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Construction of a quantum field theory in four dimensions

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We summarise our recent construction of the $\lambda\phi^4_4$-model on four-dimensional Moyal space. In the limit of infinite noncommutativity, this model is exactly solvable in terms of the solution of a non-linear integral equation. Surprisingly, this limit describes Schwinger functions of a Euclidean quantum field theory on standard $\mathbb{R}^4$ which satisfy the easy Osterwalder-Schrader axioms boundedness, invariance and symmetry. The decisive reflection positivity axiom is, for the 2-point function, equivalent to the question whether the solution of the integral equation is a Stieltjes function. A numerical investigation confirms this for coupling constants $\lambda_c < \lambda \leq 0$ with $\lambda_c \approx -0.39$.

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1. Introduction

The construction of a 4D quantum field theory [1] is a major open problem of mathematical physics. In this note we review a sequence of papers [2, 3, 4] in which we successfully used symmetry and fixed point methods to exactly solve a toy model for a 4D QFT.

We follow the Euclidean approach, starting from a partition function with source term $\mathcal{Z}[J]$. This involves the action functional of the model, but regularised in *finite volume* V and with *finite energy cut-off* Λ. Mostly, these regularisations destroy the symmetries of the model and have to be restored in the end. Our toy model is characterised by a huge symmetry group even in presence of regularisation. The resulting constraints lead to a complete solution of the model.

We start from the usual $\lambda\phi_4^4$-model with action $\int_{\mathbb{R}^4} dx\big(\frac{1}{2}\phi(-\Delta+\mu^2)\phi+\frac{\lambda}{4}\phi^4\big)(x)$. Finite volume is achieved through a harmonic oscillator potential. The energy cut-off Λ, or a minimal length scale $\frac{1}{\Lambda}$, typically makes the model *non-local*. A convenient choice is to replace the pointwise product by the Moyal product $(f\star g)(x)=\int_{\mathbb{R}^4\times\mathbb{R}^4}\frac{dk\,dy}{(2\pi)^4}f(x+\frac{1}{2}\Theta k)\,g(x+y)\,e^{iky}$, where Θ is a skew-symmetric $4{\times}4$-matrix. Schwartz class functions with Moyal product can be mapped to infinite matrices with rapidly decaying entries, and the energy cutoff Λ consists in a finite size $\mathcal{N}$ of these matrices. The regulated action thus reads

$$S[\phi]=\frac{1}{64\pi^2}\int d^4x\Big(\frac{Z}{2}\phi\star\big(-\Delta+\Omega^2\|2\Theta^{-1}x\|^2+\mu_{bare}^2\big)\phi+\frac{\lambda_{bare}Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)\,, \tag{1.1}$$

where $Z,\lambda_{bare},\mu_{bare}$ are functions of renormalised values λ,μ and of the regulators $\Omega,\Theta,\mathcal{N}$ encoded in the oscillator potential and the $\star$-product. Several limits can be discussed:

- $\Omega,\Theta,\frac{1}{\mathcal{N}}\to 0$: This is the pertubatively renormalisable, but trivial, $\lambda\phi_4^4$-model.
- $\Theta\neq 0$ fixed; $\Omega=0$: This is often called "noncommutative $\lambda\phi_4^4$-theory", which is not renormalisable due to the UV/IR-mixing problem.
- $\Theta,\Omega_{ren}\neq 0$ fixed: A perturbatively renormalisable model [5] with ultraviolet fixed point $\Omega=1$ at which the β-function vanishes [6].
- $\Omega=1$ fixed; $\Theta,\mathcal{N}\to\infty$: The limit studied here, giving rise to an exactly solvable model.

