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Integral reduction via algebraic curves

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We show that for a class of two-loop diagrams, the on-shell part of the integration-by-parts (IBP) relations correspond to exact meromorphic one-forms on algebraic curves. Since it is easy to find such exact meromorphic one-forms from algebraic geometry, this idea provides a new highly efficient algorithm for integral reduction. We demonstrate the power of this method via several complicated two-loop diagrams with internal massive legs. No explicit elliptic or hyperelliptic integral computation is needed for our method.

12th International Symposium on Radiative Corrections (Radcor 2015) and LoopFest XIV (Radiative Corrections for the LHC and Future Colliders)
15-19 June, 2015
UCLA Department of Physics & Astronomy Los Angeles, USA

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PoS(RADCOR2015)085

1. Introduction

The study of high precision scattering amplitudes in Quantum Chromodynamics and the Standard Model is important for the Run II of the Large Hadron Collider (LHC). However, the precise computation suffers from problems of the large number of loop Feynman diagrams and difficult loop integrations. This work aims at developing a new method of reducing loop integrals to the minimal set of integrals, i.e., master integrals (MIs).

Traditionally, integral reduction can be achieved by applying integration-by-parts (IBP) identities [1]. There are several implements of IBPs generating codes AIR [2], FIRE [3–5] and Reduze [6, 7], based on Laporta algorithm [8], or LiteRed [9], based on a heuristic search of the symbolic IBP reduction. For multi-loop diagrams with high multiplicities or many mass scales, it may take a lot of computer time to finish the integral reduction. There are also several new approaches for integral reduction, based on the study of the Lie algebra structure of IBPs [10], Syzygy computation [11, 12], reductions over finite fields [13], and differential geometry [14]. Besides, the number of master integrals can be determined by the critical points [15].

We present a new method of integral reduction, for a class of multi-loop diagrams, based on unitarity [16–33] and the analysis of *algebraic curves* [34–36]. We show that for a D-dimensional L-loop diagram, if the unitarity cut solution V is an irreducible algebraic curve, then the on-shell IBPs correspond to *exact meromorphic 1-forms* on V. For an algebraic curve, it is very easy to find the exact meromorphic 1-forms. Hence we get the on-shell IBP easily.

We consider some complicated diagrams to show the power of our method: (1) $D = 4$ planar double box with internal massive legs. The unitarity cut for this diagram is an elliptic curve. (2) $D = 4$ non-planar crossed box with internal massive legs. The unitarity cut for this diagram is a genus-3 hyperelliptic curve. For these examples, we get all the on-shell IBPs analytically. in the time order of minutes.

2. Integral Reduction via the Analysis of Algebraic Curves

Generically, for a quantum field theory, the L-loop amplitude can be written as [16, 17],

$$A_n^{L\text{-loop}} = \sum_k c_k I_k + \text{rational terms}\,, \tag{2.1}$$

The set $\{I_k\}$ is called the master integral (MI) basis. Traditionally, the integral reduction is done by using IBP identities [1],

$$\int \frac{d^D l_1}{(2\pi)^D} \cdots \frac{d^D l_L}{(2\pi)^D} \frac{\partial}{\partial l_i^\mu} \frac{v_i^\mu}{D_1^{\alpha_1} \dots D_k^{\alpha_k}} = 0, \tag{2.2}$$

if there is no boundary term.

We present a new way of integral reduction, based on maximal unitarity method and algebraic curves. Given a Feynman integral with k propagators, maximal unitarity splits (2.1) as [18–33],

$$\text{Int} = \sum_i c_i I_i + \big(\text{integrals with fewer-than-k propagators}\big) + \text{rational terms} \tag{2.3}$$

where the first sum is over the master integral with exact k propagators.

