

Testing UV-filtered (“fat-link”) clover fermions

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We investigate filtered clover fermions, built from fat gauge links, both in one-loop perturbation theory and in numerical simulations. We use a variety of filtering recipes (APE, HYP, EXP, HEX), some of which are suitable for a HMC with dynamical fermions. A generic filtering together with a (fat-link) clover term yields fermions with much reduced chiral symmetry breaking.

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1. Overview

Standard Wilson fermions are fairly fast to simulate, since the pertinent Dirac operator

$$D_W(x,y) = \frac{1}{2} \sum_{\mu} \left\{ (\gamma_{\mu} - I) U_{\mu}(x) \delta_{x+\hat{\mu},y} - (\gamma_{\mu} + I) U_{\mu}^{\dagger}(x - \hat{\mu}) \delta_{x-\hat{\mu},y} \right\} + \frac{1}{2\kappa} \delta_{x,y} \quad (1.1)$$

is sparse, and they have a strong point by preserving flavor. The main disadvantage is that they break chiral symmetry. There are two established procedures that can ameliorate the latter

- “ $O(a)$ -improvement”: $D_W \rightarrow D_{SW} = D_W - \frac{c_{SW}}{2} \sum_{\mu < \nu} \sigma_{\mu\nu} F_{\mu\nu} \delta_{x,y}$
- “UV-filtering” or “link-fattening”: $U_{\mu}(x) \rightarrow U_{\mu}^{\text{APE,HYP},\dots}$ and $F_{\mu\nu}(x) \rightarrow F_{\mu\nu}^{\text{APE,HYP},\dots}$

and the purpose of this talk is to highlight the fact that by *combining* both approaches, one can considerably reduce the amount of chiral symmetry breaking. It is easy to obtain residual quark masses of the order of 30 MeV to 100 MeV without any need for tuning.

In [1] we extend and systematize earlier studies [2, 3, 4, 5, 6, 7]. In particular, Ref. [3] has proven useful to us, since it is the very first account on fat-link perturbation theory.

2. UV-filtering (“smearing”, “link-fattening”) recipes

The idea of any UV-filtered fermion action is that one would carry on a smoothed copy of the actual gauge field and evaluate the Dirac operator on that background. This yields a new fermion action which differs from the old one by terms which are simultaneously *ultralocal* and *irrelevant*. The term “UV-filtered” indicates that such an action is less sensitive to the UV fluctuations of the gauge background. One may also speak of “fat-link” actions, but one should avoid the word “improved”, since the Symanzik class with typically $O(a^2)$ cut-off effects is maintained.

Obviously, there is a large amount of freedom. One needs to decide on the smoothing recipe (APE, HYP, etc.), on the parameter ($\alpha^{\text{APE}}, \alpha_{1,2,3}^{\text{HYP}}$) and on the number of iterations, n^{iter} . With D_{SW} one may either build just the clover term from smoothed links (FLIC fermions [5]), or use the same type of smoothing in the covariant derivative, too (as we do). In any case, with fixed (α, n^{iter}) the filtered (“fat-link”) action is in the *same universality class* as the usual (“thin-link”) version.

We compare a total of 16 actions. This comes, since we start from the Wilson ($c_{SW} = 0$) and the Sheikholeslami-Wohlert (“clover”, $c_{SW} = 1$) actions. Then we investigate four recipes, APE, HYP, EXP, HEX. The first two are well known, the third is the “stout” recipe of Ref. [8], and the fourth one is the hypercubic nesting idea with this EXP/stout inside – see [1] for details. Finally, we choose $n^{\text{iter}} = 1, 3$. To prevent further proliferation, we stay with one parameter per recipe

$$\begin{aligned} \text{APE with } \alpha^{\text{APE}} = 0.6 & \xleftrightarrow{\text{PT}} \text{EXP with } \alpha^{\text{EXP}} = 0.1 \text{ [“stout”]} \\ \text{HYP with } \alpha^{\text{HYP}} = (0.75, 0.6, 0.3) & \xleftrightarrow{\text{PT}} \text{HEX with } \alpha^{\text{HEX}} = (0.125, 0.15, 0.15) \end{aligned} \quad (2.1)$$

thereby exploiting a one-to-one relationship (in 1-loop PT) $\text{APE} \leftrightarrow \text{EXP}$ and ditto for $\text{HYP} \leftrightarrow \text{HEX}$.

