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Study of Anomalous Mass Generation in $N_f = 1$ QCD

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The $U(1)$ axial symmetry in QCD is anomalously broken, and in the case of one flavor, a fermion mass is generated by instanton-like gauge field configurations. Conventional continuum analysis shows that this anomalously generated mass term is “soft” and goes away at large momentum due to the low density of small instantons, distinguishing it from a normal mass term. However, it is possible that there are enough lattice-scale instantons / dislocations to generate a “hard” fermion mass, at least for a class of lattice gauge actions, leading to the mass ambiguity suggested by Creutz. We conclude that, for $N_f = 1$ QCD, there can be an additive renormalization of the fermion mass generated by lattice-scale instantons for a class of lattice actions.

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1. Introduction

If the up quark has zero mass, the strong *CP* problem would be solved automatically. Some people argue that it is indeed the case, while phenomenology studies suggest that it is not. In reference [1], however, Creutz argued that the mass of the up quark is ambiguous, because of confinement and chiral anomaly. We followed his idea futher in $N_f = 1$ QCD, and expand on it.

On one hand, because of confinement, we can not define the up quark mass based on the pole position of its propagator. There are many alternatives; we choose a generally accepted, regularization independent way to define a fermion mass. It is call the RI/MOM [6] scheme. The fermion mass definition is based on the Landau-gauge-fixed fermion propagator evaluated in momentum space. The renormalization factor $Z_q(\mu)$ and the renormalized mass $m_R(\mu)$ could be extracted by

$$Z_q(\mu) = \left.\frac{\mathrm{Tr}\left[\not{p} S^{-1}(p)\right]}{ip^2}\right|_{p^2=\mu^2} \qquad m_R(\mu) = \frac{\mathrm{Tr}\left[S^{-1}(p)\right]\big|_{p^2=\mu^2}}{Z_q(\mu)}. \tag{1.1}$$

On the other hand, we will show that the chiral anomaly plays an important role in the fermion mass. Because of the anomaly, in some regularization schemes, just like a scalar particle, an additive mass renormalization term is needed for the fermion. Normally, we do not have an additive renormalization term for the fermion mass because the fermion mass is protected by chiral symmetry and thus can receive only multiplicative quantum corrections. However, when interaction are included, axial $U(1)$ symmetry is anomalously broken, there is no a priori reason that the fermion mass does not receive additive quantum corrections.

2. ’t Hooft Effective Lagrangian

’t Hooft has demonstrated a mechanism for such mass generation by studying the effect of a classical instanton. A mass term will indeed be generated by instantons with radius between R and $R + \mathrm{d}R$ given by [7][3]

$$\frac{R^2}{m}\rho(R)\,\mathrm{d}R \tag{2.1}$$

where $\rho(R)$ is the density of instantons of radius R per unit space-time volume and per unit radius.

This result can be understood in a naive way. The fermion zero mode $u_0(x)$ of an instanton of radius R at the origin would contribute to the fermion Green’s function evaluated outside the instanton as:

$$\langle q(x)\overline{q}(y)\rangle = u_0(x)\frac{1}{m}\overline{u}_0(y) = \frac{R\gamma^\mu x^\mu}{x^4}\frac{1}{m}\frac{R\gamma^\nu y^\nu}{y^4} \tag{2.2}$$

In momentum space, for $p \ll \frac{1}{R}$ the contribution to the propagator from this fermion zero modes is:

$$S(p) = \int \mathrm{d}^4x e^{-ip\cdot x}\langle q(x)\overline{q}(0)\rangle = \frac{1}{\not{p}}\frac{R^2}{m}\rho(R)\,\mathrm{d}R\frac{1}{\not{p}} \tag{2.3}$$

One can then immediately spot the anomalous mass term given in Eq. (2.1). We are interested in a “hard” fermion mass which acts like a normal fermion mass term at all scales. This mass term cannot be generated by intantons of any fixed physical size, it can only come from instantons on the cut-off scale or lattice scale if we use a lattice regularization:

$$m_{\text{anom}} \sim \frac{a^2}{m} \rho_a \tag{2.4}$$

where a is the lattice spacing, m is the input quark mass and ρ_a is the density of the lattice-scale instantons.

’t Hooft has also calculated the density of instantons. For $N_f = 1$, the density of instantons $\rho(R)$ of radius R is approximately [7]

$$\rho(R)\,\mathrm{d}R \sim \frac{\mathrm{d}R}{R^5}(mR)\exp\left(-\frac{8\pi^2}{g(R)^2}\right) \tag{2.5}$$

The above formula should be most accurate when R is small. Thus, one might expect the density of lattice-scale instantons to be

$$\rho_a \sim \frac{1}{a^4}(ma)\exp\left(-\frac{8\pi^2}{g_a^2}\right). \tag{2.6}$$

Here, g_a is the coupling constant at the lattice scale. However, ρ_a and m_{anom} will vanish in the continuum limit $a \to 0$, if g_a follows the renormalization group equation.