The condition the all internal legs are on-shell, is called the maximal unitarity cut,

$$V: \quad D_1 = \ldots = D_k = 0, \tag{2.4}$$

and the solution set for this equation system is an *algebraic variety* V. V can be a set of discrete points, algebraic curves or surfaces. (See [37, 38] for the detailed mathematical study.) Maximal unitarity replaces the original integral with contour integrals [26–33], schematically,

$$\int \frac{d^D l_1}{(2\pi)^D} \cdots \frac{d^D l_L}{(2\pi)^D} \frac{N(l_1,\ldots l_L)}{D_1^{\alpha_1}\ldots D_k^{\alpha_k}} \to \oint \frac{d^D l_1}{(2\pi)^D} \cdots \frac{d^D l_L}{(2\pi)^D} \frac{N(l_1,\ldots l_L)}{D_1^{\alpha_1}\ldots D_k^{\alpha_k}} = \sum_j w_j \oint_{\mathscr{C}_j} \omega \tag{2.5}$$

where ω is a differential form on V, and contours c_j's are around the poles of ω and also the *nontrivial cycles* of V [30, 32]. w_j are weights of these contours. In particular, to extract the coefficients c_i in (2.1), we can find a special set of weights $w_j^{\{i\}}$ [26–32] such that,

$$c_i = \sum_j w_j^{\{i\}} \oint_{\mathscr{C}_j} \omega \tag{2.6}$$

Our observation is that if a differential form ω on V is integrated to zero, around all singular points on V, poles of ω and non-trivial cycles of V.

$$\oint_{\mathscr{C}_j} \omega = 0, \quad \forall j \tag{2.7}$$

then from (2.6) and (2.3), the integral corresponding to ω can be reduced to integrals with fewer propagators.

We focus on the cases for which the number of propagators equals $DL-1$ and the maximal unitarity cut gives one irreducible variety. In such a case, the cut solution V is a smooth *algebraic curve* with well defined complex structure. The condition (2.7) implies that ω is an exact meromorphic form on V, since the integral

$$F(P) = \int_O^P \omega, \quad \forall P \in V \tag{2.8}$$

is independent of the path and $dF = \omega$. Then from the study of meromorphic functions on V, we can list generators for F and then derive all forms which satisfy (2.7). Explicitly, for this class of diagram, we find that the scalar integral on the cut becomes a holomorphic form on V.

$$\int \frac{d^D l_1}{(2\pi)^D} \cdots \frac{d^D l_L}{(2\pi)^D} \frac{1}{D_1\ldots D_k}\bigg|_{\text{cut}} = \oint \Omega \tag{2.9}$$

where the 1-form Ω is globally holomorphic on V. On the cut, the components of l_i's become meromorphic functions. Let $F(l_1,\ldots l_L)$ be a polynomial in the components of loop momenta, then take the derivative of F,

$$dF = f\Omega. \tag{2.10}$$

The resulting $f\Omega$ is an exact meromorphic 1-form. From the analysis above, we get that,

$$\int \frac{d^D l_1}{(2\pi)^D} \cdots \frac{d^D l_L}{(2\pi)^D} \frac{f}{D_1\ldots D_k} = 0 + (\text{integrals with fewer propagators}), \tag{2.11}$$

so we obtain an integral reduction relation.

3. Elliptic Example: Double Box with Internal Masses

The method explained in the previous section can be used for integral reduction for various topologies, for instance, the double-box (Fig. 1) with three different masses for the internal propagators.

