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Lattice Landau gauge via Stereographic Projection

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The complete cancellation of Gribov copies and the Neuberger $0/0$ problem of lattice BRST can be avoided in modified lattice Landau gauge. In compact $U(1)$, where the problem is a lattice artifact, there remain to be Gribov copies but their number is exponentially reduced. Moreover, there is no cancellation of copies there as the sign of the Faddeev-Popov determinant is positive. Applied to the maximal Abelian subgroup this avoids the perfect cancellation amongst the remaining Gribov copies for $SU(N)$ also. In addition, based on a definition of gauge fields on the lattice as stereographically-projected link variables, it provides a framework for gauge fixed Monte-Carlo simulations. This will include all Gribov copies in the spirit of BRST. Their average is not zero, as demonstrated explicitly in simple models. This might resolve present discrepancies between gauge-fixed lattice and continuum studies of QCD Green's functions.

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1. Introduction

The Green's functions of QCD provide a basis for hadron phenomenology [1]. Their infrared behaviour is also known to contain information about the realisation of confinement in Landau gauge QCD. Dyson-Schwinger equation (DSE) studies [2] have established that the gluon propagator alone does not provide long-range interactions of a strength sufficient to confine quarks, and that the infrared dominant correlations are instead mediated by the Faddeev-Popov ghosts of this formulation, whose propagator was found to be infrared enhanced. This can be understood in terms of confinement in QCD [1, 3, 4], as a consequence of the Kugo-Ojima criterion which is based on the realisation of the unfixed global gauge symmetries of the covariant continuum formulation. In order to distinguish confinement from Coulomb and Higgs phases it requires: (a) The massless single particle singularity in the transverse gluon correlations of perturbation theory must be screened non-perturbatively to avoid long-range fields and charged superselection sectors as in QED. (b) The global gauge charges must remain well-defined and unbroken to avoid the Higgs mechanism. In Landau gauge, in which the (Euclidean) gluon and ghost propagators are parametrised by the two invariant functions Z and G, respectively, this entails that

$$\text{(a):} \quad \lim_{p^2\to 0} Z(p^2)/p^2 < \infty \, ; \qquad \text{(b):} \quad \lim_{p^2\to 0} G^{-1}(p^2) = 0 \, . \tag{1.1}$$

The translation of (b) into the infrared enhancement of the ghost propagator thereby rests on the ghost/anti-ghost symmetry of the Landau gauge or the symmetric Curci-Ferrari gauges. It represents an additional boundary condition on DSE solutions which then lead to the prediction of a conformal infrared behaviour for the gluonic correlations in Landau gauge QCD [4]. In fact, this behaviour is directly tied to the validity and applicability of the framework of local quantum field theory for non-Abelian gauge theories beyond perturbation theory. The subsequent verification of this infrared behaviour with a variety of different functional methods in the continuum meant a remarkable success. These methods which all lead to the same prediction include studies of their Dyson-Schwinger Equations (DSEs) [4], Stochastic Quantisation [5], and of the Functional Renormalisation Group Equations (FRGEs) [6]. This prediction amounts to infrared asymptotic forms

$$Z(p^2) \sim (p^2/\Lambda^2_{\text{QCD}})^{2\kappa_Z} \, , \quad \text{and} \;\; G(p^2) \sim (p^2/\Lambda^2_{\text{QCD}})^{-\kappa_G} \, , \tag{1.2}$$

for $p^2 \to 0$, which are both determined by a unique critical infrared exponent $\kappa_Z = \kappa_G \equiv \kappa$, with $0.5 < \kappa < 1$. Under a mild regularity assumption on the ghost-gluon vertex [4], the value of this exponent is furthermore obtained as $\kappa = (93 - \sqrt{1201})/98 \approx 0.595$ [4, 5].

The conformal nature of this infrared behaviour in the pure Yang-Mills sector of Landau gauge QCD is evident in the generalisation to arbitrary gluonic correlations [7] which has furthermore been shown to represent a unique *scaling solution* [8]. In particular, the ghost-gluon vertex is then infrared finite and the non-perturbative running coupling of [2] approaches an infrared fixed-point,

$$\alpha_S(p^2) = \frac{g^2}{4\pi} Z(p^2) G^2(p^2) \to \alpha_c \quad \text{for} \;\; p^2 \to 0 \, . \tag{1.3}$$

If the ghost-gluon vertex is regular at $p^2 = 0$, its value is maximised and given by $\alpha_c \approx 8.9/N_c$ [4]. However, the uniqueness of infrared scaling does not rule out solutions with infrared finite gluon propagator and a ghost propagator with a free massless-particle singularity, *i.e.*, $Z(p^2) \sim p^2/M^2$, and $G(p^2) \sim$ const., for $p^2 \to 0$. This solution corresponds to $\kappa_Z = 1/2$ and $\kappa_G = 0$. It does not satisfy the scaling relation $\kappa_Z = \kappa_G$ in (1.2) because transverse gluons decouple for momenta

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$p^2 \ll M^2$, and it is therefore called the *decoupling solution* [9]. The interpretation of (1.3) as a running coupling does not make sense in the infrared in this case, in which there is no infrared fixed-point and no conformal infrared behaviour.