In fact, filtered (“fat-link”) PT is not much harder than naive (“thin-link”) PT. For APE [3]

$$A_{\mu}^{(n)}(q) = \sum_{\nu} \left(\left[1 - \frac{\alpha}{2(d-1)} \hat{q}^2 \right]^n (\delta_{\mu,\nu} - \frac{\hat{q}_{\mu} \hat{q}_{\nu}}{\hat{q}^2}) + \frac{\hat{q}_{\mu} \hat{q}_{\nu}}{\hat{q}^2} \right) A_{\nu}(q) \quad (2.2)$$

and this means that in d dimensions the transverse part of the gauge field gets multiplied with a form-factor $f_{\text{APE}}^{(n)}(\hat{q}^2) = \left[1 - \frac{\alpha}{2(d-1)} \hat{q}^2 \right]^n$ for n iterations [3], where $\hat{q}_{\mu} = \frac{2}{a} \sin(\frac{a}{2} q_{\mu})$.

	thin link	1 APE	2 APE	3 APE	1 HYP
$c_{\text{SW}}=0$	51.43471	13.55850	7.18428	4.81189	6.97653
$c_{\text{SW}}=1$	31.98644	4.90876	1.66435	0.77096	1.98381
$c_{\text{SW}}=2$	1.10790	-7.11767	-5.48627	-4.23049	-4.41059

Table 1: Additive mass shift S in 1-loop PT for standard Wilson or clover fermions on Wilson glue and after APE/EXP or HYP/HEX filtering. The uncertainty is of order one in the last digit quoted.

$c_{\text{SW}}=0$	thin link	1 APE	1 HYP	$c_{\text{SW}}=1$	thin link	1 APE	1 HYP
z_S	12.95241	1.12593	-1.78317	z_S	19.30995	4.11106	-0.03678
z_P	22.59544	5.28288	0.51727	z_P	22.38259	4.80364	0.12845
z_V	20.61780	6.39810	3.38076	z_V	15.32907	3.31243	1.38517
z_A	15.79628	4.31963	2.23054	z_A	13.79274	2.96614	1.30255
$(z_P - z_S)/2$	4.82152	2.07848	1.15022	$(z_P - z_S)/2$	1.53632	0.34629	0.08262
$z_V - z_A$	4.82152	2.07847	1.15022	$z_V - z_A$	1.53633	0.34629	0.08262

Table 2: Coefficient z_X in formula (4.2) for the renormalization factor Z_X with $X = S, P, V, A$ after 1 step of APE or HYP filtering. Entries are for $c_{\text{SW}}=0$ Wilson fermions (left) or $c_{\text{SW}}=1$ clover fermions (right).

3. Critical mass in 1-loop PT

One way to assess the amount of chiral symmetry breaking is to consider the magnitude of the additive mass renormalization. In PT it has the expansion [with $C_F = 4/3$ for $SU(3)$]

$$am_{\text{crit}} = \Sigma_0 = -\frac{g_0^2}{16\pi^2} C_F S + O(g_0^4) \quad [< 0] \quad (3.1)$$

with S given in Tab. 1. Focusing on the numbers in bold print, one sees that standard clover improvement diminishes S by a factor 1.6. On the other hand, one HYP step reduces it by a factor 7.4. The interesting news is that by combining both strategies one obtains a factor 26, that is more than the product of the two. Hence we reach the conclusion that (at least in 1-loop PT) the two ingredients clover-improvement and link-fattening pile up to suppress $-am_{\text{crit}}$ quite drastically.