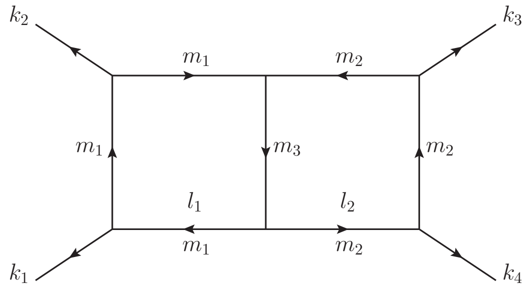

Figure 1: Planar double box diagram with 3 internal mass scales

3.1 Maximal unitarity

The denominators for double box diagrams are

$$\begin{aligned}
&D_1 = l_1^2 - m_1^2\,, \qquad && D_2 = (l_1 - k_1)^2 - m_1^2\,, \qquad && D_3 = (l_1 - k_1 - k_2)^2 - m_1^2\,, \\
&D_4 = l_2^2 - m_2^2\,, && D_5 = (l_2 - k_4)^2 - m_2^2\,, && D_6 = (l_2 - k_3 - k_4)^2 - m_2^2\,, \\
&D_7 = (l_1 + l_2)^2 - m_3^2\,. && &&
\end{aligned} \tag{3.1}$$

We parametrize the loop momenta as,

$$\begin{aligned}
l_1^\mu &= \alpha_1 k_1^\mu + \alpha_2 k_2^\mu + \alpha_3 \frac{s}{2} \frac{\langle 1|\gamma^\mu|2]}{\langle 14\rangle\,[42]} + \alpha_4 \frac{s}{2} \frac{\langle 2|\gamma^\mu|1]}{\langle 24\rangle\,[41]}, \\
l_2^\mu &= \beta_1 k_3^\mu + \beta_2 k_4^\mu + \beta_3 \frac{s}{2} \frac{\langle 3|\gamma^\mu|4]}{\langle 31\rangle\,[14]} + \beta_4 \frac{s}{2} \frac{\langle 4|\gamma^\mu|3]}{\langle 41\rangle\,[13]}.
\end{aligned} \tag{3.2}$$

The solutions for the maximal unitarity cut,

$$D_1 = D_2 = \ldots = D_7 = 0. \tag{3.3}$$

defines an elliptic curve. To see this, we first solve for the variables α_1, α_2, α_3, β_1, β_2 and β_3 in terms of α_4 and β_4,

$$\begin{aligned}
&\alpha_1 = 1\,, \qquad && \alpha_2 = 0\,, \qquad && \alpha_3 = \frac{m_1^2 t(s+t)}{\alpha_4 s^3}, \\
&\beta_1 = 0\,, && \beta_2 = 1\,, && \beta_3 = \frac{m_2^2 t(s+t)}{\beta_4 s^3}\,,
\end{aligned} \tag{3.4}$$

Then the remaining one equation relates α_4 and β_4,

$$K(\alpha_4, \beta_4) = A(\alpha_4)\beta_4^2 + B(\alpha_4)\beta_4 + C(\alpha_4) = 0, \tag{3.5}$$

Here $A(\alpha_4)$, $B(\alpha_4)$ and $C(\alpha_4)$ are quadratic polynomials of α_4, whose coefficients depend on kinematic variables. Formally, β_4 depends on α_4 as,

$$\beta_4 = \frac{-B(\alpha_4) \pm \sqrt{\Delta(\alpha_4)}}{2A(\alpha_4)}, \quad \Delta = B^2 - 4AC, \tag{3.6}$$

where Δ is a quartic polynomial in α_4 with *four distinct roots*. Hence the maximal unitarity cut defines an elliptic curve, i.e., algebraic curve with genus one,

On the cut, by a short calculation, the scalar double box integral on the cut, becomes

$$\int \frac{d^4 l_1}{(2\pi)^4} \frac{d^4 l_2}{(2\pi)^4} \frac{1}{D_1 \ldots D_7}\bigg|_{\text{cut}} = \frac{s^2 t}{16} \oint \frac{d\alpha_4}{\sqrt{\Delta}}, \tag{3.7}$$

where the overall factor is not important for the following discussion. As [32], it is remarkable that $\frac{d\alpha_4}{\sqrt{\Delta}}$ is the only *holomorphic one-form* associated with the elliptic curve. On the cut, the loop-momentum components α_i, β_i become elliptic functions.