At this point, one might conclude that the chiral anomaly will generate an anomalous mass, but this mass term is “soft” and goes away at large momentum due to the low density of small instantons, distinguishing it from a normal mass term. In this paper, we are going to prove that, generally, this naive use of ’t Hooft’s result for a lattice-scale instanton need not be correct. The above formula is certainly right for an instanton with a small but fixed physical size. However, the density of instantons with a size that shrinks as we take the continum limit depends on how we regularize the theory or, in the case of a lattice regularization, on the form of the lattice action.

3. Density of Lattice Scale Instanton

Since the density depends on the details of the regularization, to calculate it, we need to make two assumptions about the form of the lattice action, so we would not consider some lattice action which is too strange.

- The gauge action should be local, and in the continuum limit, the gauge links should become very smooth. Thus, if we divide the infinite lattice into sub-blocks of fixed size in lattice unit (e.g. 16^4), the probabilities of having an instanton in each block $p_{\text{inst}}^{16^4}$ should be almost independent. This would contribute a factor of $\frac{1}{a^4}$ to the density of lattice-scale instantons.
- For an instanton-like gauge configuration, the fermion determinent should contribute only a factor of ma to the probability. This factor comes from the fermion zero mode generated by the instanton, which requires that the lattice fermion action possess good chiral symmetry. Also, the presence of an instanton should not affect the other fermion modes too much, as is the case in ’t Hooft’s calculation.

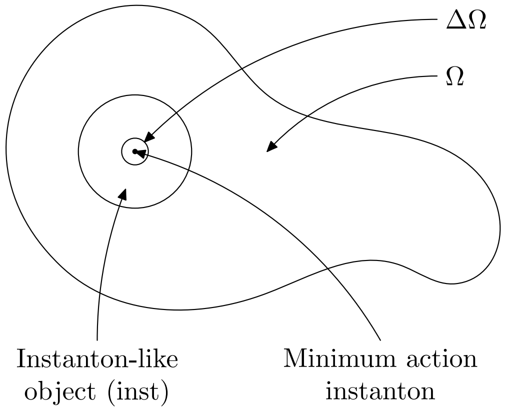

Figure 1: The diagram of gauge field space used to obtain our lower bound

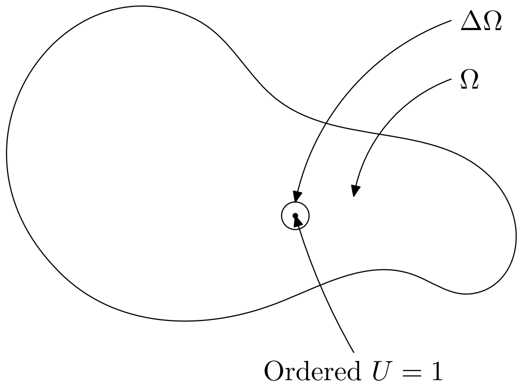

Figure 2: The diagram of gauge field space used to obtain our upper bound

Based on these two assumptions, we only need to calculate the probability of having one lattice instanton in a sub-block of size 16^4 for a pure gauge theory $p^{16^4}_{\text{inst pure gauge}}$. Then the density would be

$$\rho_a \sim \frac{1}{a^4} p^{16^4}_{\text{inst}} \sim \frac{1}{a^4} (ma)\, p^{16^4}_{\text{inst pure gauge}}. \tag{3.1}$$

The probability $p^{16^4}_{\text{inst pure gauge}}$ depends on the integration volume and the action

$$p^{16^4}_{\text{inst pure gauge}} = \frac{\int_{\text{inst}} [\mathscr{D}U] \exp(-\mathscr{A}[U])}{\int [\mathscr{D}U] \exp(-\mathscr{A}[U])}. \tag{3.2}$$

We can notice from the above formula that the integration volume does not depend on g_a at all. Since we are only interested in the density in the continuum limit where $g_a \to 0$, the action is the more important quantity, and the least action of a lattice scale instanton is the most important. So we define a g_a independent parameter α to parameterize the least action of a lattice-scale instanton

$$\mathscr{A}_{\text{inst}} = \alpha \frac{8\pi^2}{g_a^2}. \tag{3.3}$$

The lattice-scale instanton with the least action will dominate in the $g_a \to 0$ limit, because other kinds of instantons will be exponentially suppressed. So the probability of having this least-action instanton alone can be used as a lower bound of the probability in the $g_a \to 0$ limit:

