

Corrections to the Banks-Casher relation with Wilson quarks

CERN-PH-TH/2013-016

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The Banks-Casher relation links the spectral density of the Dirac operator with the existence of a chiral condensate and spontaneous breaking of chiral symmetry. This relation receives corrections from a finite value of the quark mass, a finite space-time volume and, if evaluated on a discrete lattice, from the finite value of the lattice spacing a . We present a status report of a determination of these corrections for Wilson quarks.

*The 7th International Workshop on Chiral Dynamics,
August 6 -10, 2012
Jefferson Lab, Newport News, Virginia, USA*

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1. Introduction

The Banks-Casher relation [1] relates the spectral density ρ_D of the Hermitian Dirac operator $-iD$ with the chiral condensate Σ

$$\rho_D(\gamma, m) = \frac{\Sigma}{\pi} [1 + O(\{|\gamma|, m\}/\Lambda_{QCD})], \quad (1.1)$$

where γ_k are the eigenvalues of the massless operator and m is the sea quark mass. In principle the Banks-Casher relation provides a tool to determine the chiral condensate [2] with a lattice QCD computation.

A more standard way to determine the chiral condensate with lattice QCD computations is to study the quark mass dependence of the pion mass and comparing it with the predictions of chiral perturbation theory (χ PT). Recent lattice calculations are performed close to the physical values of the quark masses (see ref. [3, 4] for recent reviews). To control accurately the light quark mass dependence of hadronic quantities it is important, if possible, to have independent determinations of leading order (LO) low energy constants (LECs), as for example the chiral condensate.

In the plot in fig. 1 we show the light quark mass dependence of the squared pion mass computed by the European Twisted Mass Collaboration (ETMC) with $N_f = 2$ dynamical light quarks [5]. Despite the rather good description of the lattice results with χ PT, it is obviously desirable being able to predict, in an independent way, the behaviour close to the chiral limit (red ellipse) to better constrain the light quark mass dependence. A constraint of the chiral fit will be beneficial for a more accurate determination of the next-to-leading order LECs and a better confidence on the chiral fits.

An example of independent determination of LO LECs is provided by lattice calculations in the so called epsilon-regime, where first estimates of the chiral condensate and decay constant are rather encouraging [6, 7]. Using Wilson-type fermions as a QCD discretization, calculations are affected by potentially large cutoff effects, thus it is important to have a theoretical analysis of quantities such as the spectral density, based on Wilson chiral perturbation theory (W χ PT) [10, 11].¹ These proceedings are a status report of an ongoing attempt to understand cutoff and finite size effects affecting the spectral density of the Wilson operator.

2. Chiral condensate from the mode number

With Wilson fermions it is advantageous to consider the Hermitean Wilson-Dirac operator $Q = \gamma_5 D_m$, where D_m is the massive Wilson operator. To compute the spectral density ρ_Q in W χ PT

¹The analysis of two-point functions in the epsilon-regime within the framework of W χ PT has been discussed in refs. [12, 13] and more recently extended in ref. [14].

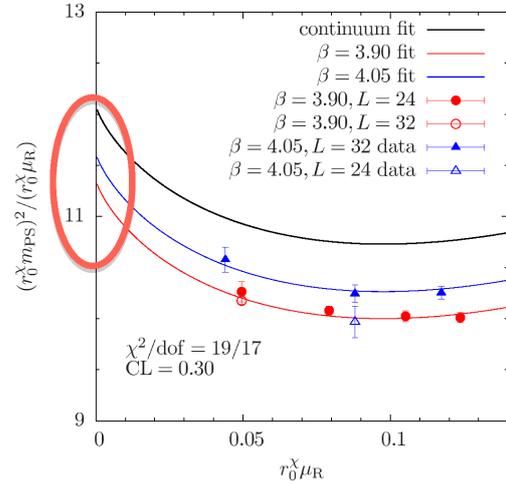


Figure 1: Light quark mass dependence of the ratio between the pseudoscalar meson mass squared and the renormalized light quark mass in units of the Sommer parameter r_0 .