3.2 Integral reduction

We now focus on the double box integral with numerator N,

$$I[N] = \int \frac{d^4 l_1}{(2\pi)^4} \frac{d^4 l_2}{(2\pi)^4} \frac{N}{D_1 \ldots D_7}, \tag{3.8}$$

Integrand reduction method via Gröbner basis method [39, 40] determines that the integrand basis contains 32 terms. On the cut, the integral becomes a meromorphic one-form,

$$I[N]|_{\text{cut}} \propto \oint \frac{d\alpha_4}{\eta} N(\alpha_3, \alpha_4, \beta_3, \beta_4) \tag{3.9}$$

where N is a polynomial in α_3, α_4, β_3 and β_4, and therefore also an elliptic function. If two integrals on the cut, differ by the contour integrals of an exact meromorphic one-form ω, then

$$I[N_1] - I[N_2]\bigg|_{\text{cut}} = \oint \omega = 0 \tag{3.10}$$

where the second equality holds for all contours, i.e., two fundamental cycles and small contours around the poles, because ω is exact. Then the integral reduction between $I[N_1]$ and $I[N_2]$ is achieved at the level of double box diagram,

$$I[N_1] - I[N_2] = 0 + (\text{integrals with} < 7 \text{ propagators}) \tag{3.11}$$

Note the α_4 and β_4 generate *all* elliptic functions on this elliptic curve. In practice, we find that to find such ω's, it is sufficient to consider the exterior derivatives of polynomials in α_3, α_4, β_3 and β_4, So we need to find the one forms $\{d\alpha_3, d\alpha_4, d\beta_3, d\beta_4\}$ and then use the chain rule to generate integral reduction relations. We can start by calculating $d\alpha_4$ in terms of the holomorphic one-form,

$$d\alpha_4 = \eta \frac{d\alpha_4}{\eta} = (2A(\alpha_4)\beta_4 + B(\alpha_4)) \frac{d\alpha_4}{\eta}, \tag{3.12}$$

where we used the definition $\eta = \sqrt{\Delta}$ and (3.6) to rewrite η in function of β_4. The purpose of this step is to get the a polynomial form of f.

We can now easily find $d\alpha_3$,

$$d\alpha_3 = d\left(\frac{\lambda_1}{\alpha_4}\right) = -\lambda_1 \frac{1}{\alpha_4^2} d\alpha_4 = -\frac{\alpha_3^2}{\lambda_1} d\alpha_4\,, \quad \lambda_1 \equiv \frac{m_1^2 t(s+t)}{s^3} \tag{3.13}$$

To generate the remaining 1-forms, we again use the form of elliptic curve. Recall that,

$$K(\alpha_4,\beta_4) = A(\alpha_4)\beta_4^2 + B(\alpha_4)\beta_4 + C(\alpha_4) = 0\,. \tag{3.14}$$

The identity $dK = 0$ reads,

$$d\beta_4 = -\left(A'(\alpha_4)\beta_4^2 + B'(\alpha_4)\beta_4 + C'(\alpha_4)\right)\frac{d\alpha_4}{\eta}\,. \tag{3.15}$$

Finally we can easily calculate $d\beta_3$,

$$d\beta_3 = d\left(\frac{\lambda_2}{\beta_4}\right) = -\lambda_2 \frac{1}{\beta_4^2} d\beta_4 = -\frac{\beta_3^2}{\lambda_2} d\beta_4\,, \quad \lambda_2 \equiv \frac{m_2^2 t(s+t)}{s^3} \tag{3.16}$$