4. Renormalization factors in 1-loop PT

Another way to assess the amount of chiral symmetry breaking is to consider

$$\begin{aligned} \langle \cdot | O_j^{\text{cont}}(\mu) | \cdot \rangle &= \sum_k Z_{jk}(a\mu) \langle \cdot | O_k^{\text{latt}}(a) | \cdot \rangle \\ Z_{jk}(a\mu) &= \delta_{jk} - \frac{g_0^2}{16\pi^2} (\Delta_{jk}^{\text{latt}} - \Delta_{jk}^{\text{cont}}) = \delta_{jk} - \frac{g_0^2}{16\pi^2} C_F z_{jk} \end{aligned} \quad (4.1)$$

for the point-like scalar/pseudoscalar densities and vector/axialvector currents. Then

$$\begin{aligned} Z_S(a\mu) &= 1 - \frac{g_0^2}{4\pi^2} \left[\frac{z_S}{3} - \log(a^2 \mu^2) \right] + \dots & Z_V &= 1 - \frac{g_0^2}{12\pi^2} z_V + \dots \\ Z_P(a\mu) &= 1 - \frac{g_0^2}{4\pi^2} \left[\frac{z_P}{3} - \log(a^2 \mu^2) \right] + \dots & Z_A &= 1 - \frac{g_0^2}{12\pi^2} z_A + \dots \end{aligned} \quad (4.2)$$

with z_S, z_P, z_V, z_A given in Tab. 2. A first check whether $(z_P - z_S)/2$ equals $z_V - z_A$ is successful.

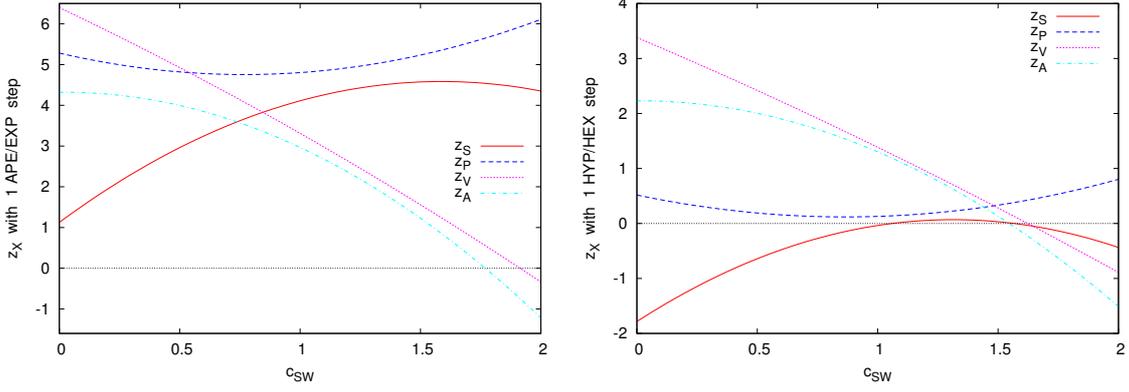


Figure 1: Finite pieces $z_{S,P,V,A}$ of the Z_X for 1 APE and 1 HYP fermions as a function of c_{SW} .

	0.12	0.24	0.36	0.48	0.6	0.72	0.84	0.96
1	6.99558	5.19536	3.77414	2.73191	2.06867	1.78443	1.87918	2.35292
2	5.44459	3.26311	2.05185	1.39240	1.02644	0.85574	0.94215	1.50761
3	4.32095	2.22832	1.31922	0.89113	0.66450	0.55028	0.63677	1.39614
4	3.49281	1.62650	0.94513	0.64620	0.48918	0.40469	0.49903	1.64011
5	2.87228	1.25138	0.72799	0.50519	0.38709	0.32019	0.42898	2.29060

Table 3: Tadpole diagram in Landau gauge [value to be multiplied with $g_0^2 C_F / (16\pi^2)$] in 1-loop PT for various $(n^{\text{iter}}, \alpha^{\text{APE}})$. The corresponding “thin-link” value is 9.174788.

Furthermore, in view of the overlap action satisfying $z_S^{\text{ov}} = z_P^{\text{ov}}, z_V^{\text{ov}} = z_A^{\text{ov}}$, it is clear that this (common) figure [in the last two lines of Tab. 2] quantifies the amount of chiral symmetry breaking. Focusing on the numbers in bold print, one sees that standard clover improvement diminishes $z_V - z_A$ by a factor 3.14. On the other hand, one HYP step reduces it by a factor 4.19. The interesting news is that by combining both strategies one obtains a factor 58.4, that is more than the product of the two. Hence we reach the conclusion that (at least in 1-loop PT) the two ingredients clover-improvement and link-fattening pile up to suppress $\frac{1}{2}(z_P - z_S) = z_V - z_A$ quite drastically.

