

## Weinberg sum rules at NLO in $1/N_C$

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We study the relevance of different renormalization schemes in Resonance Chiral Theory. The  $SS - PP$  correlator is explicitly computed at the one-loop level. Demanding the operator product expansion behaviour at short distances produces a new set of constraints, as some logarithmic terms are absent at high energies. Likewise, the loops induce subleading corrections in  $1/N_C$  to the leading-order constraints, the Weinberg sum rules. We find that the short-distance conditions from a minimally subtracted scheme generate large uncertainties which, alternatively, can be largely simplified in other schemes.

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## 1. Introduction

The effective field theory (EFT) approach is a very powerful tool for the investigation of Quantum Chromodynamics (QCD) at long distances. Chiral Perturbation theory ( $\chi$ PT) [1, 2] is the EFT for the description of the chiral (pseudo) Goldstones in the low energy domain  $E \ll \Lambda_H \sim 1 \text{ GeV}$ , with  $\Lambda_H$  typically the scale of the lowest resonance masses. Recent progress has allowed to carry  $\chi$ PT up to  $\mathcal{O}(p^6)$ , i.e., up to the two-loop level [3].

In the intermediate resonance region,  $\Lambda_H \lesssim E \lesssim 2 \text{ GeV}$ ,  $\chi$ PT stops being valid and one must explicitly include the resonance fields in the Lagrangian description. Resonance Chiral Theory ( $R\chi$ T) describes the interaction of resonance and pseudo-Goldstones within a general chiral invariant framework [4, 5]. Alternatively to the chiral counting, it uses the  $1/N_C$  expansion of QCD in the limit of large number of colours [6] as a guideline to organize the perturbative expansion. At leading order (LO), just tree-level diagrams contribute while loop diagrams yield higher order effects.

The infinite tower of mesons contained in large- $N_C$  QCD is often truncated to the lowest states in each channel, the so called single resonance approximation (SRA). This approximation has led to successful predictions of  $\mathcal{O}(p^4)$  and  $\mathcal{O}(p^6)$  low-energy constants (LECs) [4, 5, 7, 8, 9]. However, the study of Regge models with an infinite number of mesons has shown that if one keeps just the lightest states with exactly the same couplings and masses of the full model then one get wrong values for the LECs [10]. Likewise, that analysis finds that the truncated theory do not produce the right short-distance (SD) behaviour. Thus, in a matching with the OPE power behaviour the parameters of the truncated theory will become shifted in order to accommodate the right short-distance dependence. Chiral symmetry ensures the proper low-momentum structure of the  $R\chi$ T amplitudes around  $p^2 = 0$  but their high energy behaviour is not fixed by symmetry alone. Nevertheless, one knows that for large Euclidean momenta,  $(-p^2) \gtrsim 2 \text{ GeV}^2$  the  $SS - PP$  correlator is expected to follow a vanishing behaviour prescribed by the OPE. In that sense, the matched amplitude can be understood with the help of Padé approximants as a rational interpolator between the deep Euclidean  $p^2 = -\infty$  and the low-energy domain around  $p^2 = 0$  [11, 12]. The Weinberg sum-rules (WSR) [13] yield the most convenient parameters for the interpolation rather than accurate determinations of the resonance couplings.

Not much is known about the extension of  $R\chi$ T beyond the tree level approximation. Although some theoretical issues on the renormalizability of  $R\chi$ T still need further clarification [14], several chiral LECs have been already computed up to NLO in  $1/N_C$  through QFT one-loop calculations [15], dispersion relations [16] and even analyzed with the help of renormalization group techniques [17]. Here we present the basic ideas of the work in Ref. [18], where the  $SS - PP$  correlator is computed up to next-to-leading order in  $1/N_C$  (NLO). The one-loop amplitude is then taken as an improved interpolator between long and short distances and the corresponding modifications to the former WSRs are extracted. The amplitude is first computed within the subtraction scheme of  $\chi$ PT [2]. However, though equivalent at low energies, some appropriate schemes are found to be more convenient and to introduce less uncertainties in the SD constraints.