Then use the chain rule, we get all the on-shell IBPs. For example, we analytically obtain this relation,

$$\begin{aligned}
I_{\text{dbox}}[\alpha_4^3] \;=\; & \frac{1}{2s^4(4m_2^2-s)}\Bigg(3s^3\left(m_1^2 s - m_2^2 s - m_3^2 s - 4m_2^2 t + st\right) I_{\text{dbox}}[\alpha_4^2] \\
& + \; s(4m_1^2 s^2 t - 2m_2^2 s^2 t - 2m_3^2 s^2 t + m_1^4 s^2 - 2m_2^2 m_1^2 s^2 - 2m_3^2 m_1^2 s^2 + m_2^4 s^2 + m_3^4 s^2 \\
& \quad -2m_2^2 m_3^2 s^2 + 2m_1^2 s t^2 - 4m_2^2 s t^2 - 8m_2^2 m_1^2 s t - 8m_2^2 m_1^2 t^2 + s^2 t^2) I_{\text{dbox}}[\alpha_4] \\
& + \; m_1^2 t(s+t)\left(m_1^2 s - m_2^2 s - m_3^2 s - 4m_2^2 t + st\right) I_{\text{dbox}}[1]\Bigg) + \ldots
\end{aligned} \tag{3.17}$$

where ... stands for integrals with fewer than 7 propagators. Consider all polynomials whose exterior derivative satisfy the renormalizability conditions, we obtain 23 integral relations. Furthermore, consider Levi-Civita insertions which integrate to zero,

$$\varepsilon(l_2,k_1,k_2,k_3)\; l_2\cdot k_1\,, \quad \varepsilon(l_2,k_1,k_2,k_3)\; l_1\cdot k_4\,, \quad \varepsilon(l_1,l_2,k_1,k_2)\,, \quad \varepsilon(l_1,l_2,k_1,k_3)\,. \tag{3.18}$$

we get 4 more integral relations. So the total number of MIs is $\#\text{MI}_{\text{dbox}} = 32 - 23 - 4 = 5$, and explicitly the MIs can be chosen as,

$$\text{MI}_{\text{dbox}} = \left\{I_{\text{dbox}}[\alpha_4\beta_3], I_{\text{dbox}}[\alpha_4^2], I_{\text{dbox}}[\alpha_4], I_{\text{dbox}}[\beta_3], I_{\text{dbox}}[1]\right\}\,. \tag{3.19}$$

The whole computation takes about 120 seconds with our Mathematica code. The relations are numerically verified by FIRE [3,4].

4. Hyperelliptic Example: Nonplanar Crossed Box with Internal Masses

We now proceed in studying the integral reduction of the massive nonplanar double box (Fig. 2). Unlike the previous examples, this diagram's maximal unitarity cut provides a genus-3 hyperelliptic curve [37, 38]. To illustrate our method, we consider the two-loop crossed box diagrams with massless external legs and three internel masses scales $\{m_1, m_2, m_3\}$.

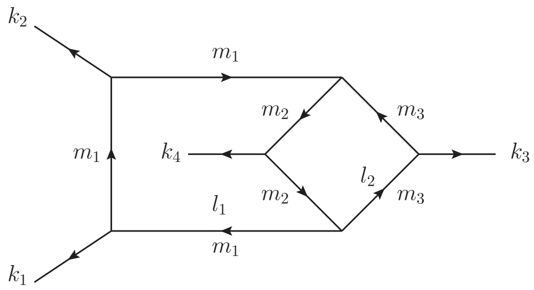

Figure 2: Nonplanar double box

4.1 Maximal Unitarity and geometric properties

The denominators for the Fig. 2 are,

$$\begin{aligned} &D_1 = l_1^2 - m_1^2\,, \qquad D_2 = (l_1-k_1)^2 - m_1^2\,, \qquad D_3 = (l_1-k_1-k_2)^2 - m_1^2\,, \\ &D_4 = l_2^2 - m_3^2\,, \qquad D_5 = (l_2-k_3)^2 - m_3^2\,, \qquad D_6 = (l_1-l_2+k_4)^2 - m_2^2\,, \\ &D_7 = (l_1+l_2)^2 - m_2^2\,. \end{aligned} \tag{4.1}$$