2. Matrix model, Ward identity and Schwinger-Dyson equations

At $\Omega=1$ the action (1.1) becomes self-dual under Langmann-Szabo transform and can be expressed as a quartic matrix model

$$S[\phi]=V\bigg(\sum_{\underline{m},\underline{n},\underline{k}\in\mathbb{N}^2_{\mathcal{N}}}E_{\underline{m}\underline{n}}\Phi_{\underline{n}\underline{k}}\Phi_{\underline{k}\underline{m}}+\frac{Z^2\lambda}{4}\sum_{\underline{k},\underline{l},\underline{m},\underline{n}\in\mathbb{N}_{\mathcal{N}}}\Phi_{\underline{k}\underline{l}}\Phi_{\underline{l}\underline{m}}\Phi_{\underline{m}\underline{n}}\Phi_{\underline{n}\underline{k}}\bigg)\,, \tag{2.1}$$

where $E_{\underline{m}\underline{n}}=E_{|\underline{m}|}\delta_{\underline{m}\underline{n}}$, $E_{|\underline{m}|}:=Z(\frac{|\underline{m}|}{\sqrt{V}}+\frac{\mu_{bare}^2}{2})$ and $V:=(\frac{\theta}{4})^2$. Under $\mathbb{N}^2_{\mathcal{N}}$ we understand the set of pairs $\underline{m}=(m_1,m_2)\in\mathbb{N}^2$ with $|\underline{m}|:=m_1+m_2\leq\mathcal{N}$. The resulting partition function $\mathcal{Z}[J]=\int\mathcal{D}[\Phi]\,\exp(-S[\Phi]+V\,\mathrm{tr}(\Phi J))$ is covariant under the unitary transformation $\Phi\mapsto U^*\Phi U$. This covariance gives rise to the following Ward identity [6]:

$$0 = \sum_{\underline{n}\in\mathbb{N}^2_{\mathcal{N}}} \Big(\frac{(E_{|\underline{a}|} - E_{|\underline{a}|})}{V} \frac{\partial^2 \mathcal{Z}}{\partial J_{\underline{a}\underline{n}} \partial J_{\underline{n}\underline{p}}} + J_{\underline{p}\underline{n}} \frac{\partial \mathcal{Z}}{\partial J_{\underline{a}\underline{n}}} - J_{\underline{n}\underline{a}} \frac{\partial \mathcal{Z}}{\partial J_{\underline{n}\underline{p}}} \Big) . \tag{2.2}$$

Perturbatively, Feynman graphs in matrix models are *ribbon graphs* which encode a genus-g Riemann surface with B boundary components. The k^{th} boundary face is characterised by $N_k \geq 1$ external double lines to which we attach the source matrices J. Since E is diagonal, the matrix index is conserved along each strand of the ribbon graph. Therefore, the right index of $J_{\underline{a}\underline{b}}$ coincides with the left index of another $J_{\underline{b}\underline{c}}$, or of the same $J_{\underline{b}\underline{b}}$. Accordingly, the k^{th} boundary component carries a cycle $J_P \equiv J^{N_k}_{\underline{p}_1\dots\underline{p}_{N_k}} := \prod_{j=1}^{N_k} J_{\underline{p}_j\underline{p}_{j+1}}$ of N_k external sources, with $N_k+1 \equiv 1$. Therefore, the logarithm of the partition function has the following expansion ($S_{N_1\dots N_B}$ is a symmetry factor):

$$\log \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} = \sum_{B=1}^{\infty} \sum_{1\leq N_1 \leq \dots \leq N_B}^{\infty} \sum_{\underline{p}_1^{\beta},\dots,\underline{p}^{\beta}_{N_\beta} \in I} \frac{V^{2-B}}{S_{N_1\dots N_B}} G_{|\underline{p}_1^1\dots\underline{p}^1_{N_1}|\dots|\underline{p}_1^B\dots\underline{p}^B_{N_B}|} \prod_{\beta=1}^{B} \Big(\frac{1}{N_\beta} J^{N_\beta}_{\underline{p}_1^\beta\dots\underline{p}^\beta_{N_\beta}} \Big) . \tag{2.3}$$

The cycle expansion (2.3) provides for external matrices E of compact resolvent the kernel of multiplication by $E_{|\underline{a}|} - E_{|\underline{p}|}$ in (2.2):