## 2. Weinberg sum rules at leading and next-to-leading order

The two-point Green function  $SS - PP$  we are interested in is defined by

$$\Pi_{S-P}^{ab}(p) = i \int d^4x e^{ip \cdot x} \langle 0 | T [S^a(x) S^b(0) - P^a(x) P^b(0)] | 0 \rangle = \delta^{ab} \Pi(p^2), \quad (2.1)$$

with  $S^a = \bar{q} \frac{\lambda_a}{\sqrt{2}} q$  and  $P^a = i \bar{q} \frac{\lambda_a}{\sqrt{2}} \gamma_5 q$ , being  $\lambda_a$  the Gellmann matrices ( $a = 1, \dots, 8$ ).

For convenience, the  $R\chi T$  Lagrangian can be organized in the form  $\mathcal{L} = \mathcal{L}_{GB} + \mathcal{L}_R + \mathcal{L}_{RR'} + \dots$ , where  $\mathcal{L}_{GB}$  contains just Goldstone bosons and external sources,  $\mathcal{L}_R$  includes operators with also one resonance field  $R$ , etc.  $\mathcal{L}_{GB}$  is provided by the  $\mathcal{O}(p^2)$   $\chi$ PT operators and the terms with one resonance field are given in the SRA by [4]

$$\mathcal{L}_R = \frac{F_V}{2\sqrt{2}} \langle V_{\mu\nu} f_+^{\mu\nu} \rangle + \frac{iG_V}{2\sqrt{2}} \langle V_{\mu\nu} [u^\mu, u^\nu] \rangle + \frac{F_A}{2\sqrt{2}} \langle A_{\mu\nu} f_-^{\mu\nu} \rangle + c_d \langle S u^\mu u_\mu \rangle + c_m \langle S \chi_+ \rangle + i d_m \langle P \chi_- \rangle, \quad (2.2)$$

where at tree-level operators with two or more resonances do not contribute.

If one computes the one-loop correlator, the perturbative result shows the form [16]

$$\frac{1}{B_0^2} \Pi(p^2) = \frac{2F^2}{p^2} + \frac{16c_m^2}{M_S^2 - p^2} - \frac{16d_m^2}{M_P^2 - p^2} + \rho(p^2), \quad (2.3)$$

with  $\rho(p^2)$  containing the renormalized loop contributions and other tree-level contributions sub-leading in  $1/N_C$  [16, 18]. The correlator has then the high-energy expansion [19],

$$\frac{1}{B_0^2} \Pi(p^2) = \sum_{k=0,2,4,\dots} \frac{1}{p^k} \left( \alpha_k^{(p)} + \alpha_k^{(\ell)} \ln \frac{-p^2}{\mu^2} \right). \quad (2.4)$$

The requirement that the amplitude follows the high energy OPE behavior  $\Pi(p^2) \xrightarrow{p^2 \rightarrow \infty} 1/p^6$  produces the SD constraints [19] for the log terms  $\alpha_0^{(\ell)} = \alpha_2^{(\ell)} = \alpha_4^{(\ell)} = 0$ , and the non-logarithmic conditions  $\alpha_0^{(p)} = 0$  and

$$\alpha_2^{(p)} = 2F^2 + 16d_m^2 - 16c_m^2 + A(\mu) = 0, \quad \alpha_4^{(p)} = 16d_m^2 M_P^2 - 16c_m^2 M_S^2 + B(\mu) = 0. \quad (2.5)$$

At LO in  $1/N_C$  there are no logs ( $\alpha_k^{(\ell)} = 0$ ). The remaining non-logarithmic constraints require the absence of local terms ( $\alpha_0^{(p)} = 0$ ) and the usual (large- $N_C$ ) Weinberg sum-rules  $8c_m^2 - 8d_m^2 - F^2 = 0$ ,  $c_m^2 M_S^2 - d_m^2 M_P^2 = 0$  [7, 13]

At NLO, the WSRs gain the subleading corrections  $A(\mu)$  and  $B(\mu)$  [16, 18]<sup>2</sup>. Notice that now

<sup>1</sup>The tiny dimension four condensate  $\frac{1}{B_0^2} \langle \mathcal{O}_{(4)}^{SS-PP} \rangle \simeq -12\pi\alpha_S F^4$  will be neglected in this work [7, 20].