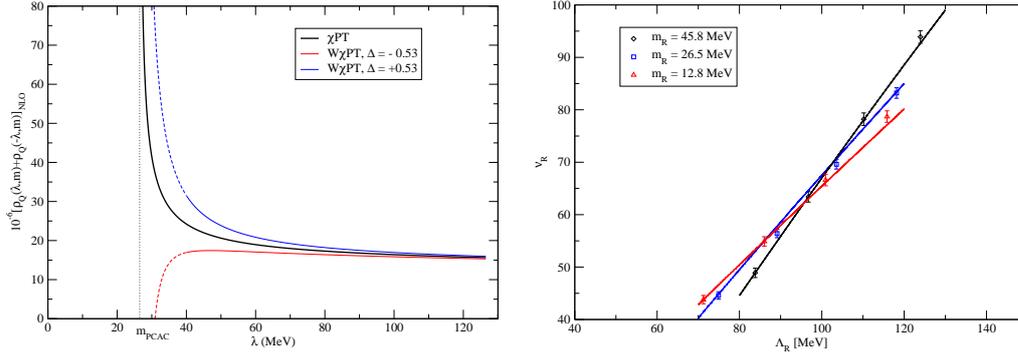


Figure 2: Left plot: the spectral density $[\rho_Q(\lambda, m) + \rho_Q(-\lambda, m)]_{NLO}$ in the infinite volume. We used the parameters $\Sigma = (275 \text{ MeV})^3$, $m_{PCAC} = 26.5 \text{ MeV}$, $F = 90 \text{ MeV}$, $\bar{L}_6 = 5$, $\mu = 139.6 \text{ MeV}$. The solid black line corresponds to the continuum χ PT prediction, while the red (blue) lines correspond to the lattice W χ PT prediction ($O(a)$ -improved) on eq. 2.3 with $\hat{a}^2 W'_8 = \pm 5 \cdot 10^6 \text{ MeV}^4$, corresponding to $\Delta = \mp 0.53$. Right plot: result of the global fit of the data published in [2] with our formula for the renormalised mode number (eqs. 2.4 and 2.3). The fit parameters are Σ , Δ and \bar{L}_6 and we obtain $\chi^2/\text{dof} = 0.91$.

one introduces a flavour doublet of valence fermions χ_v with a Wilson twisted mass action and twisted mass μ_v . The spectral density is related to the discontinuity of the valence pseudoscalar condensate along the imaginary axis in the twisted mass plane [15]

$$\text{Disc} [(\bar{\chi}_v \gamma_5 \tau^3 \chi_v)]|_{\mu_v = i\lambda} = 2i\pi [\rho_Q(\lambda, m) + \rho_Q(-\lambda, m)], \quad (2.1)$$

thus the valence pseudoscalar condensate is a tool to compute the spectral density in W χ PT². After matching the continuum Symanzik effective theory with the generalized effective chiral Lagrangian one needs to choose a proper power counting for the scales involved in the problem. The scales are the sea quark mass, m , the valence twisted mass μ_v (directly related to λ), the lattice spacing a and the linear size of the space-time volume L . In the following we always consider the sea quark mass in the so-called p -regime. For the other scales we consider $m \sim \mu_v \sim a \sim 1/L \sim O(p^2)$. The result of the calculation [16] in terms of the PCAC quark mass m_{PCAC} is given by

$$\begin{aligned} [\rho_Q(\lambda, m_{PCAC}) + \rho_Q(-\lambda, m_{PCAC})]_{NLO} &= 2[\rho_Q(\lambda, m_{PCAC})]_{NLO, cont} \\ &+ \frac{2\Sigma\lambda}{\pi\sqrt{\lambda^2 - m_{PCAC}^2}} \left[\frac{m_{PCAC}^2 \Delta}{\lambda^2 - m_{PCAC}^2} + \frac{16\hat{a}}{F^2} W_6 \right]. \end{aligned} \quad (2.2)$$

$$(2.3)$$

In these formula $\Delta = -\frac{16\hat{a}}{F^2} \left(\frac{W_8}{2} + \frac{W_{10}}{4} + \frac{\hat{a}W'_8}{M_{ss}^2} \right)$, M_{ss} is the pseudoscalar meson mass (made of two sea quarks), Σ and F are the LO LECs, $\hat{a} = 2W_0 a$ and W_0 as the others W s are the LECs parametrizing $O(a)$ and $O(a^2)$ effects. Details on the calculation and a discussion on the applicability of this formula can be found in refs. [16, 17].