Theorem 1 ([2])

$$\begin{aligned}
&\sum_{\underline{n}\in\mathbb{N}^2_{\mathcal{N}}} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{\underline{a}\underline{n}} \partial J_{\underline{n}\underline{p}}} \\
&= \delta_{\underline{a}\underline{p}} \Big\{ V^2 \sum_{(K)} \frac{J_{P_1}\cdots J_{P_K}}{S_{(K)}} \Big(\sum_{\underline{n}\in\mathbb{N}^2_{\mathcal{N}}} \frac{G_{|\underline{a}\underline{n}|P_1|\dots|P_K|}}{V^{|K|+1}} + \frac{G_{|\underline{a}|\underline{a}|P_1|\dots|P_K|}}{V^{|K|+2}} + \sum_{r\geq 1} \sum_{\underline{q}_1,\dots,\underline{q}_r \in \mathcal{N}^2_{\mathcal{N}}} \frac{G_{|\underline{q}_1\underline{a}\underline{q}_1\dots\underline{q}_r|P_1|\dots|P_K|} J^r_{\underline{q}_1\dots\underline{q}_r}}{V^{|K|+1}} \Big) \\
&\quad + V^4 \sum_{(K),(K')} \frac{J_{P_1}\cdots J_{P_K} J_{Q_1}\cdots J_{Q_{K'}}}{S_{(K)} S_{(K')}} \frac{G_{|\underline{a}|P_1|\dots|P_K|}}{V^{|K|+1}} \frac{G_{|\underline{a}|Q_1|\dots|Q_{K'}|}}{V^{|K'|+1}} \Big\} \mathcal{Z}[J] \\
&+ \frac{V}{E_{|\underline{p}|} - E_{|\underline{a}|}} \sum_{\underline{n}\in\mathbb{N}^2_{\mathcal{N}}} \Big(J_{\underline{p}\underline{n}} \frac{\partial \mathcal{Z}[J]}{\partial J_{\underline{a}\underline{n}}} - J_{\underline{n}\underline{a}} \frac{\partial \mathcal{Z}[J]}{\partial J_{\underline{n}\underline{p}}} \Big) .
\end{aligned} \tag{2.4}$$

Formula (2.4) is the core of our approach. It is a consequence of the unitary group action and the cycle structure of the partition function. The possibility to kill two J-derivatives via (2.4) lets the usually infinite hierarchy of Schwinger-Dyson equations collapse [2]:

Proposition 2. *In a scaling limit $V \to \infty$ with $\frac{1}{V}\sum_{\underline{n}\in\mathbb{N}^2_{\mathcal{N}}}$ finite, the ($B=1$)-sector of* $\log \mathcal{Z}$ *reads*

$$G_{|\underline{a}\underline{b}|} = \frac{1}{E_{|\underline{a}|} + E_{|\underline{b}|}} - \frac{\lambda}{E_{|\underline{a}|} + E_{|\underline{b}|}} \frac{1}{V} \sum_{\underline{p}\in\mathbb{N}^2_{\mathcal{N}}} \Big(G_{|\underline{a}\underline{b}|} G_{|\underline{a}\underline{p}|} - \frac{G_{|\underline{p}\underline{b}|} - G_{|\underline{a}\underline{b}|}}{E_{|\underline{p}|} - E_{|\underline{a}|}} \Big) , \tag{2.5}$$

$$G_{|\underline{b}_0\underline{b}_1\dots\underline{b}_{N-1}|} = (-\lambda) \sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|\underline{b}_0\underline{b}_1\dots\underline{b}_{2l-1}|} G_{|\underline{b}_{2l}\underline{b}_{2l+1}\dots\underline{b}_{N-1}|} - G_{|\underline{b}_{2l}\underline{b}_1\dots\underline{b}_{2l-1}|} G_{|\underline{b}_0\underline{b}_{2l+1}\dots\underline{b}_{N-1}|}}{(E_{|\underline{b}_0|} - E_{|\underline{b}_{2l}|})(E_{|\underline{b}_1|} - E_{|\underline{b}_{N-1}|})} . \tag{2.6}$$

