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Designing the sound of a cut-off drum

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The spectral action in noncommutative geometry naturally implements an ultraviolet cut-off, by counting the eigenvalues of a (generalized) Dirac operator lower than an energy of unification. Inverting the well known question "how to hear the shape of a drum", we ask what drum can be designed by hearing the truncated music of the spectral action ? This makes sense because the same Dirac operator also determines the metric, via Connes distance. The latter thus offers an original way to implement the high-momentum cut-off of the spectral action as a short distance cut-off on space. This is a non-technical presentation of the results of [8].

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1. Introduction

Cut-off are generally used to avoid undesirable divergencies occurring at small or very large scales. The scale is usually an energy scale E and the cut-off is implemented either on momentum p or on the wavelength λ. In various senses, these two implementations are dual to each other: a high momentum cut-off is equivalent to a short wavelength cut-off and vice-versa, as can be read e.g. in de Broglie relation $p = \frac{h}{\lambda}$. In this note we explore another possibility to give sense to this duality, based on the double nature of the (generalized) Dirac operator D which is at the heart of Connes approach to noncommutative geometry [6]. This is indeed the same operator D which

- provides an action which naturally incorporates a high energy cut-off. This is the *spectral action* [1]

$$\mathrm{Tr}\, f\left(\frac{D}{\Lambda}\right) \tag{1.1}$$

where f is the characteristic function of the interval $[0,1]$ and Λ a energy scale of unification. We refer to [2] for details on the choice of the operator D and how the asymptotic expansion $\Lambda \to \infty$ yields the standard model of elementary particles minimally coupled with (Euclidean) general relativity (see also [15] for an highly readable introduction to the subject, [3, 4] and [10, 11] for recent developments).

- defines a metric on the space $\mathscr{S}(\mathscr{A})$ of states[1] of an algebra $\mathscr{A}$, provided the later acts on the same Hilbert space $\mathscr{H}$ as D in such a way that

$$L_D(a) := ||[D, \pi(a)]|| \tag{1.2}$$

is finite for any $a \in \mathscr{A}$ (π denotes the representation of $\mathscr{A}$ as bounded operators on $\mathscr{H}$). This is the *spectral distance* [5]

$$d_{\mathscr{A},D}(\varphi,\psi) := \sup_{a\in\mathscr{A}} \{|\varphi(a) - \psi(a)|, L_D(a) \leq 1\} \tag{1.3}$$

for any φ, ψ in $\mathscr{S}(\mathscr{A})$. For $\mathscr{A} = C^\infty(\mathscr{M})$ the algebra of smooth functions on a manifold $\mathscr{M}$ and $D = \partial\!\!\!/$ the usual Dirac operator of quantum field theory, the spectral distance computed between pure states (which are nothing but the points of $\mathscr{M}$, viewed as the application $\delta_x : f \to f(x)$) gives back the geodesic distance [7],

$$d_{C^\infty(\mathscr{M}),\partial\!\!\!/}(\delta_x, \delta_y) = d_{\mathrm{geo}}(x,y). \tag{1.4}$$

The action (1.1) counts the eigenvalues of the Dirac operator smaller than the energy scale Λ, which amounts to cut-off the Fourier modes with energy greater than Λ. In other terms the spectral action naturally implements an ultraviolet cut-off as a high momentum cut-off. The question we adress in this note is: can this be read as a short wave-length cutoff in the distance formula (1.3) ? More precisely, by cuting-off the spectrum of the Dirac operator in the distance formula, does one transform the high-momentum cut-off Λ in a short distance cut-off λ ? Reversing the well known question on how to retrieve the shape of a drum from its vibration modes, our point here is to understand what drum can one design from hearing the spectral action ? We have shown in [8] that the answer is not obvious, and asks for a careful discussion on the nature of the points of the "cut-off drum". We report here some of these results, in a non technical manner.