<sup>2</sup>If one considers just the  $R\chi T$  Lagrangian  $\mathcal{L}_{GB} + \mathcal{L}_R$  [4], the NLO terms  $A(\mu)$  and  $B(\mu)$  result [18]

$$\begin{aligned} A(\mu) &= -\frac{3d_m^2 M_P^2}{\pi^2 F^2} \left( \ln \frac{M_P^2}{\mu^2} - 1 \right) + \frac{3c_m^2 M_S^2}{\pi^2 F^2} \left( \ln \frac{M_S^2}{\mu^2} - 1 \right) + \frac{6c_d^2 c_m^2 M_S^2}{\pi^2 F^4} \\ &\quad - \frac{6c_d c_m M_S^2}{\pi^2 F^2} \left( \ln \frac{M_S^2}{\mu^2} + \frac{1}{4} \right) + \frac{9c_d^2 M_S^2}{4\pi^2 F^2} \left( \ln \frac{M_S^2}{\mu^2} + \frac{1}{2} \right) + \frac{9G_V^2 M_V^2}{8\pi^2 F^2} \left( \ln \frac{M_V^2}{\mu^2} + \frac{1}{2} \right), \\ B(\mu) &= -\frac{3d_m^2 M_P^4}{2F^2 \pi^2} + \frac{9c_d^2 c_m^2 M_S^4}{F^4 \pi^2} + \frac{3c_m^2 M_S^4}{2F^2 \pi^2} - \frac{6c_d c_m M_S^4}{\pi^2 F^2} - \frac{9c_d^2 M_S^4}{4\pi^2 F^2} \left( \ln \frac{M_S^2}{\mu^2} - \frac{1}{2} \right) \\ &\quad + \frac{3c_d c_m M_S^4}{\pi^2 F^2} \ln \frac{M_S^2}{\mu^2} - \frac{9G_V^2 M_V^4}{8\pi^2 F^2} \left( \ln \frac{M_V^2}{\mu^2} - \frac{1}{2} \right). \end{aligned}$$

the couplings in (2.5) are the renormalized ones.

One can then consider a different renormalization scheme for  $\kappa = c_m, d_m, M_S, M_P$  (denoted with hat in the new scheme). The difference between the two schemes would be provided by the shifts  $\kappa = \hat{\kappa} + \Delta\kappa$ , with  $\Delta\kappa$  a finite constant formally subleading. Since  $A(\mu)$  and  $B(\mu)$  are already NLO in (2.5), their variation is sub-subdominant and can be neglected, leaving

$$\begin{aligned}\alpha_2^{(p)} &= 2F^2 + 16\hat{d}_m^2 - 16\hat{c}_m^2 + [32\hat{d}_m\Delta d_m - 32\hat{c}_m\Delta c_m + A(\mu)] = 0, \\ \alpha_4^{(p)} &= 16\hat{d}_m^2\hat{M}_P^2 - 16\hat{c}_m^2\hat{M}_S^2 + [32\hat{M}_P^2\hat{d}_m\Delta d_m + 16\hat{d}_m^2\Delta M_P^2 - 32\hat{M}_S^2\hat{c}_m\Delta c_m - 16\hat{c}_m^2\Delta M_S^2 + B(\mu)] = 0.\end{aligned}\quad (2.6)$$

The terms within the brackets,  $[\dots]$ , correspond to the finite renormalized contributions from the one-loop diagrams in the new scheme. In general, one finds that the expressions in the brackets suffer from large numerical uncertainties, depending on the precise values of the resonance couplings. However, there is a convenient scheme where the expressions in the brackets become zero. In that case, (2.6) shows the same structure of the large- $N_C$  WSRs [7], though now in terms of renormalized parameters  $\hat{\kappa}$ . Furthermore, the change of scheme does not change the low-energy prediction for the LECs [18]. It just removes the former uncertainty in the NLO high-energy constraints (2.5).

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