Potentially even with Wilson fermions one can use the spectral density, or equivalently the renormalization group invariant (RGI) mode number [2]

$$\nu(\Lambda, m) = V \int_{-\Lambda}^{\Lambda} d\lambda \rho_Q(\lambda, m), \quad (2.4)$$

²We recall that in the continuum the two spectral densities are connected by the relation $\rho_Q(\lambda, m) = \frac{\lambda}{\sqrt{\lambda^2 - m^2}} \rho_D(\sqrt{\lambda^2 - m^2}, m)$.

to determine the chiral condensate. To test our formula we compared the resulting mode number (2.4) with the numerical data published in [2]. We have fixed $F = 90$ MeV and the renormalisation scale $\mu = m_\pi$; we have performed a global fit at all the 3 masses available and all the values of Λ_R with 3 fit parameters: Σ , Δ and \bar{L}_6 .³ From the global fit we obtain

$$\Sigma^{1/3} = 266(7) \text{ MeV}, \quad \Delta = -0.62(80), \quad \bar{L}_6 = 6(1). \quad (2.5)$$

The numerical data and our global fit are shown in the right plot of fig. 2. For Σ we obtain a perfectly consistent result with Giusti and Lüscher [2] without performing any chiral extrapolation. We have also performed a fit fixing $F = 80$ MeV obtaining compatible results within errors. Lattice determinations of the chiral condensate using the mode number can be found in [2, 8, 9].

3. Cutoff effects close to the threshold

To improve the theoretical description of the spectral density of the Wilson operator for $\lambda \simeq m$ one needs to consider two important points. The first one is that finite size effects diverge for $\lambda \rightarrow m$ [16] and the second one is that when $\lambda \simeq m$ the power counting for the valence quark masses need to be reconsidered, i.e. the treatment of the cutoff effects in a perturbative fashion might not be adequate.

To overcome this difficulties we opt for the following power counting

$$m \sim O(p^2), \quad m_P = \sqrt{m_v^2 + \mu_v^2} \sim O(p^4), \quad 1/L, 1/T \sim O(p), \quad a \sim O(p^3), \quad (3.1)$$

that implies cutoff effects affecting the spectral density at NLO order. The framework is the so-called *mixed Chiral Effective Theory* [18], where some masses obey the p -regime counting, and others are in the epsilon-regime. As an intermediate step of our calculation we introduce a θ -term solely in the mass-term of the action as follows

$$\mathcal{L}_2 = \frac{F^2}{4} \text{Tr} [\partial_\mu U \partial_\mu U^\dagger] - \frac{\Sigma}{2} \text{Tr} [U_\theta^\dagger U(x)^\dagger \mathcal{M} + \mathcal{M}^\dagger U(x) U_\theta] - \frac{\hat{a} F^2}{4} \text{Tr} [U + U^\dagger], \quad (3.2)$$

where the mass matrix in the replica formalism [19] is

$$\mathcal{M} = \mathcal{M}^\dagger = \text{diag}(\underbrace{m, \dots, m}_{N_s}, \underbrace{m_v + i\mu_v \tau^3, \dots, m_v + i\mu_v \tau^3}_{N_r}), \quad U_\theta = \text{diag}(e^{i\frac{\theta}{N_s} \mathbb{1}_s}, \mathbb{1}_r). \quad (3.3)$$

While in the continuum it is not important how the θ -term is introduced in the parametrization of the U -field [18, 20], adding a θ -term in the sea sector only, becomes relevant at finite lattice spacing. With this choice, even with a θ -term in the action, we can reabsorb the leading $O(a)$ cutoff effects in a redefinition of the quark mass, because with our choice of power counting (3.1), in the sea sector the leading $O(a)$ effects appear at NNLO, i.e. the sea quarks are effectively in the continuum up to higher order corrections. With this particular choice of power counting and parametrization of the θ -term we can achieve, as in the continuum [18, 20], a factorization of the partition function for the zero and non-zero modes. The periodicity in θ of the chiral Lagrangian allows us to write the partition function in standard fashion

$$\mathcal{Z}(\theta) = \sum_{v=-\infty}^{v=+\infty} e^{-iv\theta} \mathcal{Z}_v, \quad \mathcal{Z}_v = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{iv\theta} \mathcal{Z}(\theta). \quad (3.4)$$

³In this proceedings with Σ we denote its value renormalised in the \overline{MS} scheme at a scale of 2 GeV.