[1] A state φ of $\mathscr{A}$ is a linear application $\mathscr{A} \to \mathbb{C}$ which is positive ($\varphi(a^*a) \in \mathbb{R}^+$) and of norm 1 (in case $\mathscr{A}$ is unital, this means $\varphi(\mathbb{I}) = 1$).

2. Cutting-off the geometry

We implement a cut-off by the conjugate action of a projection P_Λ acting on $\mathscr{H}$:

$$D \to D_\Lambda := P_\Lambda D P_\Lambda. \tag{2.1}$$

Typically P_Λ is a projection on the eigenspaces of D, but for the moment it is simply a projection acting on $\mathscr{H}$. We are interested in computing the spectral distance on a manifold $\mathscr{M}$ induced by this cut-off, that is formula (1.3) for $\mathscr{A} = C^\infty(\mathscr{M})$ but with D substituted with D_Λ.

Suppose that D_Λ is a bounded operator with norm $\Lambda \in \mathbb{R}^+$. Then one has [8, Prop. 5.1]

$$d_{C^\infty(\mathscr{M}),D_\Lambda}(\delta_x, \delta_y) \geq \Lambda^{-1}. \tag{2.2}$$

This seems to be precisely the answer one was expecting: by cutting-off the spectrum of D below Λ, one is not able to probe space with a resolution better than Λ^{-1}. Unfortunately (2.2) is an inequality, not an equality. There is as expected a lower bound to the resolution on the position space, but nothing guarantees that this bound is optimal. In particular if D_Λ has finite rank, then the distance is actually infinite [8, Prop.5.4]. For $\mathscr{M}$ compact, this happens for instance when $P_\Lambda = P_N$ is the projection on the first N Fourier modes for some $N \in \mathbb{N}$. Then $D_\Lambda = D_N := P_N D P_N$ has finite rank and the distance between any two points is infinite,

$$d_{C^\infty(\mathscr{M}),D_N}(\delta_x, \delta_y) = \infty. \tag{2.3}$$

In other terms, cutting-off all but a finite number of Fourier modes destroys the metric structure of the manifold. It is an open question whether the distance remains finite for a bounded D_Λ with infinite rank.

Eq. (2.3) illustrates the tension between truncating the Dirac operator while keeping the usual notion of points. A solution to maintain a metric structure is to truncate point as well. This actually makes sense in full generality, that is for $\mathscr{A}$ non-necessarily commutative, acting on some Hilbert space $\mathscr{H}$ together with an operator D such that $L_D(a)$ is finite for any $a \in \mathscr{A}$. Given a finite rank projection P_N in $\mathscr{B}(\mathscr{H})$, we then define

$$\mathscr{O}_N := P_N\, \pi(\mathscr{A}_{sa})\, P_N \tag{2.4}$$

where $\mathscr{A}_{sa}$ is the set of selfadjoint elements of $\mathscr{A}$. The set $\mathscr{O}_N$ has no reason to be an algebra but it has the structure of *ordered unit space*, which is sufficient to define its state space $\mathscr{S}(\mathscr{O}_N)$ and to give sense to formula (1.3) (substituting the seminorm L_D with the seminorm

$$L_N := ||[D_N, \cdot]|| \tag{2.5}$$

and $\mathscr{A}$ with $\mathscr{O}_N$). In addition to the original distance $d_{\mathscr{A},D}$, one thus inherits from the cut-off two "truncated" distances: $d_{\mathscr{A},D_N}$ on $\mathscr{S}(\mathscr{A})$ and $d_{\mathscr{O}_N,D_N}$ on $\mathscr{S}(\mathscr{O}_N)$. To make the comparison between these distances possible, we use the injective map $\varphi^\sharp := \varphi \circ \mathrm{Ad}\, P_N$ that sends a state φ of $\mathscr{O}_N$ to a state $\varphi^\sharp$ of $\mathscr{A}$. By pull back, one obtains three distances on $\mathscr{S}(\mathscr{O}_N)$:

$$d_{\mathscr{O}_N,D_N}(\varphi,\psi), \quad d^\flat_{\mathscr{A},D}(\varphi,\psi) := d_{\mathscr{A},D}(\varphi^\sharp,\psi^\sharp), \quad d^\flat_{\mathscr{A},D_N}(\varphi,\psi) := d_{\mathscr{A},D_N}(\varphi^\sharp,\psi^\sharp). \tag{2.6}$$

