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$\Lambda_{\overline{MS}}$ from renormalization group optimized perturbation

Jean-Loïc KNEUR* **and André NEVEU**
CNRS, Laboratoire Charles Coulomb UMR 5221, F-34095, Montpellier, France
Université Montpellier 2, Laboratoire Charles Coulomb UMR 5221, F-34095, Montpellier, France
E-mail: `jean-loic.kneur@univ-montp2.fr`, `andre.neveu@univ-montp2.fr`

A variationally optimized perturbation, combined with renormalization group properties, can give approximations to certain nonperturbative quantities like the chiral symmetry breaking order parameters. We evaluate, up to third order in this modified perturbation, $F_\pi/\Lambda_{\overline{MS}}$, where F_π is the pion decay constant and $\Lambda_{\overline{MS}}$ the basic QCD scale in the $\overline{MS}$ scheme. We obtain $\Lambda_{\overline{MS}}^{n_f=2} \simeq 255^{+40}_{-15}$ MeV, including rather conservative estimates of theoretical uncertainties of the method. This compares reasonably well with some recent lattice determinations.

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*Speaker.

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In the massless quarks limit, the strong coupling $\alpha_S(\mu)$ at a given reference scale μ is the only QCD parameter. Equivalently the Renormalization-Group (RG) invariant scale

$$\Lambda_{\overline{MS}}^{n_f} \equiv \mu e^{-\frac{1}{\beta_0 \alpha_S}} (\beta_0 \alpha_S)^{-\frac{\beta_1}{2\beta_0^2}} (\cdots) , \quad (1)$$

in a specified renormalization scheme, is the fundamental QCD scale (β_0, β_1 are one- and two-loop RG beta function coefficients, and ellipsis denote higher RG orders corrections). $\Lambda_{\overline{MS}}^{n_f}$ depends on the number of active quark flavors n_f, with perturbative matching at the quark mass thresholds[1]. Although the present α_S World average is impressively accurate [1]: $\alpha_S(m_Z) \simeq .118 \pm .001$, it remains of great interest to estimate $\Lambda_{\overline{MS}}$ from other observables and other theoretical approaches, specially to access the infrared QCD regime for $n_f = 2(3)$, where a perturbative extrapolation from $\alpha_S(m_Z)$ is unreliable. Indeed $\Lambda_{\overline{MS}}$ determination from Lattice calculations is a very active topics.

Here we explore an alternative determination of $\Lambda_{\overline{MS}}$, exploiting the fact that the precisely known pion decay constant F_π should be entirely determined by $\Lambda_{\overline{MS}}$ in the strict chiral limit. A problem, however, is how to calculate $F_\pi/\Lambda_{\overline{MS}}$ in the nonperturbative regime at the relevant scale close to $\Lambda_{\overline{MS}}$. Moreover, the standard F_π perturbative series, being proportional to light quark masses m_q, vanishes in the strict chiral limit $m_q \to 0$. One can circumvent both problems by a modification of the ordinary perturbative expansion. The basic idea [2] is to introduce in the standard QCD Lagrangian a new expansion parameter $0 < \delta < 1$, interpolating between $\mathscr{L}_{free}$ and $\mathscr{L}_{interaction}$, such that the (current) quark mass m_q becomes an arbitrary parameter. It is perturbatively equivalent to taking a standard renormalized series in $g \equiv 4\pi\alpha_S$, re-expanded in powers of δ after substitution:

$$m_q \to m\,(1-\delta)^a, \;\; g \to \delta\, g \, . \quad (2)$$

This procedure is consistent with renormalizability and gauge invariance. The extra parameter a in (2) reflects some freedom in the interpolating form, allowing to impose further physical constraints. The $\delta \to 1$ limit is taken after δ-expansion, to recover the original *massless* theory, but leaves a remnant m-dependence at finite δ^k-order, and m is typically fixed by an optimization (OPT) prescription[3]. The convergence of this procedure, which can be seen as a particular case of "order-dependent mapping"[4], has been proven[5] for the $D = 1$ $\lambda\phi^4$ oscillator. In $D > 1$ renormalizable models, the large δ-orders behavior is quite involved (see e.g. [6]), but anyhow the method provides well-defined approximations to certain nonperturbative quantities beyond the mean field approximations. Some previous applications in QCD [7] implied a complicated resummation of renormalization group (RG) dependence, not easy to generalize beyond the first few orders. In contrast our recent approach[8, 9] introduces a straightforward marriage of OPT and RG properties. Start from a standard perturbative expansion for a physical quantity $P(m,g)$, after applying (2) and expanding in δ at order k. In addition to the OPT equation:

$$\frac{\partial}{\partial m} P^{(k)}(m,g,\delta=1)|_{m\equiv\tilde{m}} \equiv 0 \, , \quad (3)$$

we require the (δ-modified) series to satisfies a standard RG equation: $\mu \frac{d}{d\mu}\left(P^{(k)}(m,g,\delta=1)\right) = 0$, where the usual RG operator is $\mu\frac{d}{d\mu} = \mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \gamma_m(g)\, m\frac{\partial}{\partial m}$. (Our normalization is $\beta(g) \equiv dg/d\ln\mu = -2b_0 g^2 - 2b_1 g^3 + \cdots$, $\gamma_m(g) = \gamma_0 g + \gamma_1 g^2 + \cdots$ with b_i, γ_i up to 4-loop given in [10]).

Combined with Eq. (3), the usual RG equation takes a reduced form:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right]P^{(k)}(m,g,\delta=1)=0 \tag{4}$$

such that Eqs. (4), (3) together completely fix optimized values $m\equiv\tilde{m}$ and $g\equiv\tilde{g}$.

We now illustrate this concretely on the perturbative series relevant for the pion decay constant F_π. Start from the definition of F_π from the axial current correlator:

$$i\langle 0|TA^i_\mu(p)A^j_\nu(0)|0\rangle \equiv \delta^{ij}g_{\mu\nu}F^2_\pi+\mathscr{O}(p_\mu p_\nu) \tag{5}$$

where the axial current is $A^i_\mu \equiv \bar{q}\gamma_\mu\gamma_5\frac{\tau_i}{2}\,q$, and $F_\pi\sim 92.3$ MeV [1]. The perturbative expansion of (5) in the $\overline{MS}$ scheme is known up to 4-loop orders [11]:

$$F^2_\pi = 3\frac{m^2}{2\pi^2}\left[-L+\frac{\alpha_S}{4\pi}(8L^2+\frac{4}{3}L+\frac{1}{6})+(\frac{\alpha_S}{4\pi})^2[f_{30}L^3+f_{31}L^2+f_{32}L+f_{33}]+\mathscr{O}(\alpha_S^3)\right]+\mathrm{div} \tag{6}$$

where $L\equiv\ln\frac{m}{\mu}$, and the coefficients f_{ij} are extracted from [11]. One subtlety, however, is that *e.g.* in dimensional regularization (6) requires an extra subtraction renormalization, of the form $m^2\times 1$ on dimensional grounds. To obtain a finite RG-invariant expression this subtraction should be performed consistently with RG properties, and fixing its perturbative expansion at order k needs knowing the coefficient of the L term at order $k+1$ [7]. We thus apply to the (subtracted)

Table 1: Combined OPT+RG results at successive δ-order

δ-order k	$\frac{F_\pi^{(k)}(\tilde{m},\tilde{g})}{\Lambda_{\overline{MS}}}$	$\tilde{L}$	$\tilde{\alpha}_S$
1	$0.372\pm 0.16i$	$-0.45\pm 0.11i$	$1.01\pm 0.08i$
2	$0.353\pm 0.03i$	$-0.52\mp 0.69i$	$0.73\pm 0.02i$
3 ($s_4=\mathrm{PA}[1,2]$)	$0.341\pm 0.07i$	$-0.23\mp 0.04i$	$0.59\pm 0.31i$