Because of the inevitable finite-volume effects, early lattice studies of the gluon and ghost propagators could have been consistent with both, the scaling solution or the decoupling solution. Recently, the finite-volume effects have been analysed carefully in the Dyson-Schwinger equations to demonstrate how the scaling solution is approached in the infinite volume limit there [10]. This is clearly not what is being observed, however, in more recent $SU(2)$ lattice data on impressively large lattices [11, 12]. Present lattice data is fully consistent with the decoupling solution which poses the question whether we are perhaps comparing apples with oranges when comparing the minimal lattice Landau gauge correlations with those of local quantum field theory in the infrared?

The latter is based on a cohomology construction of a physical Hilbert space over the indefinite metric spaces of covariant gauge theory from the representations of the Becchi-Rouet-Stora-Tyutin (BRST) symmetry. But do we have a non-perturbative definition of a BRST charge in presence of Gribov copies? In the most direct translation of BRST symmetry on the lattice, there is a perfect cancellation among these gauge copies which gives rise to the famous Neuberger $0/0$ problem. It asserts that the expectation value of any gauge invariant (and thus physical) observable in a lattice BRST formulation will always be of the indefinite form $0/0$ [13] and therefore prevented such formulations for more than 20 years now. In present lattice implementations of the Landau gauge this problem is avoided because the numerical procedures are based on minimisations of a gauge-fixing potential w.r.t. gauge transformations. To find absolute minima is not feasible on large lattices as this is a non-polynomially hard computational problem. One therefore settles for local minima which in one way or another, depending on the algorithm, samples gauge copies of the first Gribov region among which there is no cancellation. For the same reason, however, this is not a BRST formulation. The emergence of the decoupling solution can thus not be used to dismiss the Kugo-Ojima criterion of covariant gauge theory in the continuum.

2. Lattice BRST and the Neuberger $0/0$ Problem

In principle, a BRST symmetry could be implemented on the lattice by inserting the partition function of a topological model with BRST exact action into the gauge invariant lattice measure. Because of its topological nature, this gauge-fixing partition function Z_{GF} will be independent of gauge orbit and gauge parameter. The problem is that in the standard formulation this partition function calculates the Euler characteristic χ of the lattice gauge group which vanishes [14],

$$Z_{\text{GF}} = \chi(SU(N)^{\#\text{sites}}) = \chi(SU(N))^{\#\text{sites}} = 0^{\#\text{sites}} . \tag{2.1}$$

Neuberger's $0/0$ problem of lattice BRST arises because we have then inserted zero instead of unity (according to the Faddeev-Popov prescription) into the measure of lattice gauge theory. On a finite lattice, such a topological model is equivalent to a problem of supersymmetric quantum mechanics with Witten index Z_{GF}, except that for gauge-fixing we need a model with non-vanishing Witten index to avoid the Neuberger $0/0$ problem. Then however, just as the supersymmetry of the corresponding quantum mechanical model, such a lattice BRST cannot break.

In Landau gauge the Neuberger zero, $Z_{\text{GF}} = 0$, arises from the perfect cancellation of Gribov copies via the Poincaré-Hopf theorem. The gauge-fixing potential for a generic link configuration thereby plays the role of a Morse potential for gauge transformations and the Gribov copies are its critical points (the global gauge transformations need to remain unfixed so that there are strictly speaking only (#sites−1) factors of $\chi(SU(N)) = 0$ in (2.1)). The Morse inequalities then immediately imply that there are at least $2^{(N-1)(\#\text{sites}-1)}$ such copies in $SU(N)$ on the lattice, or $2^{\#\text{sites}-1}$ in compact $U(1)$, and equally many with either sign of the Faddeev-Popov determinant.

The topological origin of the zero originally observed by Neuberger in a certain parameter limit due to uncompensated Grassmann ghost integrations in standard Faddeev-Popov theory [13] becomes particularly evident in the ghost/anti-ghost symmetric Curci-Ferrari gauge with its quartic ghost self-interactions. In this gauge the same parameter limit leads to computing the zero in (2.1) from a product of independent Gauss-Bonnet integral expressions, for each site of the lattice [15], corresponding to the Gauss-Bonnet limit of the equivalent supersymmetric quantum mechanics model in which only constant paths contribute [16]. The indeterminate form of physical observables as a consequence of (2.1) can be regulated by a Curci-Ferrari mass term. While such a mass m decontracts the double BRST/anti-BRST algebra, which is known to result in a loss of unitarity, observables can then be meaningfully defined in the limit $m \to 0$ via l'Hospital's rule [15].