Equation (2.5) was first obtained in [7] by the graphical method proposed by [6]. The non-linearity of (2.5) was successfully addressed in [2, 4]. The purely algebraic formula (2.6) for $N \geq 4$ relies, apart from (2.4), on the reality $\Phi = \Phi^*$ of the matrix model. Absence of index summations in (2.6) means that the β-function of the QFT defined by (1.1) vanishes identically, as proved perturbatively in [6]. The Schwinger-Dyson equations for functions $G_{|\underline{p}_1^1\dots\underline{p}_{N_1}^1|\dots|\underline{p}_1^B\dots\underline{p}_{N_B}^B|}$ with $B>1$ are similar in the following sense: The basic functions with all $N_i \leq 2$ satisfy a complicated, but linear, equation. All higher functions with at least one $N_i \geq 3$ are purely algebraic.

3. Renormalisation and integral representation

The scaling limit $V\to\infty$ with $\frac{1}{V}\sum_{\underline{n}\in\mathbb{N}^2_{\mathcal{N}}}$ finite turns discrete matrix indices into continuous variables and sums into integrals. These integrals diverge and therefore require an energy cutoff $a,b,\dots\in[0,\Lambda^2]$. Normalisation conditions on the lowest Taylor terms of the two-point function $G_{|\underline{a}\underline{b}|}\mapsto G_{ab}$ express the bare quantities Z,μ_{bare} in terms of renormalised values $\mathscr{Y},\mu$ and of the cut-off Λ^2. Eliminating Z,μ_{bare} by their normalisation equations leads to a highly non-linear equation for the renormalised two-point function. The non-linearity cancels for the difference $G_{ab}-G_{a0}$ if the finite wavefunction renormalisation is $1+\mathscr{Y}=-\frac{dG_{0b}}{db}\big|_{b=0}$. These steps turn (2.5) into a linear singular integral equation of Carleman type. The solution theory of such equations gives:

Theorem 3 ([4]) *The matrix 2-point function G_{ab} of the $\lambda\phi_4^{\star 4}$-model is in infinite volume limit and for coupling constants $\lambda<0$ given in terms of the boundary 2-point function G_{0a} by*

$$G_{ab}=\frac{\sin(\tau_b(a))}{|\lambda|\pi a}\exp\Big(\mathrm{sign}(\lambda)\big(\mathscr{H}_0^\Lambda[\tau_0(\bullet)]-\mathscr{H}_a^\Lambda[\tau_b(\bullet)]\big)\Big)\,, \tag{3.1}$$

where $\tau_b(a):=\underset{[0,\pi]}{\arctan}\left(\dfrac{|\lambda|\pi a}{b+\frac{1+\lambda\pi a\mathscr{H}_a^\Lambda[G_{0\bullet}]}{G_{0a}}}\right)$ *and* $\mathscr{H}_a^\Lambda[f(\bullet)]:=\frac{1}{\pi}\lim_{\varepsilon\to0}\Big(\int_0^{a-\varepsilon}+\int_{a+\varepsilon}^{\Lambda^2}\Big)\frac{f(q)\,dq}{q-a}$ *denotes the finite Hilbert transform. The boundary function satisfies the fixed point equation*

$$G_{0b}=\frac{1}{1+b}\exp\left(-\lambda\int_0^b dt\int_0^{\Lambda^2}\frac{dp}{(\lambda\pi p)^2+\Big(t+\frac{1+\lambda\pi p\mathscr{H}_p^\Lambda[G_{0\bullet}]}{G_{0p}}\Big)^2}\right)\,. \tag{3.2}$$

For positive coupling constants $\lambda>0$ the angle function $\tau_b(a)$ ranges from 0 to π and therefore gives rise to a winding number which manifests in an ambiguity in the formulae for G_{ab} and G_{0b}. A perturbative solution of (3.2) reproduces the Feynman graph expansion. However, for any $\lambda>0$ one leaves the radius of convergence of the arctan series so that the perturbative expansion does not converge. A better strategy is to solve (3.2) by iteration (and exactly in $\lambda<0$). This iteration converges numerically, and according to Figure 1 we find evidence for a second-order phase transition at critical coupling constant $\lambda_c\approx-0.39$.