Finally, one may consider how the z_X (with $X = S, P, V, A$) depend on c_{SW} . In 1-loop PT one finds quadratic polynomials with details given in Fig. 1. For the action with one APE step $c_{SW} = 1.2648$ is the point where z_S is closest to z_P (and hence z_V closest to z_A). For the action with one HYP step the minimal amount of chiral symmetry breaking is realized through $c_{SW} = 1.1653$. This gives some hope that a one-loop improved action ($c_{SW} = 1 + \text{const} g_0^2$) or even a tree-level improved action ($c_{SW} = 1$) with some vigorous filtering might have decent chiral properties.

5. Irrelevance of tadpole resummation

For “thin-link” actions it is customary to split S in Landau gauge into two contributions

$$S/(16\pi^2) = -\Sigma_0/(g_0^2 C_F) = -[\text{sunset}]_0/(g_0^2 C_F) - [\text{tadpole}]_0/(g_0^2 C_F) \quad (5.1)$$

β	5.846	6.000	6.136	6.260	6.373
L/a	12	16	20	24	28
a^{-1} [GeV]	1.590	2.118	2.646	3.177	3.709
n_{conf}	64	32	16	8	4

Table 4: Matched $(\beta, L/a)$ combinations; n_{conf} is the number of configurations per filtering and mass.

where the sunset piece carries (in 1-loop PT) all dependence on c_{SW} and is (at least for $0 \leq c_{\text{SW}} \leq 2$) “small”, while the tadpole piece is “large”. Hence it makes sense to resum the latter [9].

With filtering the situation is different, as a glimpse at Tab. 3 reveals. For the APE/EXP recipes (the columns simultaneously refer to $\alpha^{\text{APE}} = 0.12, 0.24, \dots$ and $\alpha^{\text{EXP}} = 0.02, 0.04, \dots$) any intermediate choice of $(\alpha, n^{\text{iter}})$ renders the tadpole contribution much smaller than in the unfiltered case (where it is 9.174788). With the tadpole and hence S being small for a generic filtering, there is no need to resum the tadpole contributions. In summary, the irrelevance of tadpole resummation for “fat-link” actions gives us hope that PT in g_0^2 might converge nicely for UV-filtered actions.

6. Non-perturbative tests

Given the good chiral properties of UV-filtered actions in 1-loop PT, it is useful to check to which extent they are realized non-perturbatively. To this end we performed a quenched study with a^{-1} ranging from 1.6 GeV to 3.7 GeV in a fixed physical volume $V = (1.5 \text{ fm})^4$. We use the Wilson gauge action with the parameters of Tab. 4. The bare Wilson and PCAC masses are defined as

$$m^{\text{W}} = m_0 - m_{\text{crit}} \quad \text{where} \quad am_0 = \frac{1}{2\kappa} - 4, \quad am_{\text{crit}} = \frac{1}{2\kappa_{\text{crit}}} - 4 \quad (6.1)$$

$$m^{\text{PCAC}} = \frac{\langle \bar{\partial}_\mu [A_\mu^a(x) + ac_A \bar{\partial}_\mu P^a] O^a(0) \rangle}{2 \langle P^a(x) O^a(0) \rangle} \quad (6.2)$$

with $\bar{\partial}$ the symmetric derivative, and the renormalized VWI and AWI quark masses follow via

$$m^{\text{VWI}}(\mu) = Z_m(a\mu)(1 + b_m am^{\text{W}})m^{\text{W}} \quad (6.3)$$

$$m^{\text{AWI}}(\mu) = \frac{Z_A}{Z_P(a\mu)} \frac{1 + b_A am^{\text{W}}}{1 + b_P am^{\text{W}}} m^{\text{PCAC}}. \quad (6.4)$$

We use $c_{\text{SW}} = 0, 1$ and the tree-level improvement coefficients $b_X = 1, c_{V,A} = 0$. It turns out that $b_m = -\frac{1}{2}$ leads to unacceptable fits. Since we cannot afford one more free parameter, we choose to set $b_m = 0$. From considering the m^{PCAC} versus m_0 we extract the inverse slope $\tilde{Z}_A = Z_A Z_S / Z_P$ and the horizontal offset $-am_{\text{crit}}$. The idea is now to compare them to predictions from 1-loop PT.

