

Kaon bag parameter B_K at the physical mass point

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[Budapest-Marseille-Wuppertal Collaboration]

A summary of the recent determination of the kaon bag parameter B_K by the Budapest-Marseille-Wuppertal collaboration is given. We use a subset of our new 2+1 flavor ensembles with tree-level Symanzik glue and 2-step HEX-smearred clover fermions, featuring pion masses between 120 MeV and 380 MeV, a total of 4 lattice spacings (at 2 of which the physical mass point is reached) and large volumes (up to 6 fm). We use the RI/MOM scheme with trace subtraction to renormalize the data non-perturbatively. We find $B_K^{\text{RI}}(3.5 \text{ GeV}) = 0.5308(56)(23)$ at the physical mass point, in the continuum and in infinite volume, with all systematics accounted for. Finally, the details of the conversion to RGI and $\overline{\text{MS}}$ conventions are discussed.

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1. Introduction

In the Standard Model of particle physics the Cabibbo-Kobayashi-Maskawa (CKM) matrix contains a complex phase which is the only source of CP violation in hadronic matter. In the kaon system, it manifests itself as “direct” (in $K_L \rightarrow 2\pi$ decays) and “indirect” (through $K^0 \leftrightarrow \bar{K}^0$ mixing prior to the decay) CP violation, the latter being quantified by the parameter ε_K . By combining an experimental value of ε_K with a lattice determination of the bag parameter

$$B_K = \frac{\langle \bar{K}^0 | O^{\Delta S=2} | K^0 \rangle}{\frac{8}{3} \langle \bar{K}^0 | A_\mu | 0 \rangle \langle 0 | A_\mu | K^0 \rangle} \quad (1.1)$$

a constraint on the unitarity triangle parameters $(\bar{\rho}, \bar{\eta})$ is obtained (see e.g. [1] for details). Here $O^{\Delta S=2} = [\bar{s}\gamma_\mu(1-\gamma_5)d][\bar{s}\gamma^\mu(1-\gamma_5)d]$ denotes a local 4-fermion operator with $\Delta S=2$, and this leads to the contraction topologies shown in the left panel of Fig. 1.

2. Action details

A precise determination of B_K with Wilson fermions is usually considered tricky, because the Wilson action breaks chiral symmetry, and in consequence the renormalization pattern of $O_1 = VV + AA$ is not the same as in the continuum, but involves the other dimension six operators $O_2 = VV - AA$, $O_{3/4} = SS \mp PP$, $O_5 = TT$, too. Here V, A, S, P, T denote vector, axial-vector, scalar, pseudoscalar and tensor $\Delta S = 1$ bilinears, respectively.

Based on the observation of [2, 3] that the *combination* of link smearing and (even tree-level) $O(a)$ -improvement drastically reduces the amount of chiral symmetry breaking [both in perturbation theory and non-perturbatively], one may attempt a precise determination of B_K , with the hope that the mixing coefficients Δ_{1k} (see below) turn out to be small.

Our determination of B_K is based on the $N_f = 2+1$ and $N_f = 3$ ensembles that were generated for our study of light quark masses [4, 5]. We used a tree-level Symanzik action for the gluons and a 2-fold HEX [3] smeared clover action with $c_{SW} = 1$ for the fermions. This, together with our choice to opt for large space-time volumes, resulted in extremely favorable algorithmic properties [6], excellent scaling of hadron masses and reliable data in the RI/MOM renormalization scheme [7]. The parameters $\beta = 3.5, 3.61, 3.7, 3.8$ correspond to the lattice spacings $a = 0.093, 0.077, 0.065, 0.054$ fm, respectively, the range of pion masses is $M_\pi = 120 - 380$ MeV, and the spatial box sizes extend up to 6 fm. The final result for B_K appeared in [8].

3. Non-perturbative renormalization

The $N_f = 3$ ensembles have the same β -values as the $N_f = 2+1$ ensembles, with several (degenerate) quark masses to extrapolate to the chiral limit (see [5, 8] for details). They are used to renormalize the operators O_i ($i = 1, \dots, 5$) in the RI/MOM scheme [9]. We denote their bare matrix elements $\langle \bar{K}^0 | O_i | K^0 \rangle$ by Q_i . The renormalization pattern is then given by [10]

$$Q_i^{\text{ren}} = \tilde{Z}_{ik} Q_k = Z_{ij} (\delta_{jk} + \Delta_{jk}) Q_k. \quad (3.1)$$

which means that we need to calculate $Z_{B_K} \equiv Z_{11}/Z_A^2$ and the mixing coefficients Δ_{1k} for $k = 2, \dots, 5$.