4. Schwinger functions and reflection positivity

By reverting the matrix representation we convert the matrix correlation functions $G_{|\dots|}$ to Schwinger functions in position space. Under conditions identified by Osterwalder-Schrader [8], the Fourier-Laplace transform of Schwinger functions gives rise to Wightman functions of a relativistic quantum field theory [1]. These conditions are [OS0] growth conditions, [OS1] Euclidean invariance, [OS2] reflection positivity, and [OS3] permutation symmetry. An additional axiom [OS4] clustering would give a unique vacuum state.

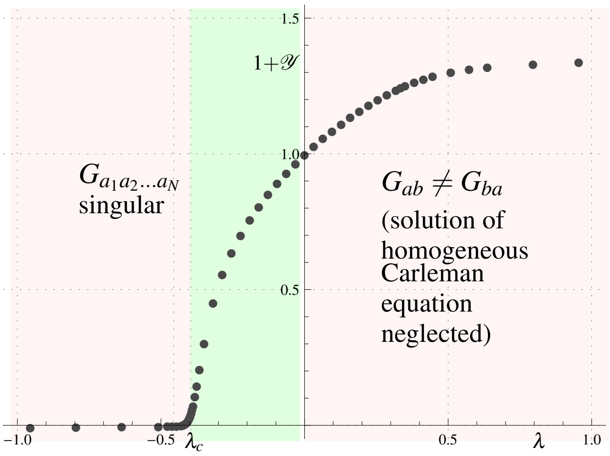

Figure 1: $1+\mathscr{Y} := -\frac{dG_{0b}}{db}\big|_{b=0}$ as function of λ, based on G_{0b} for $\Lambda^2{=}10^7$ with 2000 sample points.

Since the initial action (1.1) badly violates [OS1], it was completely clear to us that our model has no chance to satisfy the Osterwalder-Schrader axioms. To our enormous surprise, the infinite volume limit $\Theta \to \infty$ restored full Euclidean invariance:

Theorem 4 ([4]) *The connected N-point Schwinger functions of the $\lambda\phi^4_4$-model on extreme Moyal space $\theta \to \infty$ are given by*

$$S_c(\mu x_1,\dots,\mu x_N) = \frac{1}{64\pi^2} \sum_{\substack{N_1+\ldots+N_B=N\\ N_\beta \text{ even}}} \sum_{\sigma\in\mathscr{S}_N} \Bigg(\prod_{\beta=1}^{B} \frac{4^{N_\beta}}{N_\beta} \int_{\mathbb{R}^4} \frac{dp_\beta}{4\pi^2\mu^4}\, e^{\mathrm{i}\left\langle \frac{p_\beta}{\mu}, \sum_{i=1}^{N_\beta}(-1)^{i-1}\mu x_{\sigma(N_1+\ldots+N_{\beta-1}+i)}\right\rangle}\Bigg)$$
$$\times\, G_{\underbrace{\frac{\|p_1\|^2}{2\mu^2(1+\mathscr{Y})},\dots,\frac{\|p_1\|^2}{2\mu^2(1+\mathscr{Y})}}_{N_1}\big|\ldots\big|\underbrace{\frac{\|p_B\|^2}{2\mu^2(1+\mathscr{Y})},\dots,\frac{\|p_B\|^2}{2\mu^2(1+\mathscr{Y})}}_{N_B}}\,. \tag{4.1}$$

Permutation symmetry [OS3] is trivially realised, and growth estimates [OS0] can be deduced from the integral equation (3.2). Clustering [OS4] is violated.

