2M\right)^2 \quad ,$$

where $g = -e/\kappa$, $G = 1-(e^2/\kappa^2)\phi^2$, and $p_\zeta = -\kappa C^2/2$ [2]. The system of iterated second-order equations may readily be derived from $H_{CAN.}$. We do not quote them here.

Again, for the Painlevé test, the asymptotic relations are found: $\phi:\ \beta = 0$ or $\beta = 1/2$. If one takes the $\beta = 0$ case, one is left with two problematic outputs, as the equation for M is considered: either $\gamma = \beta = 0$, or $\beta = 0$, $b^2 = 1/g^2$, γ undetermined. So the signature of lack of strong Painlevé property remains.

5. Analysis of chaos

Since the analytical approaches suggest that the system may not be integrable, we now turn to a numerical study to verify if such a non-integrability feature is presented in a chaotic form.

5.1 SALI method

The most well–known method used to detect whether a system is chaotic or not is the maximal Lyapunov Characteristic Exponent (LCE), σ_1. If $\sigma_1 > 0$ the flow is chaotic. The σ_1 is computed [21, 22] from

$$L_t = \frac{1}{t}\ln\frac{|\vec{w}(t)|}{|\vec{w}(0)|}\ , \text{ performing the limit } \quad \sigma_1 = \lim_{t\to\infty} L_t\,,$$

where $\vec{w}(0)$, $\vec{w}(t)$ are deviation vectors and the time evolution of $\vec{w}$ is given by solving the *equations of motion* and associated *variational equations*.

Since these vectors tend to acquire an exponential growth in short time intervals, many calculations of L_{T_1}, as $\vec{w}(t)$ evolves for a short time t_1, are carried out after each $\vec{w}(t)$ is normalized. With this procedure, the mean value of L_{T_1} is computated as

$$\sigma_1 = \frac{1}{N}\sum_{i=1}^{N} L_{T_i}$$

For Hamiltonian sytems, this computation becomes very lengthy with poor convergence, and this long procedure may point to a false chaos diagnosis.

[2] In fact, such a relation between the conserved quantities p_ζ and A^2 ($\equiv C^2$) must be imposed to ensure first-order $\dot{\zeta} = -e^2\phi^2/\kappa G$ equation of motion

We have chosen to adopt the method developed by Skokos, Antonopoulos, Bountis and Vrahatis, the so-called Smaller Alignment Index, SALI, for brevity[23, 24]. The reason for this choice is that the SALI method is computationally faster and less unstable than the Lyapunov exponent analysis, improving the adequacy of the former for the system we investigate. The SALI is a indicator of chaos that tends to zero for chaotic orbits, while it exhibits small fluctuations around non-zero values for ordered ones. So, the SALI is defined as:

$$\mathrm{SALI}(t) = \min\left\{ \left\| \frac{\vec{w}_1(t)}{\|\vec{w}_1(t)\|} + \frac{\vec{w}_2(t)}{\|\vec{w}_2(t)\|} \right\|, \left\| \frac{\vec{w}_1(t)}{\|\vec{w}_1(t)\|} - \frac{\vec{w}_2(t)}{\|\vec{w}_2(t)\|} \right\| \right\}, \tag{5.1}$$

where $\vec{w}_1(t)$ and $\vec{w}_2(t)$ are the evolutions of two deviations vectors with different initial conditions, $\|\cdot\|$ is the Euclidean norm and t is the time. The authors of SALI method showed that SALI can be approximated by means of the difference of the two largest Lyapunov characteristic exponents, σ_1 and σ_2.

The main advantage of the SALI in chaotic regions is that it uses two deviation vectors and exploits at every step the convergence from all previous steps. The SALI value tends to zero for chaotic flows at a rate which is a function of the difference of the two largest Lyapunov characteristic exponents σ_1, σ_2: SALI $\propto e^{-(\sigma_1-\sigma_2)t}$. As usually done in numerical computations, we need to define a threshold to indicate that a computed number must be considered null. In most situations, the selected value is $< 10^{-5}$. So, similarly to the Lyapunov exponent analysis, such a criterion shall here be used, in the context of the SALI method, to distinguish between order and chaos.

5.2 Equations of motion and requirements

The integration of the system was performed by means of a Gear algorithm, in a variable step mode, starting from the minimal step size h = 0.0001, and eventually getting reduced in order to preserve the value of the Hamiltonian and p_ζ, known to be constants of motion. Another constraint maintained along the integration was that G ($G \equiv 1 - g^2\phi^2$) should be greater than zero. The first order equations of motion used in the numerical integration are those quoted in (2.4).

For the sake of simplicity, we have adopted the following notation:

$\zeta = q_1,\ A = q_2,\ \phi = q_3,\ M = q_4,\ p_\zeta = p_1, p_A = p_2,\ p_\phi = p_3,\ P_M = p_4.$

For each set of parametric inputs the numerical integration was performed and SALI method used after transient damped. In the following, we display representative samples of our findings.