7. Rational fits for $-am_{\text{crit}}$

We know that asymptotically $-am_{\text{crit}} \rightarrow \frac{S}{12\pi^2} g_0^2$ with S given in Tab. 1. Fitting our data with

$$-am_{\text{crit}} = \frac{c_1 g_0^2 + c_2 g_0^4}{1 + c_3 g_0^2} \quad (7.1)$$

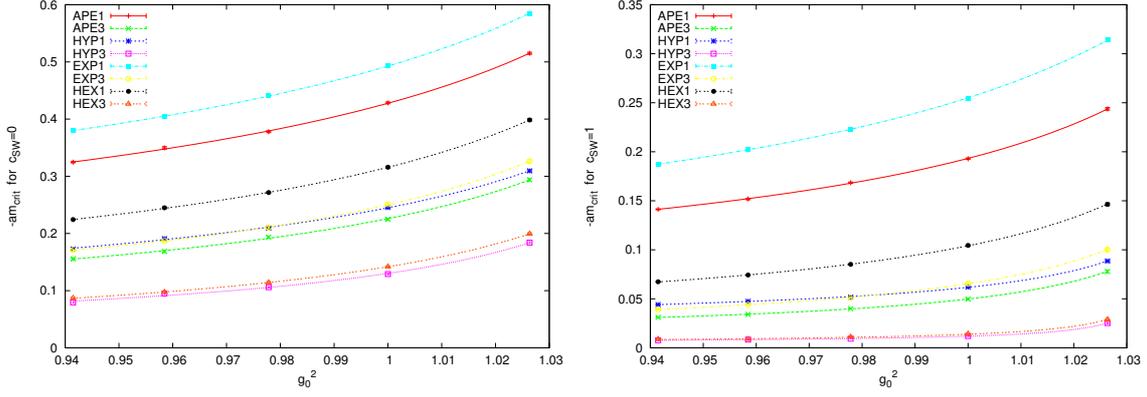


Figure 2: $-am_{\text{crit}}$ vs. g_0^2 for Wilson ($c_{\text{SW}}=0$, left) and clover ($c_{\text{SW}}=1$, right) fermions with our 8 recipes.

	$c_{\text{SW}} = 0$		$c_{\text{SW}} = 1$	
pert.	0.114480		0.0414467	
1 APE 1 EXP	0.213(12)	0.252(12)	0.0909(28)	0.1094(20)
pert.	0.040629		0.0065096	
3 APE 3 EXP	0.077(14)	0.083(07)	0.0172(15)	0.0171(09)
pert.	0.058906		0.0167502	
1 HYP 1 HEX	0.095(14)	0.121(04)	0.0338(12)	0.0332(16)
pert.	—		—	
3 HYP 3 HEX	0.034(15)	0.026(01)	0.0060(02)	0.0060(15)

Table 5: Fitted c_1 values in (7.1) compared to the 1-loop prediction $S/(12\pi^2)$ with the S taken from Tab. 1.

we have two options. We may set c_1 to its perturbative value and adjust c_2, c_3 . Or we may fit all three coefficients and compare the fitted c_1 to its perturbative prediction. It turns out that the first option leads (at our couplings) to unacceptable fits, hence we are left with the second one. In Fig. 2 we plot our non-perturbative values of $-am_{\text{crit}}$ versus g_0^2 for our 16 actions. The results of the rational fits (7.1) are included and one sees that they give a decent description of the data.

The pertinent c_1 coefficients are collected in Tab. 5, along with the 1-loop prediction with S taken from Tab. 1. Comparing the perturbative and the non-perturbative values, one may say that they are close on an absolute scale (set by the unfiltered action), since all c_1 are small. However, on a relative scale, the two differ significantly — typically, the non-perturbative c_1 is larger than the perturbative one by a factor 2–3. In spite of this disagreement, the non-perturbative data still show a consistency $c_1^{\text{APE}} \simeq c_1^{\text{EXP}}$ and ditto for $c_1^{\text{HYP}} \simeq c_1^{\text{HEX}}$, as predicted in PT. We find this amusing, in particular in view of the fact that the pertinent curves in Fig. 2 are not close at all. In summary, we would say that there are some encouraging signs, but there is no quantitative agreement of c_1 with 1-loop PT in our range of couplings.

