PROCEEDINGS OF SCIENCE

On the Generalized Rainich Algebra in Scalar-Tensor Gravities

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We obtain exact solutions for a static and charged cosmic string in a Einstein-Maxwell-Dilaton theory of a scalar-tensor type in (3+1)-Dimensions. This theory is specified by the dilaton field ϕ, the graviton field $g_{\mu\nu}$ and the electromagnetic field $F_{\mu\nu}$, and one post-Newtonian parameter $\alpha(\phi)$. It contains three different cases, each of them corresponding to a particular solution of the Rainich algebra for the Ricci tensor.

Fifth International Conference on Mathematical Methods in Physics — IC2006
April 24-28 2006
Centro Brasilerio de Pesquisas Fisicas, Rio de Janeiro, Brazil

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PoS(IC2006)061

1. Introduction

Current-carrying strings have been considered in the framework of Einstein's gravity. Here, we intend to generalize these results considering a scalar-tensor gravity. The motivation for this relies on the fact that, theoretically, the possibility that gravity might not be fundamentally Einsteinian is gathering credence. This is in part a consequence of superstring theory [1], which is consistent in ten dimensions (or M-theory in eleven dimensions), but also the more phenomenological recent developments of "braneworld" scenarios [2, 3] have motivated the study of other gravitational theories in four-dimensions. In fact, the origin of the (gravitational) scalar field can be many: the scalar field arising from the size of the compactified internal space in the Kaluza-Klein theory; the zero mode (dilaton field) described by a symmetric second-rank tensor behaving as space-time metric at low energy level in the closed string theory; the scalar field in a brane world scenario; and more [4]. In any case, clearly, if gravity is essentially scalar-tensorial there will be direct implications on observed effects both in the small scale scenarios of alternative theories of gravity [5, 6, 7, 8] and in the large scale cosmological scenarios from modified gravity [9].

Our purpose in this work is to generalize previous results [10, 11, 12, 13] and to study static and charged cosmic strings as particular solutions of a Einstein-Maxwell-Dilaton theory of a scalar-tensor type in (3+1)-dimensions by proposing a generalization of the Rainich algebra for the Ricci tensor. We obtain three exact solutions corresponding to an electric, magnetic and "mixed" strings. This manuscript is organized as follows. In the section 2, we review briefly the model for a current-carrying cosmic string in the scalar-tensor theory. In section 3, we generalize the Rainich algebra and we obtain the three exact solutions, each of them corresponding to the metric of a current-carrying string. In section 4, we summarize our main results.

2. Current-Carrying Strings

In this section we will study the gravitational field generated by three cases of a string carrying a current. We start with the action in the Jordan-Fierz Frame

$$\mathscr{S} = \frac{1}{16\pi}\int d^4x\sqrt{-\tilde{g}}[\tilde{\Phi}\tilde{R} - \frac{\omega(\tilde{\Phi})}{\tilde{\Phi}}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\Phi}\partial_\nu\tilde{\Phi}] + \mathscr{S}_m[\psi_m, \tilde{g}_{\mu\nu}] \tag{2.1}$$

$\widetilde{g}_{\mu\nu}$ is the physical metric which contains both scalar and tensor degrees of freedom, $\widetilde{R}$ is the curvature scalar associated to it and $\mathscr{S}_m$ is the action for general matter field which, at this point, is left arbitrary. The metric signature is assumed to be (-,+,+,+).

In what follows, we will concentrate our attention to superconducting vortex configuration which arise from spontaneous breaking of the symmetry $U(1)\times U_{em}(1)$. Therefore, the action for the matter fields will be composed by two pairs of coupled complex scalars and gauge fields (φ, B_μ) and (σ, A_μ).