5.3 Case with g=0

Since our model comes from a supersymmetric version of Maxwell-Chern-Simons-Higgs with non-minimal coupling, it is not clear whether we shall recover dynamical properties similar to the ones observed in other studies [8, 7] when $g = 0$ is adopted, and the model is so reduced to the minimal coupling case. The two main properties found in the cited papers were the existence of chaos in the presence of Maxwell term and the asymptotic evolution - order versus chaos - sensibility to initial conditions. To verify these properties in our model, we have chosen the initial conditions defining a fixed point of the system, and then we varied q_3 from 0 to 2 with parameters set as $e = 2$, $k = 2$, $v = 2$, and as mentioned above, $g = 0$. The initial conditions are $q_1(0) = 1$, $q_2(0) = 1$, $q_4(0) = 2$, $p_1(0) = 0$, $p_2(0) = -1$, $p_3(0) = 0$ and $p_4(0) = 0$. The results are presented in Figure 1, where we display a graph of SALI as a function of q_3.

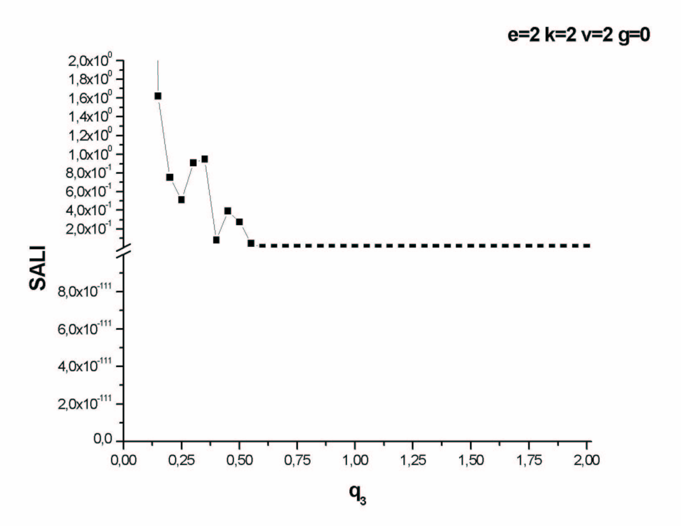

Figure 1:

We can see from Figure 1, with the help of the break in the vertical axis, that for $q_3 \gtrsim 0.55$ the behavior of the system becomes chaotic with SALI assuming values between 10^{-89} and 10^{-33}.

5.4 Case with g $\neq 0$ but outside critical coupling regime

Now, we use $g \neq 0$, but outside critical coupling regime, to check whether the inclusion of this kind of coupling may turn some configuration dynamics into chaotic, with initial conditions that, in the case $g = 0$, lead to regularity. To do that, we fix the parameters and the initial conditions as $e = 2$, $k = 2$, $v = 2$ and $q_1(0) = 1$, $q_2(0) = 1$, $q_3(0) = 0.25$, $q_4(0) = 2$, $p_1(0) = 0$, $p_2(0) = -1$, $p_3(0) = 0$ and $p_4(0) = 0$ and we analyze the model with the following values of g: 0.1, 0.7, 1.5 and 2.5. For all these cases, the behavior remained the same, indicating that the variation of g does not change the behavior of the system from regular to chaotic.

5.5 Case with g in the critical coupling regime

Now, we explore the critical coupling regime where $g = -\dfrac{e}{k}$. We fix the parameters $k = 2$, $v = 2$ and vary e , keeping the initial conditions as a perturbation case of the fixed point; but, in this case, we shall have a different set of initial conditions for each e, since the general expression for fixed point element $p_4(0)$ depends on e. With this in mind, we keep the same values for $q_1(0) = 1$, $q_2(0) = 1$, $p_1(0) = 0$, $p_2(0) = -1$, $p_3(0) = 0$, $p_4(0) = 0$ and set $q_3(0) = 0.7$, a value that makes the system chaotic in the case $g = 0$ and $q_4(0) = e$. In this case, $q_3(0) = 0.7$ is the perturbation, since in the fixed point $q_3(0) = 0.0$. As in the case g=0, we plotted a graph of e versus SALI (figure 2). In the SALI graphs, we break the vertical axis to show that the minimal SALI values are above the threshold of chaos, according to the expectations of the SALI method, that is, SALI $< 10^{-5}$.