A sufficient condition that makes the truncated distance $d^{\flat}_{\mathscr{A},D_N}$ equivalent to the "bi-truncated" distance $d_{\mathscr{O}_N,D_N}$ is that [8, Prop. 3.5] the seminorm $L_N := |||[D_N,\cdot]|||$ is *Lipschtiz* [14], meaning that $L_N(a) = 0$ if and only if $a = \mathbb{C}\mathbf{1}$. If in addition the non-truncated seminorm L_D is Lipschitz, or P_N is in the commutant $\mathscr{A}'$ of the algebra $\mathscr{A}$, or P_N commutes with D, then one also has that $d^{\flat}_{\mathscr{A},D}$ is equivalent to $d_{\mathscr{O}_N,D_N}$.

In the commutative case, the set $\mathscr{S}(\mathscr{O}_N)$ permits to give a precise meaning to the notion of *truncated points*. By this we mean a sequence of states of $\mathscr{O}_N$ that tends to some δ_x as $N \to \infty$. For instance on the circle, that is $\mathscr{A} = C^\infty(S^1)$, the Fejer transform

$$\Psi_{x,N}(f) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) f_n e^{inx}, \quad N \in \mathbb{N} \tag{2.7}$$

is a state of $\mathscr{O}_N$ for P_N the projection on the first N negative and N positive Fourier modes [8, Lem. 5.10]. It is an approximation of the point $x \in S^1$ in that

$$\lim_{N\to\infty} \Psi_{x,N}(f) = f(x) \quad \forall f \in C^\infty(S^1). \tag{2.8}$$

Moreover this approximation deforms the metric structure of the circle but does not destroy it, since - with D the usual Dirac operator of S^1 - the bi-truncated distance between any two Fejer transforms is finite for any N [8, Prop. 5.11],

$$d_{\mathscr{O}_N,D_N}(\Psi_{x,N},\Psi_{y,N}) \leq d_{\text{geo}}(x,y), \tag{2.9}$$

and tends towards the geodesic distance for large N,

$$\lim_{N\to\infty} d_{\mathscr{O}_N,D_N}(\Psi_{x,N},\Psi_{y,N}) = d_{\text{geo}}(x,y) \qquad \forall x,y \in S^1. \tag{2.10}$$

A similar example has been worked out on the real line [8, Prop. 5.7].

3. Convergence of truncations

Let us study in a more systematic way the idea introduced in the previous section of approximating a state by a sequence of truncated states. To this aim, take $\mathscr{A}$, $\mathscr{H}$ and D satisfying the conditions of the precedent section, and let us consider a sequence $\{P_N\}_{N\in\mathbb{N}}$ of increasing finite-rank projections, weakly converging to 1. Under which conditions can states of $\mathscr{A}$ be approximated by states of $\mathscr{O}_N$ in such a way that the metric structure is preserved ?

For normal states [2], the answer is simple in case the topology induced by the spectral distance coincides with the weak* topology. Then any normal states φ with density matrix R is the limit of its truncation [8, Prop. 4.2],

$$\lim_{N\to\infty} d_{\mathscr{A},D}(\varphi,\varphi_N^{\sharp}) = 0 \tag{3.1}$$

where φ_N is the state with density matrix $Z_N^{-1}R$ where $Z_N := \text{Tr}(P_N R)$. One also has the convergence in the sense of metric spaces[8, Prop. 4.3]: $(\mathscr{S}(\mathscr{O}_N), d^{\flat}_{\mathscr{A},D})$ converges to $(\overline{\mathscr{N}(\mathscr{A})}, d_{\mathscr{A},D})$ for the Gromov-Hausdorff distance.