Also, for technical purposes, it is preferable to work in the so-called Einstein (or conformal) frame, in which the scalar and tensor degrees of freedom do not mix

$$\mathscr{S} = \frac{1}{16\pi G_*}\int d^4x\sqrt{-g}[R - 2\partial_\mu\phi\partial^\mu\phi]$$

$$+\int d^4x\sqrt{-g}\{-\frac{1}{2}\Omega^2(\phi)[(D_\mu\phi)^*(D^\mu\phi)+(D_\mu\sigma)^*(D^\mu\sigma)]$$
$$-\ \frac{1}{16\pi}(F_{\mu\nu}F^{\mu\nu}+H_{\mu\nu}H^{\mu\nu})-\Omega^2(\phi)V(|\varphi|,|\sigma|)\}\ , \tag{2.2}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu, H_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu$ and the potential is suitably chosen in order that the pair (φ,B_μ) breaks one symmetry $U(1)$ in vacuum (given rise to the vortex configuration) and the second pair (σ,A_μ) breaks the symmetry $U_{em}(1)$ in the core of the vortex (giving rise to the superconducting properties)

$$V(|\varphi,|\sigma||)=\frac{\lambda_\varphi}{8}(|\varphi|^2-\eta^2)^2+f(|\varphi|^2-\eta^2)|\sigma|^2+\frac{\lambda_\sigma}{4}|\sigma|^4+\frac{m_\sigma^2}{2}|\sigma|^2\ . \tag{2.3}$$

We restrict ourselves to the configurations corresponding to an isolated, static current-carrying vortex lying in the $z-axis$. In a cylindrical coordinate system (t,r,θ,z) such that $r\geq 0$ and $0\leq\theta<2\pi$, we make the following ansatz

$$\varphi=\varphi(r)e^{i\theta}\ \text{and}\ B_\mu=\frac{1}{q}[Q(r)-1]\delta_\mu^\theta\ . \tag{2.4}$$

The pair (σ,A_μ), which is responsible for the superconducting properties of the vortex, is set in the form

$$\sigma=\sigma(r)e^{i\chi(t)}\ ,\ A_t=\frac{1}{e}[P_t(r)-\partial_t\chi]\ \text{and}\ A_z=\frac{1}{e}[P_z(r)-\partial_z\chi]\ , \tag{2.5}$$

where $P_t(P_z)$ corresponds to the electric (magnetic) field which leads to a timelike (spacelike) current in the vortex. We also require that the functions $\varphi(r),Q(r),\sigma(r)$, $P_z(r)$ and $P_t(r)$ must be regular everywhere and must satisfy the usual boundary conditions of vortex [14] and superconducting configurations [15, 16].

The action (2) is obtained from (1) by a conformal transformation

$$\widetilde{g}_{\mu\nu}=\Omega^2(\phi)g_{\mu\nu}\ , \tag{2.6}$$

and by redefinition of the quantity

$$G_*\Omega^2(\phi)=\widetilde{\Phi}^{-1}\ . \tag{2.7}$$

This last expression makes evident the feature that any gravitational phenomena will be affected by the variation of the gravitation constant G_* in the scalar tensor gravity, and by introducing a new (post-Newtonian) parameter

$$\alpha^2=\left(\frac{\partial\ln\Omega(\phi)}{\partial\phi}\right)^2=[2\omega(\widetilde{\Phi})+3]^{-1}\ . \tag{2.8}$$

Variation of the action (2) with respect to the metric $g_{\mu\nu}$ and to the dilaton field ϕ gives the modified Einstein's equations and the wave equation for the dilaton, respectively. Namely,

$$\begin{aligned} G_{\mu\nu}\ &=\ 2\partial_\mu\phi\partial_\nu\phi-g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi+8\pi G_*T_{\mu\nu}\ ,\\ \Box_g\phi\ &=\ -4\pi G_*\alpha(\phi)T\ . \end{aligned} \tag{2.9}$$

Here, $T_{\mu\nu}$ is the energy-momentum tensor which is obtained by

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta \mathscr{S}_m}{\delta g_{\mu\nu}} \; . \tag{2.10}$$