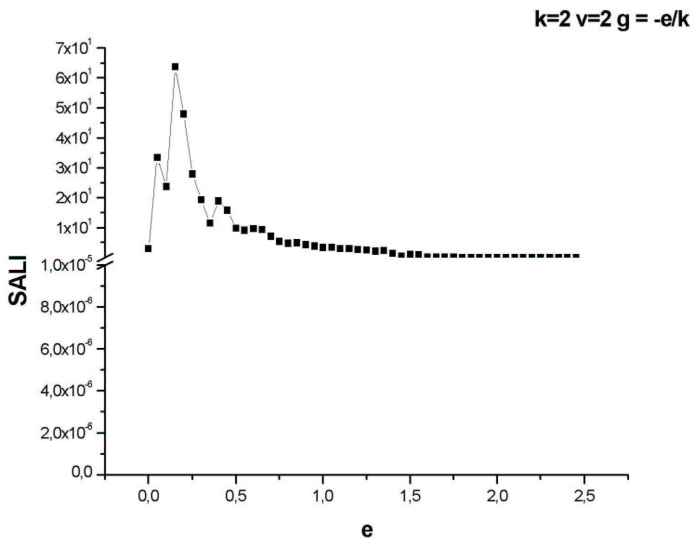

Figure 2:

6. Concluding comments

The comparison between the integrable "pure Chern-Simons" system with critical coupling, presented in the work of Ref.[8], and our extended N=2-*susy* descent model indicates that the extra *susy* is responsible [25] for the global non-integrability (in the strong Painlevé sense) situation found even in the C.S.-like regime.

In studying chaos, two main possibilities have been checked. First, we have verified if non-vanishing values of g were able to drive the system (previously with initial conditions and parameters such that regularity was achieved for $g = 0$) into a chaotic regime. The second point we have tackled concerns the opposite effect, namely, if a configuration which is chaotic for $g = 0$ may become regular whenever g becomes non-trivial.

In the case of a non-critical coupling, as g increases from zero, a configuration that is stable for $g = 0$ keeps its stability as g varies. In those cases, the difference is that, for larger values of g, the system becomes slightly more unstable, but its dynamics is still regular. Something similar happens for configurations that exhibit chaos for $g = 0$. In such cases, the system remains chaotic, but a little more unstable. For critical coupling, orbits that were chaotic for $g = 0$ become now regular; this may indicate that the critical coupling plays the rôle of a stabiliser of our model. These results may be interpreted on the basis of two points:

1. For the critical coupling regime, there occurs a partial decoupling between the variables ϕ and M, and this reduces the non-linearity of the system.

2. The quantity $G(G = 1 - g^2\phi^2)$ must be positive, with $0 < G \leq 1$. This must be so in order to ensure positivity of the energy, and the existence of a stable ground state. In Eqs. (3.5),(3.6) and (3.7), G accompanies all terms with time derivatives, and for large enough g or ϕ, G becomes small, rendering the algebraic sectors of these equations dominant. This fact may have a stabilising consequence, implying that, in the case of a non-critical coupling, the

dynamics for $g \neq 0$ is not that different from the case with $g = 0$. In the critical coupling regime, the stabilising effect could be a combination of both arguments just presented.

It is also noteworthy to point out that, for negative values of G, the phase space volume is no longer conserved, and, as a consequence, we do not have any longer a Hamiltonian system. For this reason, the results reported above make sense only if the initial conditions and the parameters ensure the positivity of G. Finally, as far as configurations whose behavior is more regular than the ones found in Ref.[8] (as g increases) show up in this model, we wonder whether special physical conditions - extended supersymmetry in the original system - might be responsible for this stabilising effect.

As we have already stated in the Introduction, we address here the question as to whether supersymmetry would have, or not, a stabilizing rôle for those field theories that, in their non-supersymmetric version, display chaotic behavior [6]. This question may actually be interpreted in two ways: (i) Should supersymmetric theories be less chaotic than their ordinary (non-supersymmetric) counterparts? (ii) Could it happen that, under specific conditions, supersymmetry renders regular a theory that, in its non-supersymmetric version, displays chaos? Our work sets out to answer these two questions. In our study, we point out that, in a supersymmetric scenario, the parameters of the theory are more severely constrained, which contributes to render the supersymmetric theory less integrable than the originally non-supersymmetric version. The loss of freedom to play with the parameters also implies that a chaotic ordinary theory should generally persist as such even after supersymmetry is brought about. In addition, we also answer the second question when we show that, in the very special critical coupling regime (a characteristic of supersymmetric theories), our model leads to a regularization of the dynamics under consideration, suppressing its chaotic behavior.

Acknowledgments

L.P.G. De Assis is grateful to FAPERJ-Rio de Janeiro for his post-doctoral fellowship. F. Haas and J.A.H.-Neto express their gratitude to CNPq-Brazil for the invaluable financial support. A.L.M.A. Nogueira thanks Lab. de Ciências Físicas - LCFIS/CCT/UENF-Campos dos Goytacazes and FAT/UERJ-Resende (where part of this work has been carried out) for the kind hospitality.

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