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Schrödinger functional renormalization schemes for Ginsparg-Wilson quarks

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A technically simple implementation of Schrödinger functional (SF) boundary conditions for Domain-wall and overlap quarks can be obtained by using a Wilson kernel with chirally rotated SF boundary conditions in the Neuberger relation. The boundary conditions of the Wilson kernel are inherited by the overlap operator and with an even number of quark flavours the theory thus obtained can be interpreted as a chirally rotated version of the standard SF. I shortly discuss the orbifold construction and identify the (exact) flavour and parity symmetries, which are partly realised à la Ginsparg-Wilson.

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1. Introduction

The Schrödinger functional [1, 2] has become a universal tool to tackle renormalisation problems in lattice QCD. It allows for the definition of finite volume renormalisation schemes (SF schemes) which, in combination with recursive finite size techniques, completely solve the problem of large scale differences [3]. In principle, the Schödinger functional can be formulated with any regularisation. On the lattice, however, it may not always be obvious how to proceed, as the required Dirichlet boundary conditions for the fermionic fields cannot always be obtained by explicitly imposing them on the fields. Rather, the boundary conditions arise dynamically, depending on the lattice action and its structure close to the boundaries. To make sure that the desired boundary conditions are indeed obtained in the continuum limit, one may have to tune some parameters, depending on the symmetries of the regularisation.

Of particular interest are fermion actions with exact chiral symmetry. A nice solution for overlap quarks has been offered by Lüscher, which relies on universality arguments [4]. Previous work [5, 6] made use of an orbifold construction, which however remains technically involved, and does not directly lead to a real fermion determinant in the single flavour case. An even number of quark flavours may remove this defect, at the expense of an exact flavour symmetry. The lack of continuum symmetries may induce undesired counterterms, which have been proven to be absent at the tree-level only. Nevertheless, this formulation has been implemented for domain wall quarks in the quenched approximation and first results have been presented at this conference [7].

Here I would like to propose a solution for even numbers of quark flavours which does enjoy exact flavour and parity symmetries, and yet is simple to implement for both overlap and Domain-wall quarks. For technical reasons, a slight detour is taken by implementing the Schrödinger functional in a chirally rotated basis, which, in the continuum limit, is equivalent to the standard Schrödinger functional. This writeup is organised as follows. I first discuss the chirally rotated SF in the continuum, which is then regularised on the lattice through an orbifold reflection applied to overlap quarks. I then discuss how the symmetries are realised and comment on its application to Domain-Wall quarks.

2. The chirally rotated SF

The basic objects of interest are correlation functions obtained from the Schrödinger functional in the chiral limit. Assuming that the flavour doublets χ' and $\bar{\chi}'$ satisfy standard homogeneous SF boundary conditions [2], the chiral rotation

$$\chi' = \exp(i\alpha\gamma_5\tau^3/2)\chi, \qquad \bar{\chi}' = \bar{\chi}\exp(i\alpha\gamma_5\tau^3/2), \tag{2.1}$$

implies that the rotated fields satisfy

$$\begin{aligned} P_+(\alpha)\chi(x)\,|_{x_0=0} &= 0, & P_-(\alpha)\chi(x)\,|_{x_0=T} &= 0, \\ \bar{\chi}(x)\gamma_0 P_-(\alpha)\,|_{x_0=0} &= 0, & \chi(x)\gamma_0 P_+(\alpha)\,|_{x_0=T} &= 0, \end{aligned} \tag{2.2}$$

with the projectors,

$$P_\pm(\alpha) = \tfrac{1}{2}[1 \pm \gamma_0 \exp(i\alpha\gamma_5\tau^3)]. \tag{2.3}$$

Performing a change of variables in the functional integral, one then obtains the formal identities

$$\langle O[\chi,\bar{\chi}]\rangle_{(P_\pm)} = \langle O[\exp(i\alpha\gamma_5\tau^3/2)\chi,\bar{\chi}\exp(i\alpha\gamma_5\tau^3/2)]\rangle_{(P_\pm(\alpha))}, \tag{2.4}$$

where the quark masses have been set to zero. The subscripts for the correlation functions indicate the boundary conditions for χ at $x_0 = 0$, and the quark and anti-quark boundary fields [11] may be included in $O[\chi,\bar{\chi}]$, by replacing