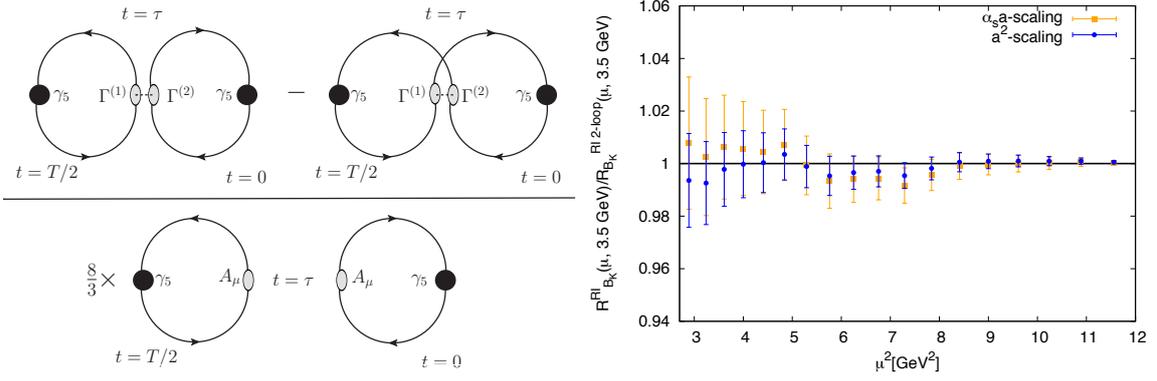


Figure 1: *Left:* Schematic view of the contraction topologies that contribute to B_K . In our study the black blobs represent wall sources at $t = 0$ and $T/2$. *Right:* Ratio of the non-perturbative evolution $R_{B_K}^{\text{RI}}(\mu, 3.5 \text{ GeV}) = Z_{B_K}^{\text{RI}}(\mu)/Z_{B_K}^{\text{RI}}(3.5 \text{ GeV})$ to the same quantity in 2-loop perturbation theory, after an extrapolation to the chiral limit and the continuum, using the $O(\alpha_s a)$ (orange) or $O(a^2)$ (blue) ansatz.

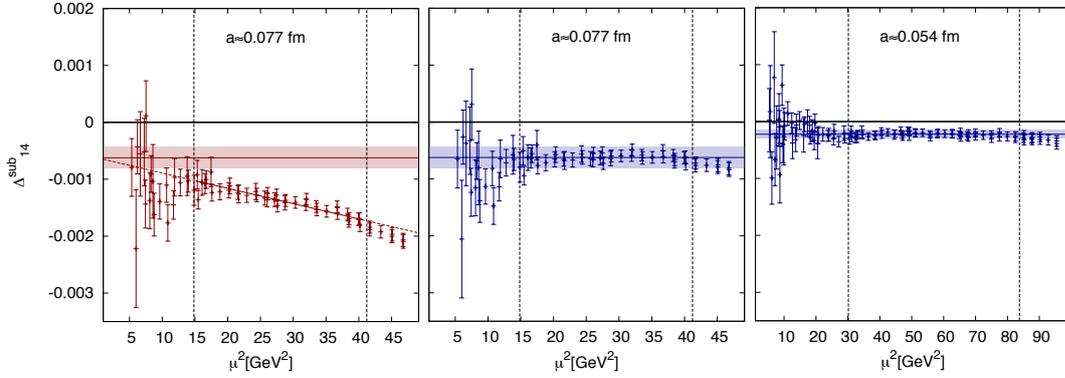


Figure 2: *Left:* Mixing coefficient $\Delta_{14}^{\text{sub}}(\mu^2)$ in the chiral limit at $\beta = 3.61$. *Center:* Ditto, after fitting the linear piece away. *Right:* Ditto, for our finest lattice ($\beta = 3.8$); here the value is $O(10^{-4})$.