We did similar fits for \tilde{Z}_A and am_{res} . In the former case the filtering achieves $\tilde{Z}_A \simeq 1$, but the deviation from 1 is not adequately described in 1-loop PT. In the latter case, the situation is analogous to $-am_{\text{crit}}$, which is no surprise, since $\tilde{Z}_A \simeq 1$ implies $|m_{\text{crit}}| \simeq m_{\text{res}}$. Our am_{res} [defined as the AWI mass at $m_0=0$] differs from the version that is standard in the domain-wall community;

	(5.846)	6.000	6.136	6.260	6.373
$m_{\text{res}}^{3\text{APE}} [\text{MeV}]$	(144)	111	107	108	113
$m_{\text{res}}^{3\text{HYP}} [\text{MeV}]$	(47)	27	25	26	27

Table 6: Residual mass [defined via m^{PCAC}] for our couplings. We estimate that the error is of order 5 MeV.

nonetheless, it might be interesting to compare. We collect our results, converted into physical units, in Tab. 6. The first striking feature is that, after abandoning the coarsest lattice, they are almost independent of the coupling. The second surprise is that the tree-level improved 3 HYP action achieves $m_{\text{res}} \simeq O(30\text{MeV})$. Clearly, this lies well above of the residual masses that can be achieved with domain-wall fermions. On the other hand, it is much smaller than the residual mass of an unfiltered Wilson or clover action, typically $O(1\text{GeV})$. Our hope is that the small m_{res} of UV-filtered clover quarks leads to good scaling properties and reduces the CPU time requirements to obtain a predefined accuracy of phenomenologically interesting observables in the continuum.

8. Summary

We close with highlighting some key properties of UV-filtered (“fat-link”) clover actions:

1. UV-filtering of D_{SW} yields a legal action for any fixed ($\alpha^{\text{recipe}}, n^{\text{iter}}$).
2. 1-loop PT suggests that the series in g_0^2 at $O(a)$ converges well, without tadpole resummation.
3. Maybe even the the non-perturbative $O(a)$ ambiguities in c_V, c_A will be gone.
4. Chiral symmetry breaking is reduced: $m^{\text{res}} \simeq 30 - 100\text{MeV}$ and $\frac{1}{2}(z_P - z_S) = z_V - z_A \ll 1$.
5. With EXP/stout filtering (and maybe 1-loop c_{SW}) D_{SW} is ready for dynamical simulations.

References

- [1] S. Capitani, S. Dürr and C. Hoelbling, *Rationale for UV-filtered clover fermions*, hep-lat/0607006.
- [2] T.A. DeGrand, A. Hasenfratz and T.G. Kovacs [MILC Collab.], *Optimizing the chiral properties of lattice fermion actions*, hep-lat/9807002.
- [3] C.W. Bernard and T. DeGrand, *Perturbation theory for fat-link fermion actions*, Nucl. Phys. Proc. Suppl. **83**, 845 (2000) [hep-lat/9909083].
- [4] M. Stephenson, C. DeTar, T.A. DeGrand and A. Hasenfratz, *Scaling and eigenmode tests of the improved fat clover action*, Phys. Rev. D **63**, 034501 (2001) [hep-lat/9910023].
- [5] J.M. Zanotti *et al.* [CSSM Collab.], *Hadron masses from novel fat-link fermion actions*, Phys. Rev. D **65**, 074507 (2002) [hep-lat/0110216].
- [6] T. DeGrand, *One loop matching coefficients for a variant overlap action and some of its simpler relatives*, Phys. Rev. D **67**, 014507 (2003) [hep-lat/0210028].
- [7] T. DeGrand, A. Hasenfratz and T.G. Kovacs, *Improving the chiral properties of lattice fermions*, Phys. Rev. D **67**, 054501 (2003) [hep-lat/0211006].
- [8] C. Morningstar and M.J. Peardon, *Analytic smearing of $SU(3)$ link variables in lattice QCD*, Phys. Rev. D **69**, 054501 (2004) [hep-lat/0311018].
- [9] G.P. Lepage and P.B. Mackenzie, *On the viability of lattice perturbation theory*, Phys. Rev. D **48**, 2250 (1993) [hep-lat/9209022].