RG-invariant perturbative series for F_π the procedure (2), at orders δ^k, then solving OPT and RG Eqs.(3), (4). Now the latter being polynomial in (L,g), at increasing δ-orders there are (too) many solutions, most being complex (conjugates). A very natural selection comes about by selecting only the solution matching asymptotically the standard perturbative RG behavior for $g\to 0$:

$$\tilde{g}(\mu\gg\tilde{m})\sim(2b_0\ln\frac{\mu}{\tilde{m}})^{-1}\,. \tag{7}$$

However, to have RG OPT solutions behaving as (7) at any δ^k-orders, requires a critical value of a in (2): $a=\gamma_0/(2b_0)$. This connection with RG anomalous dimensions was observed similarly in other theories, e.g. Φ^4 in $D=3$ [12, 13].

The RG criteria (7) appears to give unique solutions, given in Table 1. But those solutions remain complex (conjugates). Since this is unphysical, we can only expect acceptable solutions to have at least $\mathrm{Re}(\tilde{g})>0$ and $\mathrm{Im}(F_\pi)\ll\mathrm{Re}(F_\pi)$, the imaginary part indicating an intrinsic theoretical uncertainty. Comparing second and first δ-orders in Table 1, the solution has a much smaller imaginary part, and $\mathrm{Re}\,\tilde{\alpha}_S$ decreases to reasonably perturbative values as the δ-order increases. At order δ^3, the subtraction needs knowledge of the presently unknown 5-loop coefficient of L, so we have estimated it from a Padé Approximant PA$[1,2]$ from lower orders. Optimal RG solutions are remarkably stable with respect to such approximations on the 4-loop order, or RG truncations, with at most $\sim$ 2-3% differences on $\Lambda^{n_f=2}_{\overline{MS}}$. In addition we incorporate a more intrinsic error, taking the range spanned by $\mathrm{Re}(F_\pi(\tilde{g},\tilde{L})) - F_\pi(\mathrm{Re}(\tilde{g}),\mathrm{Re}(\tilde{L}))$, as this tends to maximize the uncertainty for increasing $\mathrm{Im}(\tilde{g},\tilde{L})$. Clearly the occurrence of complex solutions is the main source of theoretical uncertainties, and we adopt in this way a conservative estimate of theoretical errors. Finally we can subtract out the explicit chiral symmetry breaking effects from small $m_u, m_d \neq 0$. Denoting F as the F_π value in the chiral limit $m_u, m_d \to 0$, Lattice calculations recently obtained [14]: $F_\pi/F \sim 1.073 \pm 0.015$, which we take into account in the final $\Lambda^{n_f=2}_{\overline{MS}}$ numerical value. With all theoretical uncertainties (linearly) combined we thus obtain:

$$\Lambda^{n_f=2}_{\overline{MS}} \simeq 255 \pm 15^{+25}\ \mathrm{MeV}\ , \tag{8}$$

where the central value corresponds to $\mathrm{Re}\,F^2_\pi(\tilde{g},\tilde{L})$, the first errors include higher order and F_π/F above mentioned uncertainties, and the upper bound corresponds to $F^2_\pi(\mathrm{Re}(\tilde{g}),\mathrm{Re}(\tilde{L}))$. This compares reasonably well with different kinds of lattice calculations [15] (see [9] for a discussion).

In conclusion, a straightforward implementation of RG properties within a variationally optimized perturbation gives $\Lambda_{\overline{MS}}$ values, with a remarkable stability at successive perturbative orders. In principle one could extrapolate to $\alpha_S(\mu)$ at high (perturbative) scale μ. Now, since our RG-improved OPT modifies perturbative expansions, it should also be used consistently to extrapolate to higher scales, which can differ substantially from a standard perturbative extrapolation. We therefore leave for future work a precise determination of $\alpha_S(m_Z)$, incorporating also explicit chiral symmetry breaking effects in this framework.

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