At the coarsest β we cannot reach the desired renormalization scale $\mu = 3.5 \text{ GeV}$ without violating the constraint $\mu < \pi/(2a)$ which we impose to avoid large discretization artefacts. Hence, to circumvent the well known window problem of RI/MOM we use the same procedure as in [4, 5], i.e. we perform a separate continuum limit of the (chirally extrapolated) evolution function $R_{B_K}^{\text{RI}}(\mu, 3.5 \text{ GeV}) = Z_{B_K}^{\text{RI}}(\mu)/Z_{B_K}^{\text{RI}}(3.5 \text{ GeV})$. In the entire range $1.7 - 3.5 \text{ GeV}$ (thus covering more than a factor 2 in scale) we see no difference to the perturbative evolution (see Fig. 1). This may, at least in part, be due to the subtraction of a contact term (“trace subtraction”), see [11].

Next we calculate the mixing coefficients Δ_{1k} as in [10], with the additional subtraction [12]

$$\Delta_{1k}^{\text{sub}}(a, m_1, m_2) = \frac{m_1 \Delta_{1k}(a, m_1) - m_2 \Delta_{1k}(a, m_2)}{m_1 - m_2} \quad (3.2)$$

where the m_i denote the quark masses. This procedure removes $O(p^{-2})$ contributions coming from virtual pion exchanges. The dominant corrections are then $O((ap)^2)$ discretization errors and an $O(p^{-4})$ term attributed to double pion exchanges. The result for $\Delta_{14}^{\text{sub}}(p^2)$ at $\beta = 3.61$ ($a = 0.077 \text{ fm}$) is shown in Fig. 2 before (left) and after (middle) subtracting the linear piece in p^2 . At $\beta = 3.8$ ($a = 0.054 \text{ fm}$) the resulting value is $O(10^{-4})$, and for Δ_{12}^{sub} the resulting values are

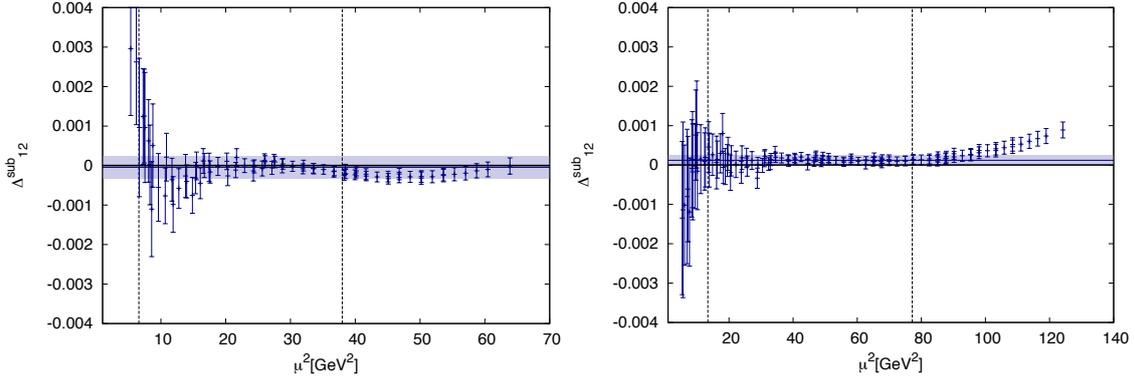


Figure 3: Mixing coefficient $\Delta_{12}^{\text{sub}}(\mu^2)$ in the chiral limit at $\beta = 3.61$ (left) and $\beta = 3.8$ (right); note the scale.