We note, in passing, that in the conformal frame this tensor is not conserved providing us with an additional equation

$$\nabla_\mu T^\mu_\nu = \alpha(\phi) T \nabla_\nu \phi \; . \tag{2.11}$$

3. The Generalized Rainich Algebra

Due to the specific properties of the Maxwell tensor

$$T_\mu{}^\mu = 0 \text{ and } T_\nu{}^\alpha T_\alpha{}^\mu = \frac{1}{4}(T^{\alpha\beta} T_{\alpha\beta})\delta_\nu{}^\mu \; , \tag{3.1}$$

the Einstein's equations may be transformed into some algebraic relations called Rainich algebra [17] which in scalar-tensor theories are modified by a term which depends explicitly on the dilaton field

$$R = R_t{}^t + R_r{}^r + R_\theta{}^\theta + R_z{}^z = 2g^{rr}(\phi')^2 \; , \tag{3.2}$$

$$(R_t{}^t)^2 = (R_r{}^r - 2g^{rr}(\phi')^2)^2 = (R_\theta{}^\theta)^2 = (R_z{}^z)^2 \; , \tag{3.3}$$

we can see from (3.3) that there exist three possible solutions

- Case I:
$$R_t{}^t = -R_\theta{}^\theta \quad R_t{}^t = R_r{}^r - 2g^{rr}(\phi')^2 \quad R_\theta{}^\theta = R_z{}^z \tag{3.4}$$

- Case II:
$$R_t{}^t = R_\theta{}^\theta \quad R_t{}^t = -R_r{}^r + 2g^{rr}(\phi')^2 \quad R_\theta{}^\theta = -R_z{}^z \tag{3.5}$$

- Case III:
$$R_t{}^t = -R_\theta{}^\theta \quad R_t{}^t = -R_r{}^r + 2g^{rr}(\phi')^2 \quad R_\theta{}^\theta = -R_z{}^z \tag{3.6}$$

For each of these cases we find the following exterior metric, respectively

$$\begin{aligned} ds_E^2 \; = & \left(\frac{r}{r_0}\right)^{2m^2-2n} W^2(r)(dr^2 + dz^2) \\ & + \left(\frac{r}{r_0}\right)^{-2n} W^2(r) B^2 r^2 d\theta^2 \\ & - \left(\frac{r}{r_0}\right)^{2n} \frac{1}{W^2(r)} dt^2 \; , \end{aligned} \tag{3.7}$$

$$
\begin{aligned}
ds_M^2 &= \left(\frac{r}{r_0}\right)^{2m^2-2n} W^2(r)(dr^2-dt^2) \\
&+ \left(\frac{r}{r_0}\right)^{-2n} W^2(r)B^2r^2d\theta^2 \\
&+ \left(\frac{r}{r_0}\right)^{2n} \frac{1}{W^2(r)}dz^2 \,,
\end{aligned} \tag{3.8}
$$

$$
\begin{aligned}
ds_{Mi}^2 &= \left(\frac{r}{r_0}\right)^{2} \frac{V^2(r)}{(n-1)}dz^2 \\
&+ \left(\frac{r}{r_0}\right)^{F+1} \frac{V^2(r)}{(n-1)}(dr^2-dt^2) \\
&+ B^2\frac{(n-1)}{V^2(r)}d\theta^2 \,,
\end{aligned} \tag{3.9}
$$

where

$$
\begin{aligned}
W(r) &= \frac{(\frac{r}{r_0})^{2n}+k}{1+k} \\
V(r) &= [-c_1\sin(\sqrt{n-1}\ln r)+c_2\cos(\sqrt{n-1}\ln r)]
\end{aligned}
$$

All the integration constants are to be determined after the inclusion of matter fields.

4. Summary

In this work we obtain exact solutions for a static and charged cosmic string in a Einstein-Maxwell-Dilaton theory of a scalar-tensor type in (3+1)-Dimensions. This theory is specified by the dilaton field ϕ, the graviton field $g_{\mu\nu}$ and the electromagnetic field $F_{\mu\nu}$, and one post-Newtonian parameter $\alpha(\phi)$. It contains three different cases, each of them corresponding to a particular solution of the Rainich algebra for the Ricci tensor.

The three solutions correspond to solutions of an electric, magnetic and “mixed" strings. Each of them, stemming from the low-energy limit of the string theory, present very distinct gravitational features providing us with useful constraints on the underlying microscopic model [18, 19].

Acknowledgements

M. Leineker Costa would like to thank CAPES for a support. A. L. Naves de Oliveira and M. E. X. Guimarães would like to thank CNPq for a support.

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