Only a restricted sector of the underlying matrix model contributes to position space: All strands of the same boundary component carry the same matrix index. The most interesting sector is $N_\beta = 2$ in every boundary component, $G_{\frac{\|p_1\|^2}{2\mu^2(1+\mathscr{Y})}\frac{\|p_1\|^2}{2\mu^2(1+\mathscr{Y})}\big|\ldots\big|\frac{\|p_B\|^2}{2\mu^2(1+\mathscr{Y})}\frac{\|p_B\|^2}{2\mu^2(1+\mathscr{Y})}}$. The corresponding matrix functions $G_{a_1a_1|\ldots|a_Ba_B}$ satisfy more complicated (but linear!) integral equations. This $(2+\ldots+2)$-sector describes the propagation and interaction of B (Euclidean) particles without any momentum exchange. This is familiar from two-dimensional integrable models, but a sign of triviality in 4D. Typical triviality proofs rely on clustering or analyticity in Mandelstam representation. The validity of these assumptions in the present case needs verification.

Reflection positivity of $S_c(\mu x_1,\mu x_2)$ is equivalent [3] to the condition that G_{aa} is a Stieltjes function, i.e. representable as $G_{aa} = \int_0^\infty \frac{d\rho(m^2)}{a+m^2}$ for a positive measure ρ. This representation, which can be checked by purely real conditions, defines a holomorphic continuation of G_{aa} to the cut plane $\mathbb{C}\setminus[-\infty,0]$ together with Minkowskian positivity $\mathrm{Im}(G_{aa}) \geq 0$ for $\mathrm{Im}(a) < 0$. A discrete approximation as in Figure 1 cannot be holomorphic, but the Stieltjes property should fail in higher order for finer resolution. This is precisely what we observe (left of Figure 2). The improvement

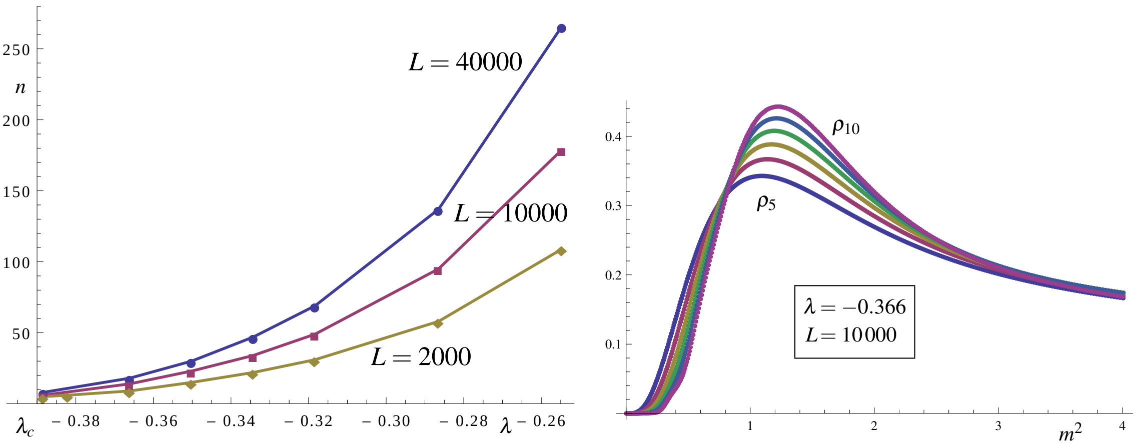

Figure 2: Left: Failure of logarithmically complete monotonicity $(-1)^n(\log G_{0b})^{(n)} \geq 0$ for various resolutions L as function of λ. Right: The sequence ρ_k of discrete approximations to the measure function $\rho(m^2)$ of G_{aa}.

slows down at precisely the same value $\lambda_c \approx -0.39$ as for the completely different problem of Figure 1. On the right of Figure 2 we show the first elements of a sequence ρ_k which would converge to the measure ρ if G_{aa} is Stieltjes. Again we confirm positivity. Details are given in [4].

All this is clear evidence, albeit no proof, of reflection positivity of the Schwinger 2-point function $S_c(\mu x_1, \mu x_2)$ precisely in the phase $\lambda_c < \lambda \leq 0$.

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