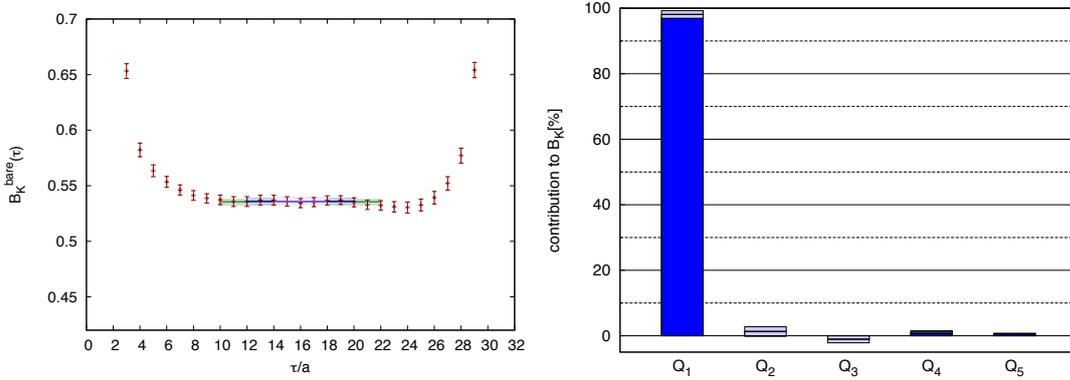


Figure 4: Left: Plateau of $B_K^{\text{bare}}(\tau)$ on a $48^3 \times 64$ lattice at $\beta = 3.7$ (symmetrized about $T/2$). Right: Relative contributions of Q_1 and $\Delta_{1k}Q_k$ ($k = 2, \dots, 5$) to B_K on our lightest ensemble ($M_\pi \simeq 120 \text{ MeV}$, $\beta = 3.61$).

even consistent with zero within errors (Fig. 3). This is in line with our expectation that the 2 HEX action features small chiral symmetry violations.

4. Matrix elements

To compute the bare matrix elements, we use random $U(1)$ wall sources, placed at $t = 0$ and $t = T/2$ of our $N_f = 2 + 1$ lattices, and vary the time slice τ of the operator insertion between 1 and $T - 1$. The matrix elements Q_i for $i = 1, \dots, 5$ are determined by performing constant fits of the time-symmetrized plateaus in τ as shown in Fig. 4 (left). We use three different ranges to estimate systematic effects due to excited states. By combining the results from these fits with the mixing coefficients Δ_{1k} and the evolution factor $R_{B_K}^{\text{RI}}(\mu, 3.5 \text{ GeV})$ that were calculated previously, we determine $B_K^{\text{RI}}(3.5 \text{ GeV})$ for each ensemble. It turns out that even for our lightest pion mass ($M_\pi = 120 \text{ MeV}$) the chirality breaking contributions from $\Delta_{1k}Q_k$ ($k = 2, \dots, 5$) are much smaller than the leading contribution from Q_1 , see Fig. 4 (right).

5. Physical B_K in RI conventions

With the renormalized $B_K^{\text{RI}}(3.5 \text{ GeV})$ for each ensemble in hand, we find the physical B_K by means of a combined interpolation in (m_{ud}, m_s) and a continuum extrapolation. For this combined

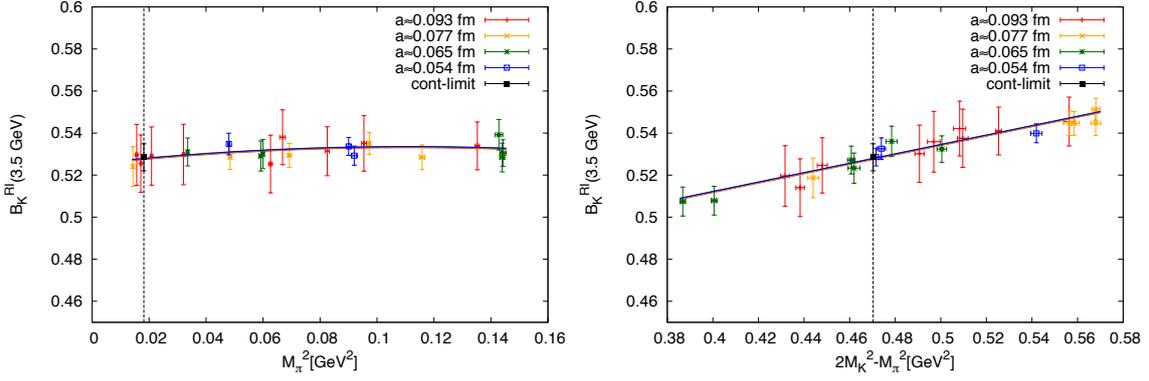


Figure 5: Continuum extrapolation and interpolation in M_π^2 (left) and $2M_K^2 - M_\pi^2$ (right) of $B_K^{\text{RI}}(3.5 \text{ GeV})$.