$$\zeta(\mathbf{x}) \to P_-\chi(0,\mathbf{x}), \qquad \bar{\zeta}(\mathbf{x}) \to \bar{\chi}(0,\mathbf{x})P_+, \tag{2.5}$$

and similarly for the fields at $x_0 = T$. As the chiral rotation is part of the non-singlet chiral symmetries of QCD, both formulations are thus equivalent in the continuum limit. At least for even numbers of flavours, the regularisation of the SF may therefore proceed at any value of the angle α. In particular, I will choose $\alpha = \pi/2$ where the projectors read

$$P_\pm(\pi/2) \equiv Q_\pm = \frac{1}{2}(1 \pm i\gamma_0\gamma_5\tau^3). \tag{2.6}$$

Furthermore, in the absence of mass terms, the distinction between flavour and chiral symmetries becomes a convention. I will stick to the convention that the standard SF boundary conditions in terms of the projectors $P_\pm$ are invariant under flavour and parity symmetries. This means that these symmetries take a somewhat unusual form when expressed for the rotated fields (s. below).

3. Orbifold construction

The basic procedure is completely analogous to the case of Wilson quarks described in [8], except for a small offset of O(a) introduced for technical reasons to become clear shortly. The starting point is the standard lattice action for a single massless overlap quark,

$$S_f[\psi,\bar{\psi},U] = a^4\sum_x \bar{\psi}(x)D_N\psi(x), \qquad aD_N = 1 - A(A^\dagger A)^{-1/2}, \quad A = 1 - aD_W, \tag{3.1}$$

where D_W is the standard massless Wilson-Dirac operator, and the fermion fields are anti-periodic with period $2(T+a)$ (rather than $2T$),

$$\psi(x_0 + 2(T+a),\mathbf{x}) = -\psi(x), \qquad \bar{\psi}(x_0 + 2(T+a),\mathbf{x}) = -\bar{\psi}(x). \tag{3.2}$$

The orbifold reflection is defined by

$$R: \psi(x) \to i\gamma_0\gamma_5\psi(-a-x_0,\mathbf{x}), \qquad \bar{\psi}(x) \to \bar{\psi}(-a-x_0,\mathbf{x})i\gamma_0\gamma_5. \tag{3.3}$$

The gauge field is extended to the interval $[-T-a, T+a]$,

$$U_k(-a-x_0,\mathbf{x}) = U_k(x_0,\mathbf{x}), \qquad U_0(-2a-x_0,\mathbf{x})^\dagger = U_0(x), \tag{3.4}$$

and then $2(T+a)$-periodically continued. The fermionic fields are then decomposed into even and odd with respect to R as in [8] and, due to $[D_N, R] = 0$, the functional integral factorises. Interpreting even and odd fields as flavour components of a doublet χ we see that the rotated SF boundary conditions are indeed obtained, albeit only up to O(a) effects, which are due to the O(a)

offset in the orbifold reflection. Note that the dynamical field variables are now $\chi(x)$ and $\bar{\chi}(x)$ for all Euclidean times $0 \le x_0 \le T$.

What has been achieved by the O(a) offset is that the overlap operator $\mathscr{D}_N$ acting on the doublet $\chi(x)$ is block diagonal in Euclidean time. The desired explicit reduction to the time-interval $[0,T]$ is then simply obtained by considering only one of the blocks. Furthermore, the same holds true in the case of the Wilson-Dirac operator itself, and the Neuberger operator can therefore still be obtained by inserting the corresponding Wilson-Dirac kernel in the Neuberger relation, viz.

$$a\mathscr{D}_N = 1 - \mathscr{A}(\mathscr{A}^\dagger \mathscr{A})^{-1/2}, \qquad \mathscr{A} = 1 - a\mathscr{D}_W, \tag{3.5}$$

with the Wilson-Dirac kernel $\mathscr{D}_W$,

$$a\mathscr{D}_W \chi(x) = -U(x,0)P_- \chi(x + a\hat{\mathbf{0}}) + (K\psi)(x) - U(x - a\hat{\mathbf{0}})^\dagger P_+ \chi(x - a\hat{\mathbf{0}}). \tag{3.6}$$

Here, I have set $\chi(x) = 0$ for $x_0 < 0$ and $x_0 > T$, and the time diagonal kernel K is given by

$$K = 1 + \frac{1}{2}\sum_{k=1}^{3}\left\{a(\nabla_k + \nabla_k^*)\gamma_k - a^2\nabla_k^*\nabla_k\right\} + \delta_{x_0,0}\, i\gamma_5\tau^3 P_- + \delta_{x_0,T}\, i\gamma_5\tau^3 P_+. \tag{3.7}$$