By performing an exact integration over the constant field θ one obtains

$$\mathcal{Z}_V \propto e^{-\frac{N_s v^2}{2z_s}} \int_{U(N_r)} d\bar{U}_0 (\det \bar{U}_0)^v e^{\frac{m_v \Sigma v}{2} \text{Tr}[\bar{U}_0^\dagger + \bar{U}_0]}, \quad z_s = mV\Sigma, \quad (3.5)$$

from which one observes that the distribution of v is Gaussian and it is controlled by the sea quarks which are in the p -regime. The computation of the spectral density is now a computation at fixed v , i.e. $\rho = \sum_v \rho_v \frac{\mathcal{Z}_v}{\mathcal{Z}}$. The sum over v can be done because we know the weight factor $\frac{\mathcal{Z}_v}{\mathcal{Z}}$ (cfr. eq. 3.5). We expand the pseudo Nambu-Goldstone field $U(x)$ around the ground state of the theory U_V

$$U_V = \text{diag}\left(\underbrace{1}_{N_s}, \underbrace{e^{i\tau^3 \omega_0}}_{N_r}\right), \quad \sin \omega_0 = \frac{\mu_v}{m_P}, \quad \cos \omega_0 = \frac{m_v}{m_P}, \quad (3.6)$$

and with this parametrization, the mass term in the chiral Lagrangian becomes like in the untwisted case, with a degenerate polar mass m_P in the valence (replicated) sector.

We compute the pseudoscalar valence condensate (cf. eq. (2.1)) in the chiral effective theory,

$$\langle P_V^3 \rangle = \frac{1}{V} \left\langle \frac{\partial}{\partial J} \int d^4x \mathcal{L}(x) \right\rangle \Big|_{J=0}, \quad (3.7)$$

where now the source term has the following form

$$\mathcal{M} \rightarrow \mathcal{M}_J = \mathcal{M} + J\hat{\tau}^3, \quad \mathcal{M}^\dagger \rightarrow \mathcal{M}_J^\dagger = \mathcal{M}^\dagger - J\hat{\tau}^3, \quad \hat{\tau}^3 = \text{diag}\left(\underbrace{0}_{N_s}, \underbrace{\tau^3}_{2}, 0, \dots, 0\right). \quad (3.8)$$

The final result of the calculation can be written as

$$\begin{aligned} \langle P_V^3 \rangle &= \frac{\Sigma_{\text{eff}}}{2} F_1 + 8B\hat{a}W_6 F_2 + 2B\hat{a}W_8 F_3 \\ &+ \frac{\Sigma}{2} [4B\hat{a}W_6 N_s mV F_2 + \hat{a}^2 V W_6' (F_4 + 4N_s F_2) + \hat{a}^2 V W_7' F_5 + \hat{a}^2 V W_8' F_6], \end{aligned} \quad (3.9)$$

where $F_{i=1..6}$ are integrals over zero modes and Σ_{eff} is an effective chiral condensate which includes NLO corrections from the sea quarks and the lattice spacing

$$\Sigma_{\text{eff}} = (\Sigma_{\text{eff}})_{\text{cont}} + 16\Sigma \frac{\hat{a}W_6}{F^2}. \quad (3.10)$$

The explicit formula for $(\Sigma_{\text{eff}})_{\text{cont}}$ can be found in [21]. In fig. (3) we show the impact of the NLO corrections to the chiral condensate [21]. One observes that in the typical range of quark masses for dynamical simulations the relative corrections induced by the presence of sea quarks can reach 30%. This is a warning in case one would like to extract the chiral condensate from fits of mode numbers where the effect of the sea quarks have been neglected. To compute the zero-modes integrals

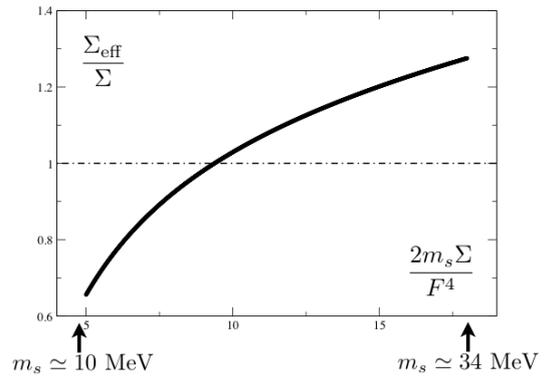


Figure 3: Significance of the NLO corrections for $\Sigma_{\text{eff}}/\Sigma$ in a typical range of sea quark masses for dynamical lattice QCD computations.

we switch to the supersymmetric formulation. All integrals at fixed v can be computed by derivating the graded $SU(2|2)$ partition function [22, 23] with respect to appropriately chosen sources. The integrals have been computed and cross-checked. The final step to determine the formula for the spectral density is the calculation of the discontinuity along the imaginary axis in the twisted valence mass plane (cfr. (2.1)). The analysis of the final result is in progress. We conclude observing that the effects of the sea quarks are twofold. They change the absolute normalization by introducing an m dependence in Σ_{eff} and they control the distribution of v , which, we remark, stays Gaussian only because the sea quarks are in the p -regime.

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