$$\begin{aligned} p^{16^4}_{\text{inst pure gauge}} = \frac{\int_{\text{inst}} [\mathscr{D}U] \exp(-\mathscr{A}[U])}{\int [\mathscr{D}U] \exp(-\mathscr{A}[U])} &> \frac{\Delta\Omega \exp\left(-\alpha \frac{8\pi^2}{g_a^2} - \frac{\varepsilon'}{g_a^2}\right)}{\Omega} \\ &> \exp\left(-(\alpha+\varepsilon) \frac{8\pi^2}{g_a^2}\right) \end{aligned} \tag{3.4}$$

where $\Omega = \int [\mathscr{D}U]$ is the total integration volume, and $\Delta\Omega$ is a small integration volume around the minimum point as shown in Fig. 1. Here $\Delta\Omega$ is choosen to be small but fixed. Within this volume, the variation of the action about $\alpha \frac{8\pi^2}{g_a^2}$ is less then $\frac{\varepsilon'}{g_a^2}$, where ε' is an arbitrarily small number. Since

$\frac{\Delta\Omega}{\Omega}$ is a constant in the limit, we can simply drop it and change the explonential a little bit by choosing another small number ε.

Although this seems to be a rather crude estimation, suprisingly, it is already good enough for our purpose since we can work out the upper bound of this probability in a similar way and find

$$\begin{aligned} p^{16^4}_{\text{inst pure gauge}} = \frac{\int_{\text{inst}} [\mathscr{D}U] \exp\left(-\mathscr{A}[U]\right)}{\int [\mathscr{D}U] \exp\left(-\mathscr{A}[U]\right)} &< \frac{\Omega \exp\left(-\alpha \frac{8\pi^2}{g_a^2}\right)}{\Delta\Omega \exp\left(-\frac{\varepsilon'}{g_a^2}\right)} \\ &< \exp\left(-(\alpha-\varepsilon)\frac{8\pi^2}{g_a^2}\right), \end{aligned} \tag{3.5}$$

where $\Omega = \int [\mathscr{D}U]$ is again the total integration volume, and $\Delta\Omega$ is a small integration volume around the vaccum point as shown in Fig. 2.

Having calculated the density of lattice-scale instantons in a quite general setting, recalling the renormalization equation

$$\frac{8\pi^2}{g_a^2} \approx \left(11 - \frac{2}{3}N_f\right) \ln\frac{1}{a}, \tag{3.6}$$

and combine Eqs. (2.4) (3.1) (3.4) and (3.5), we get the anomalous quark mass for $N_f = 1$ QCD

$$m_{\text{anom}} \sim a^{\frac{31}{3}\alpha - 1}. \tag{3.7}$$

Thus, if the generated anomalous mass term does not vanish in the continuum limit, we should have

$$\alpha \le \frac{3}{31} \approx 0.097 \tag{3.8}$$

The above criteria is a necessary and sufficient condition.

4. Exploration of the Design Space of Lattice Gauge Action

After talking so much about the α parameter, it's time for us to explore the design space of lattice actions to actually calculate this parameter for some gauge action and see how the minimum instanton action depends on the form of lattice gauge action.

The minimum instanton action or the α parameter for a given lattice action is certainly a well defined quantity if we restrict ourself to a finite space-time volume in lattice unit. However, it is not practical to perform a complete search over all possible configurations. We estimate the minimum instanton action by manually constructing an instanton based on a classical solution and then slowly smoothing the gauge field to minimize the action. Obviously, there is no topological barrier on a lattice unless we add some other restrictions [2]. If we make the field smooth enough, we would end up with a free field. As a result, there must be a topological tunneling at some point. We define the topological index as half the difference of the positive and negative eigenvalues of $\gamma_5 D_{\text{w}}(-1)$, the Hermitian Wilson Dirac operator at negative mass -1. [4] So we keep track of the lowest ten eigenvalues of $\gamma_5 D_{\text{w}}(-1)$, the tunneling happens when a eigenvalue cross zero.

We have applied this method to the Wilson action and the rectangular actions (see Fig. 3),

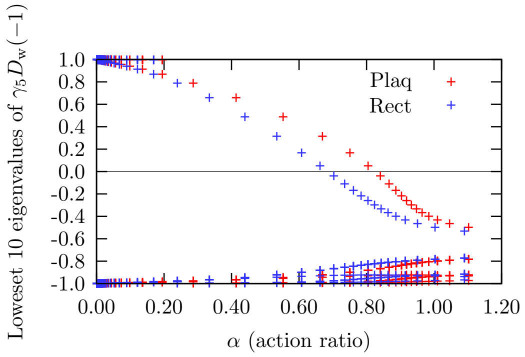

Figure 3: Trajectories evolving from right to left followed by the 10 lowest eigenvalues of $\gamma_5 D_\mathrm{w}(-1)$ (y axis) and α (x axis) as a classical instanton is smoothed.