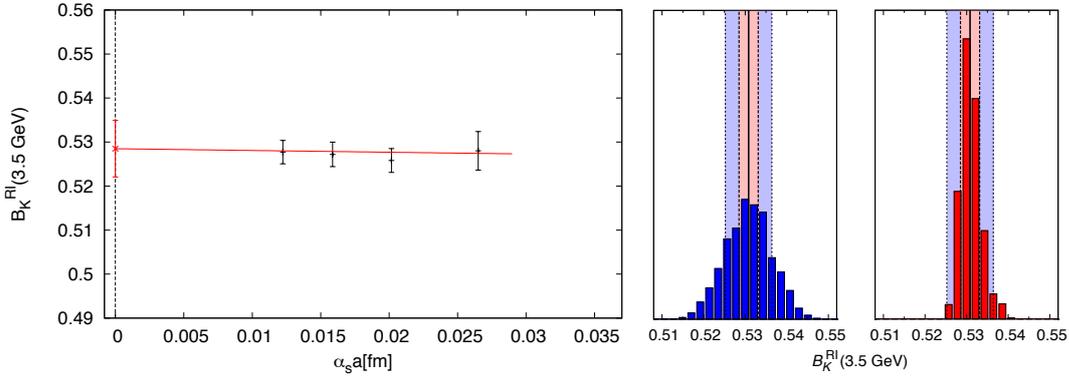


Figure 6: Left: Continuum extrapolation of $B_K^{\text{RI}}(3.5 \text{ GeV})$; here with $\alpha_s a$ discretization terms. Right: Statistical (blue histogram from 2000 bootstrap samples) and systematic (red histogram from 5760 analyses) variation of $B_K^{\text{RI}}(3.5 \text{ GeV})$. The respective 1σ variations are indicated in the background in both cases.

fit we choose the following functional ansatz

$$B_K^{\text{RI}}(3.5 \text{ GeV}, x, y, a) = B_K^{\text{RI}}(3.5 \text{ GeV}) \cdot f(x, y) + d(a) \quad (5.1)$$

where $f(x, y)$ with $x = M_\pi^2$ and $y = 2M_K^2 - M_\pi^2$ describes the quark mass dependence, and $d(a)$ denotes a discretization term. The generic form of $f(x, y)$ is

$$f(x, y) = 1 + a_{10}x + a_{20}x^2 + a_{01}y + a_{11}xy - \frac{a_\chi x}{32\pi^2 f_0^2} \log\left(\frac{x}{\mu^2}\right) \quad (5.2)$$

where we let the fit adjust a selection of the coefficients and set the remaining ones to zero, since otherwise all coefficients tend to be consistent with zero, within errors – see [8] for details. In the same spirit $d(a)$ is chosen proportional to *either* $\alpha_s a$ or a^2 , but not to both of them simultaneously. It turns out that all these fits yield acceptable $\chi^2/\text{d.o.f.}$, and hence represent reasonable ways of calculating the physical value of B_K . An example of such an interpolation in M_π^2 and $2M_K^2 - M_\pi^2$ is shown in Fig. 5. It is interesting to note that a nearly flat behavior in m_{ud} is in stark contrast to a much steeper behavior in m_s . The data show no visible a -dependence – an observation which is in line with the rather flat continuum extrapolation shown in the left panel of Fig. 6.

We also test the effect of cutting away pion masses above 340 MeV, rather than 380 MeV. This, together with different fit ranges for the pion and kaon mass determination, different fit ranges

for the matrix element, and different choices for the determination of the mixing terms and the renormalization factor Z_{B_K} yields a total of 5760 reasonable analysis procedures. There is no reason for preferring one over the other. Accordingly it makes sense to consider the overall distribution as our main result, and not a particular individual fit. We take the median as our central value of B_K , and the central 68% as our best estimate for the inherent systematic uncertainty. By repeating this entire procedure on 2000 bootstrap samples we also obtain the statistical errors of these quantities.

For the case of our analysis of B_K the statistical and systematic distributions are shown in the right panel of Fig. 6. Based on these distributions we quote as our final result

$$B_K^{\text{RI}}(3.5 \text{ GeV}) = 0.5308(56)_{\text{stat}}(23)_{\text{syst}}. \quad (5.3)$$

Evidently, the systematic error in (5.3) is reliable to the extent to which the systematic “looping” over the analysis options includes the main uncertainties. Here we carefully examined this point to make sure that the variation adequately explores all sources of systematic error. In such a case our analysis method, which was also used in [4, 5], is an automated way of transforming the spread that results from reasonable analysis options into a systematic uncertainty of the final result.

6. Conversion to RGI and $\overline{\text{MS}}$ conventions

Our main result (5.3) is given in a scheme where everything is done fully non-perturbatively. For phenomenological applications, nevertheless, often RGI or $\overline{\text{MS}}$ conventions are preferred, and we now give details of how we convert our result into these frameworks, and how we assess the extra conversion error that is to be attached to the converted result.