4. Symmetries

As stated earlier, I refer to the standard SF boundary conditions as being flavour and parity invariant. Then it is not difficult to see that the SU(2) lattice flavour symmetry in the rotated basis is realised à la Ginsparg-Wilson [9, 10]:

$$\gamma_5\tau^{1,2}\mathscr{D}_N + \mathscr{D}_N\gamma_5\tau^{1,2} \;=\; a\mathscr{D}_N\gamma_5\tau^{1,2}\mathscr{D}_N, \tag{4.1}$$

$$\tau^3\mathscr{D}_N - \mathscr{D}_N\tau^3 \;=\; 0. \tag{4.2}$$

Defining $\Gamma_5 = \gamma_5(1 - a\mathscr{D}_N)]$ (note that $\Gamma_5^2 \neq 1$), the generators are easily identified,

$$\hat{T}^1 = \Gamma_5\tau^2/2, \quad \hat{T}^2 = -\Gamma_5\tau^1/2, \quad \hat{T}^3 = \tau^3/2, \tag{4.3}$$

and the flavour algebra does indeed close,

$$[\hat{T}^a, \hat{T}^b] = i\varepsilon^{abc}\hat{T}^c. \tag{4.4}$$

On the other hand, all chiral symmetries are explicitly broken by the SF boundary conditions. In the rotated framework one finds

$$[\tau^{1,2}, \mathscr{D}_N] \neq 0, \qquad \{\gamma_5\tau^3, \mathscr{D}_N\} \neq a\mathscr{D}_N\gamma_5\tau^3\mathscr{D}_N. \tag{4.5}$$

Note that the last equation also implies that the standard Ginsparg-Wilson relation does not hold. However, the violations are expected to be exponentially small in the distance from the boundaries, and I have numerically checked that this is indeed the case at tree level.

Parity and time reversal are again realised in the Ginsparg-Wilson fashion, e.g.

$$P: \chi(x) \to i\gamma_0\gamma_5\tau^3\chi(\tilde{x}), \qquad \tilde{x} = (x_0, -\mathbf{x}), \qquad \mathscr{D}_N P + P\mathscr{D}_N = a\mathscr{D}_N P\mathscr{D}_N, \tag{4.6}$$

so that the symmetries of the two-flavour SF match exactly those of the standard SF with two flavours of Wilson quarks. In contrast to the analogous construction with Wilson quarks, the recovery of SU(2) flavour and parity as Ginsparg-Wilson-like symmetries means that there are no additional counterterms to be expected. One thus expects that renormalisation and O(a) improvement works out with the same number of counterterms as needed in the standard case. In particular, O(a) improvement of most (massless) correlation functions will require an analogue of the counterterm proportional to $\tilde{c}_t$ [11]. This counterterm will already contribute at the tree level, which is a consequence of having chosen to off-set the orbifold reflection by O(a).

5. Concluding Remarks

Chirally rotated SF boundary conditions have been successfully implemented for a doublet of quarks, and the generalisation to any even number of quark flavours seems straightforward. I have shown that the usual SF symmetries are exactly realised with overlap quarks. As in infinite volume, the overlap operator can be obtained by Neuberger's construction [12], with a corresponding Wilson kernel. The construction can therefore easily be translated to Domain-wall quarks [13, 14], where it is sufficient to use the same Wilson kernel in the 4-dimensional slices. Note also that Pauli-Villars fields do not pose a problem here, as a the addition of a standard mass term is compatible with the orbifold construction. However, Domain-wall quarks will induce exponentially small violations of flavour symmetry (as defined here), due to the finite extent of the lattice in the extra dimension. Note that the determinant of the overlap Dirac operator is real and non-negative. In fact, both flavour components of the overlap operator have real and equal determinant, so that the SF for a single flavour could be defined in the same way. However, this cannot easily be related to the corresponding standard single-flavour SF, as this relation now involves a singlet axial rotation, the Jacobian of which is expected to be non-trivial. Nevertheless, such an alternative definition of the SF could be interesting in its own right. Incidentally, it corresponds to the boundary conditions singled out by Symanzik in his celebrated paper on the Schrödinger representation in Quantum Field Theory [15].

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