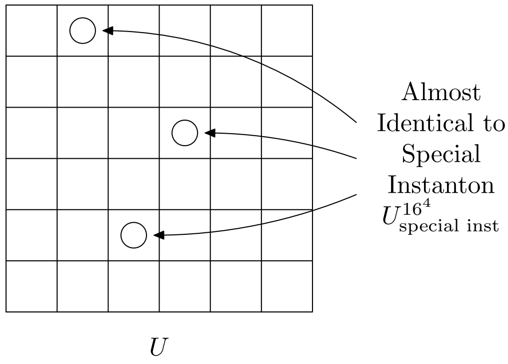

Figure 4: A sample arrangement of guage field, where 3 blocks are occupied by guage fields close to the special structure. Hence $N_\text{special inst}[U] = 3$

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- Wilson action $\mathscr{A} = \frac{\beta}{3}\sum_{x;\mu<\nu} P^{1\times 1}_{\mu\nu}$, $\alpha < 0.83$.
- Rectangular action $\mathscr{A} = \frac{\beta}{24}\sum_{x;\mu\neq\nu} P^{1\times 2}_{\mu\nu}$, $\alpha < 0.69$.

The result obtained above is only an upper bound on the minimum instanton action. If there is a lattice-scale instanton which is not close to a classical instanton, above procedure would not find it. Also, there are many different ways to smooth the field, and many parameters for the initial instanton to adjust. For Fig. 3 we use ape smearing. We have also tried other smooth methods and different initial instanton sizes. The results are not changed much. Although we are not completely sure, it seems that these two example actions do not satisfy the previous condition Eq. 3.8 and thus would not generate an anomalous mass term. However, these results do show that α can be less than one and can be different for different action. The reason is that we are free to change the lattice scale behavior without changing the continuum limit as long as we adjust the bare parameters properly. This is the spirit of a renormalizable field theory. This suggests that for some lattice action a lattice scale instanton could have a very small action and an anomalous mass term could be generated. This is indeed true. Here is an artifitial example which would satisfy Eq. (3.8) by construction. The idea is that we can design a lattice gauge action which would enhance a certain kind of lattice-scale instantons by reducing its action. To define this lattice action, we need to pick a special localized instanton configuration with a small size, say 16^4, to enhance. After a special instanton configuration is choosen, the lattice gauge action is defined by

$$\mathscr{A}[U] = \mathscr{A}_\text{Wilson}[U] - N_\text{special inst}[U]\,\Delta\mathscr{A} \tag{4.1}$$

where $\mathscr{A}_\text{Wilson}[U]$ is the common Wilson action and $\Delta\mathscr{A}$ is an adjustable parameter. $N_\text{special inst}[U]$ is tricky. As is shown in Fig. 4, we would divide the configuration into small 16^4 blocks, $N_\text{special inst}[U]$ is the number of blocks in which all the link variables are close enough to those of our chosen special instanton configuration up to some symmetric transformation, e.g. gauge transformation, CP, etc.

Note that this lattice gauge action is still local, and by construction, there is now a lattice-scale instanton, which has the same shape as the special instanton, with action

$$\mathscr{A}_{\text{inst}} = \alpha \frac{8\pi^2}{g_a^2} = \mathscr{A}_{\text{Wilson}} \left[U^{16^4}_{\text{special inst}} \right] - \Delta\mathscr{A} \tag{4.2}$$

Here $\mathscr{A}_{\text{Wilson}} \left[U^{16^4}_{\text{special inst}} \right]$ denote the Wilson action of the special configuration. Since $\Delta\mathscr{A}$ is adjustable, α can be any value we like, e.g. the value dictated by Eq. (3.8).

5. Conclusion

We have shown that the naive estimation of the density of cut-off scale instanton need not be correct, because the density is regularization dependent. Then, we have shown that for some lattice regularization scheme, there will be enough lattice scale instanton that an anomalous mass term will be generated. This mass term does not depend on the input quark mass, thus it doesn't acquire a phase when we apply a chiral rotation to the fermion field. This property provides another solution [5] to the strong *CP* problem for $N_f = 1$ QCD. Suppose that for some reason we can not add a explicit mass term to the single quark, and all the observed quark mass is generated by lattice scale instantons, then the θ term would be absorbed by applying an appropriate chiral rotation to the quark field. One difficulty for this approach is that we would need to carefully adjust α to keep the quark mass small and finite.

It is quite straight forward to apply the above calculation to multi-flavor QCD. The only difficulty is that one would need negative α to generate a non-vanishing fermion mass. This does not cause any serious problem, since each quark flavor would contribute an *ma* factor for every instanton and suppress the lattice scale instanton density.

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