3. Lattice Landau Gauge from Stereographic Projection

The 0/0 problem due to the vanishing Euler characteristic of $SU(N)$ is avoided when fixing the gauge only up to the maximal Abelian subgroup $U(1)^{N-1}$ because the Euler characteristic of the coset manifold is non-zero. The corresponding lattice BRST has been explicitly constructed for $SU(2)$ [14], where the coset manifold is the 2-sphere and $\chi(SU(2)/U(1)) = \chi(S^2) = 2$. This indicates that the Neuberger problem might be solved when that of compact $U(1)$ is, where the same cancellation of lattice Gribov copies arises because $\chi(S^1) = 0$. A surprisingly simple solution to this problem is possible, however, by stereographically projecting the circle $S^1 \to \mathbb{R}$ which can be achieved by a modification of the minimising potential [17]. The resulting potential is convex to the above and leads to a positive definite Faddeev-Popov operator for compact $U(1)$ where there is thus no cancellation of Gribov copies, but $Z_{\text{GF}}^{U(1)} = N_{\text{GC}}$, for N_{GC} Gribov copies, which follows from a simple example of a Nicolai map [16]. As compared to the standard lattice Landau gauge the number of copies is furthermore exponentially reduced. This is easily verified explicitly in low dimensional models. While N_{GC} grows exponentially with the number of sites in the standard case as expected, the stereographically projected version has only $N_{\text{GC}} = N_x$ copies on a periodic chain of length N_x, and $\ln N_{\text{GC}} \sim N_t \ln N_x$ on a $2D$ lattice of size $N_t \times N_x$ in Coulomb gauge, for example, and in both cases their number is verified to be independent of the gauge orbit.

Applying the same techniques to the maximal Abelian subgroup $U(1)^{N-1}$, the generalisation to $SU(N)$ lattice gauge theories is possible when the odd-dimensional spheres S^{2n+1}, $n = 1, \dots N-1$, of its parameter space are stereographically projected to $\mathbb{R} \times \mathbb{R}P(2n)$. In absence of the cancellation of the lattice artifact Gribov copies along the $U(1)$ circles, the remaining cancellations between copies of either sign in $SU(N)$, which will persist in the continuum limit, are then necessarily incomplete, however, because $\chi(\mathbb{R}P(2n)) = 1$. For $SU(2)$ this program is straightforward. Starting from a modified gauge-fixing potential [17] one defines stereographically-projected gauge fields on the lattice (see [18]),

$$A_{x\mu} = \frac{1}{2ia}\left(\widetilde{U}_{x\mu} - \widetilde{U}_{x\mu}^{\dagger}\right) , \quad \text{where} \quad \widetilde{U}_{x\mu} \equiv \frac{2U_{x\mu}}{1 + \frac{1}{2}\operatorname{Tr} U_{x\mu}} ,$$

such that the gauge-fixing condition is given by the usual lattice divergence but in terms of these stereographically-projected gauge fields. A particular advantage of the non-compact gauge fields is that they allow to resolve the gauge condition of the stereographically-projected lattice Landau gauge by Hodge decomposition. This provides a framework for gauge-fixed Monte-Carlo simulations which is currently being developed for $SU(2)$ in 2 dimensions [19]. In the low-dimensional models mentioned above it can furthermore be verified explicitly that the corresponding gauge-fixing partition function is indeed given by $Z_{\rm GF}^{SU(2)} = Z_{\rm GF}^{U(1)} \neq 0$, as expected from $\chi(\mathbb{R}P(2)) = 1$.

4. Conclusions and Outlook

Comparisons of the infrared behaviour of QCD Green's functions as obtained from lattice Landau gauge implementations based on minimisations of a gauge-fixing potential and from continuum studies based on BRST symmetry have to be taken with a grain of salt. Evidence of the asymptotic conformal behaviour predicted by the latter is seen in the strong coupling limit of lattice Landau gauge where such a behaviour can be observed at large lattice momenta $a^2p^2 \gg 1$ [18]. Observed deviations from scaling at small momenta in the strong-coupling limit are not finite-volume effects, but discretisation dependent and hint at a breakdown of BRST symmetry arguments beyond perturbation theory in this approach. Non-perturbative lattice BRST has been plagued by the Neuberger $0/0$ problem, but its improved topological understanding provides ways to overcome this problem. The most promising one at this point rests on stereographic projection to define gauge fields on the lattice together with a modified lattice Landau gauge. This new definition has the appealing feature that it will allow gauge-fixed Monte-Carlo simulations in close analogy to the continuum BRST methods which it will thereby elevate to a non-perturbative level.

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