[2] $\varphi \in \mathscr{S}(\mathscr{A})$ is normal if and only if there exists a positive traceclass operator R on $\mathscr{H}$ (the density matrix) such that $\varphi(a) = \text{Tr}(Ra)$, $\forall a \in \mathscr{A}$.

In case the topology of the spectral distance is not the weak* topology, the answer is more challenging. A preliminary step is to work out a class of states at finite distance from one another. In the commutative case, for $\mathscr{M}$ connected and complete, such a class is given by states with finite moment of order 1. Recall that there is a 1-to-1 correspondance between states φ of $C_0^\infty(\mathscr{M})$ and probability measures μ on $\mathscr{M}$,

$$\varphi(f) = \int_{\mathscr{M}} f(x)\, \mathrm{d}\mu(x) \quad \forall f \in C_0(\mathscr{M}).$$

For $\mathscr{M}$ connected, the finiteness of the moment of order 1 of φ,

$$\mathscr{M}_1(\varphi, x') := \int_{\mathscr{M}} d_{\text{geo}}(x,x')\, \mathrm{d}\mu(x) \tag{3.2}$$

does not depend on the choice of $x' \in \mathscr{M}$ and so is intrinsic to the state. If furthermore $\mathscr{M}$ is complete, one has that the spectral distance $d_{\partial\!\!\!/}$ between states with finite moment of order 1 is finite (see e.g. [9]).

In the noncommutative case, the correspondance between states and probability measure on the pure state space is no longer 1-1, as can be seen on easy examples such as $M_2(\mathbb{C})$. However in [8, §4.2] we proposed to give meaning to the notion of "finite moment of order 1" for normal states in the following way. Let φ be a normal state with density matrix R. Fix an orthonormal basis $\mathfrak{B} = \{\psi_n\}_{n\in\mathbb{N}}$ of $\mathscr{H}$ made of eigenvectors of R, with eigenvalues $p_n \in \mathbb{R}^+$. Denote $\Psi_n(a) := \langle \psi_n, a\psi_n\rangle$ the corresponding vector states in $\mathscr{S}(\mathscr{A})$ so that

$$\varphi(a) = \sum_{n\geq 0} p_n \Psi_n(a) \quad \forall a \in \mathscr{A}.$$

We call *moment of order* 1 *of* R with respect to the eigenbasis $\mathfrak{B}$ and to a state Ψ_k (induced by a vector $\psi_k \in \mathfrak{B}$) the quantity

$$\mathscr{M}_1(R, \mathfrak{B}, \Psi_k) := \sum_{n\geq 0} p_n\, d_{\mathscr{A},D}(\Psi_k, \Psi_n). \tag{3.3}$$

Unlike the commutative case the finiteness of (3.3) is not intrinsic to the density matrix (hence even less to the state), because for the same density matrix R one may have that $\mathscr{M}_1(R, \mathfrak{B}, \Psi_k)$ is infinite for a given basis while $\mathscr{M}_1(R, \mathfrak{B}', \Psi'_k)$ is finite for another one [8, Ex. 4.6]. However, once fixed $\mathfrak{B}$, the finiteness of $\mathscr{M}_1(R, \mathfrak{B}, \Psi_k)$ does not depend on Ψ_k. We write $\mathscr{N}_0(\mathscr{A})$ the set of normal states for which there exists at least one density matrix R with an eigenbasis $\mathfrak{B} = \{\psi_n\}$ such that

$$\mathscr{M}_1(R, \mathfrak{B}, \Psi_n) < \infty. \tag{3.4}$$

Consider then an increasing sequence $\{P_N\}_{N\in\mathbb{N}}$ of projections weakly convergent to 1. For any $\varphi \in \mathscr{N}_0(\mathscr{A})$ such that (3.4) holds for an eigenbasis $\mathfrak{B}$ in which the P_N's are all diagonal, there exists a sequence $\varphi_N \in \mathscr{S}(\mathscr{O}_N)$ such that

$$\lim_{N\to\infty} d_{\mathscr{A},D}(\varphi, \varphi_N^\sharp) = 0. \tag{3.5}$$