The master formula that relates the renormalization group invariant B_K^{RGI} , in the literature often denoted by \hat{B}_K , to $B_K(\mu)$ in a generic scheme (e.g. RI or $\overline{\text{MS}}$) reads

$$B_K^{\text{RGI}} = \left(\frac{g(\mu)^2}{4\pi} \right)^{-\gamma_0/(2\beta_0)} \exp \left\{ \int_0^{g(\mu)} dg \left(\frac{\gamma(g)}{\beta(g)} + \frac{\gamma_0}{\beta_0 g} \right) \right\} B_K(\mu) \quad (6.1)$$

$$= \left(\frac{g(\mu)^2}{4\pi} \right)^{-\gamma_0/(2\beta_0)} \left\{ 1 + \frac{g(\mu)^2}{(4\pi)^2} \left[\frac{\beta_1 \gamma_0 - \beta_0 \gamma_1}{2\beta_0^2} \right] + \dots \right\} B_K(\mu) \quad (6.2)$$

where the running coupling, as well as the β - and γ -functions, are specific to that scheme. This means that the conversion from RI to $\overline{\text{MS}}$ proceeds by using (6.1) or (6.2) twice – once with RI quantities on the right-hand side, and then backwards with $\overline{\text{MS}}$ quantities on the right-hand side.

A specific property of RI and $\overline{\text{MS}}$ is that they share the same β -function up to 4-loop order, while the γ -function of B_K is only known at 2-loop order, with a shared 1-loop part [13]. Since (6.1) and (6.2) differ by 3-loop contributions to the γ -function, they are at the practical level equivalent. On this basis, we decide to give the conversion factors and their uncertainties as follows. With the 4-loop β -function and the 2-loop γ -function the conversion to the RGI or ($\overline{\text{MS}}$, 2GeV) frameworks entails a conversion factor of 1.457 and 1.062, respectively. Using instead the 4-loop β -function and the 1-loop γ -function yields the factors 1.427 and 1.047, respectively, and hence ultra-conservative conversion errors of order 2% and 1%. In our opinion a better estimate results from using again the 2-loop γ -function, but trading (6.2) for (6.2); this suggests conversion errors in the per-mil range. Based on this we decide to assign all converted results a generous extra 1% error to reflect the perturbative uncertainty inherent in the conversion.

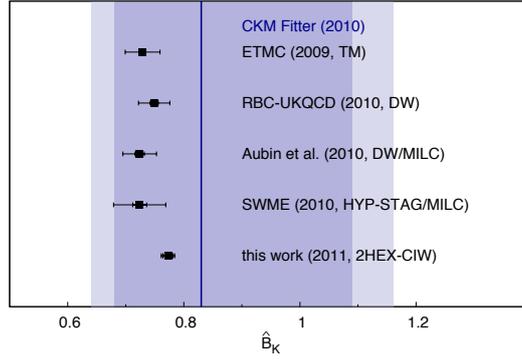


Figure 7: Our result (7.1) in comparison with a few other recent determinations from $N_f = 2$ [ETMC] or $N_f = 2 + 1$ [RBC-UKQCD, Aubin et al., SWME] QCD simulations (see [1, 8] for further details).

7. Summary

The calculation described in [8] and in this proceedings contribution yields the main result (5.3) which is fully non-perturbative and in the RI scheme at the renormalization scale $\mu = 3.5$ GeV. Using perturbation theory, this result is converted into RGI and $\overline{\text{MS}}$ conventions, in the latter case at the scale $\mu = 2$ GeV. Our converted result then reads

$$\hat{B}_K = 0.7727(81)_{\text{stat}}(34)_{\text{syst}}(77)_{\text{pert}}, \quad B_K^{\overline{\text{MS}}}(2 \text{ GeV}) = 0.5644(59)_{\text{stat}}(25)_{\text{syst}}(56)_{\text{pert}} \quad (7.1)$$

with an extra 1% conversion error to reflect the uncertainty in this last step.

A comparison with other recent determinations of \hat{B}_K is shown in Fig. 7. It seems worth pointing out that our result has a combined error of 1.5%, while ETMC has 4.1% [14], RBC-UKQCD has 3.6% [15], Aubin et al. 4.0% [16], and SWME have a combined error of 6.3% [17].

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