In other terms, any $\varphi \in \mathscr{N}_0(\mathscr{A})$ can be approximated in the metric topology by a truncation φ_N. However unlike (3.1) where the truncating-projections P_N where fixed once for all, in case the metric topology is not the weak* the truncating procedure may depend on the state.

4. An unbounded example: Berezin quantization of the plane

We conclude by an example where the truncated Dirac operator is not bounded: the Berezin quantization of the plane. We omit the details that can be found in [8, §6.2]. For other applications of noncommutative geometry to Berezin quantization, see [12].

One starts with $\mathscr{H} = L^2(\mathbb{C}, \frac{d^2z}{\pi})$ and, for $\theta > 0$, define P_θ as the projection on the subspace

$$\mathscr{H}_\theta := \text{Span}\left\{ h_n(z) := \frac{z^n}{\sqrt{\theta^{n+1} n!}} e^{-\frac{|z|^2}{2\theta}} \right\}_{n\in\mathbb{N}}. \tag{4.1}$$

For D the Dirac operator of the Euclidean plane, the truncated Dirac operator

$$D_\theta := (P_\theta \otimes \mathbb{I}_2)\, D\, (P_\theta \otimes \mathbb{I}_2) = \frac{2}{\sqrt{\theta}} \begin{pmatrix} 0 & \mathfrak{a}^\dagger \\ \mathfrak{a} & 0 \end{pmatrix} \tag{4.2}$$

is unbounded ($\mathfrak{a}^\dagger, \mathfrak{a}$ are the creation, annihilation operators on the h_n's).

Let $\mathscr{A} = \mathscr{S}(\mathbb{R}^2)$ denote the algebra of Schwartz functions on the plane, and denote $\mathscr{O}_\theta$ the order unit space generated by $P_\theta f P_\theta$ (for $f = f^* \in \mathscr{A}$). Both act on $\mathscr{H} \otimes \mathbb{C}^2$. For any states φ, ψ of $\mathscr{A}$, define

$$d^{(\theta)}_{\mathscr{A},D}(\varphi,\psi) := \sup_{f=f^*\in\mathscr{A}} \left\{ \varphi(f) - \psi(f),\, ||[D, B^\theta(f)|| \leq 1 \right\} \tag{4.3}$$

where

$$B_\theta(f) : z \to \langle \psi_z, P_\theta f P_\theta\, \psi_z \rangle \quad \text{where} \quad \psi_z = e^{-\frac{|z|^2}{2\theta}} \sum_{n=0}^{\infty} \frac{\bar{z}^n}{\sqrt{\theta^n n!}} h_n \tag{4.4}$$

is the *Berezin transform* of f. One gets [8]

$$d_{\mathscr{A},D}(\varphi^\sharp, \psi^\sharp) \leq d_{\mathscr{O}_\theta, D_\theta}(\varphi, \psi) \leq d^{(\theta)}_{\mathscr{A},D}(\varphi^\sharp, \psi^\sharp). \tag{4.5}$$

In particular, the distance between coherent states $\Psi_z, \Psi_{z'}$, $z, z' \in \mathbb{C}$, is

$$d_{\mathscr{O}_\theta, D_\theta}(\Psi_z, \Psi_{z'}) = |z - z'|. \tag{4.6}$$

A similar result was found in [13] from a completely different perspective, based on the construction of the element that attains the supremum in the distance formula. This illustrates that the cut-off procedure could be an efficient tool to make explicit calculations of the distance.

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