The on-shell constrains are $D_1 = \ldots = D_7 = 0$. We use the same loop momenta parametrization (3.2). Again, we first solve for α_1, α_2, α_3, β_1, β_2 and β_4 in terms of α_4 and β_3,

$$\begin{aligned} &\alpha_1 = 1\,, \qquad \alpha_2 = 0\,, \qquad \alpha_3 = \frac{m_1^2 t(s+t)}{\alpha_4 s^3}\,, \\ &\beta_1 = -(\alpha_4 + \alpha_3 + \frac{t}{s})\,, \qquad \beta_2 = 0\,, \qquad \beta_4 = \frac{(m_3^2)t(s+t)}{\beta_3 s^3}\,. \end{aligned} \tag{4.2}$$

The rest two variables satisfy a polynomial equation,

$$K(\alpha_4,\beta_3) = A(\alpha_4)\beta_3^2 + B(\alpha_4)\beta_3 + C(\alpha_4) = 0, \tag{4.3}$$

whose solution can be formally represented as,

$$\beta_3 = \frac{-B(\alpha_4) \pm \sqrt{\Delta(\alpha_4)}}{2A(\alpha_4)}, \quad \Delta \equiv B^2 - 4AC \tag{4.4}$$

Unlike the previous examples, $\Delta(\alpha_4)$ here is a degree-8 polynomial in α_4 with 8 distinct roots. Hence the unitarity cut of this diagram provides a genus-3 hyperelliptic curve.

4.2 Integral reduction

First, the integrand reduction via Gröbner basis [39,40] determines that, the integrand basis contains 38 terms in the numerator. Then, consider the maximal cut for the scalar integral of this diagram. The residue computation gives,

$$I_{\text{xbox}}[1]|_{7-cut} = \frac{s^3(s+t)}{16} \oint \frac{\alpha_4 d\alpha_4}{\sqrt{\Delta(\alpha_4)}}\,. \tag{4.5}$$

Note that unlike the elliptic case, on a genus-3 curve there are three holomorphic 1-forms.

$$\frac{d\alpha_4}{\sqrt{\Delta(\alpha_4)}}, \quad \frac{\alpha_4 d\alpha_4}{\sqrt{\Delta(\alpha_4)}}, \quad \frac{\alpha_4^2 d\alpha_4}{\sqrt{\Delta(\alpha_4)}} \tag{4.6}$$

the scalar integral cut corresponds to the second one.

This hyperelliptic curve have 6 fundamental cycles and 8 poles as shown in the previous subsection. By global residue theorem, only 7 poles' residues are independent. Therefore we may perform maximal unitarity by computing integrals over $6+7=13$ contours. Therefore the number of master integers must be 13.

Following what we did for elliptic cases, we calculate the differential forms $\{d\alpha_3, d\alpha_4, d\beta_3, d\beta_4\}$.

$$d\alpha_4 = \frac{\eta}{\eta} d\alpha_4 = (2A(\alpha_4)\beta_4 + B(\alpha_4))\frac{d\alpha_4}{\eta} = (2A(\alpha_4)\beta_4 - B(\alpha_3))\frac{\alpha_3}{\lambda_1}\frac{\alpha_4\, d\alpha_4}{\eta}\,, \tag{4.7}$$

where we have used the usual definition $\eta \equiv \sqrt{\Delta}$. In the second equality, we used the on-shell identity, $\alpha_3\alpha_4 = \lambda_1 \equiv \frac{m_1^2 t(s+t)}{s^3}$ to recover the form of the scalar integral cut (4.5). The step is not needed for elliptic cases. Then,

$$d\alpha_3 = d\left(\frac{\lambda_1}{\alpha_4}\right) = -\lambda_1 \frac{1}{\alpha_4^2} d\alpha_4 = -\frac{\alpha_3^2}{\lambda_1} d\alpha_4\,, \tag{4.8}$$

where again we have used (4.2) to simplify our expression. The exterior derivatives for β_i are more complicated,

$$d\beta_3 = -\left(A'(\alpha_4)\beta_3^2 + B'(\alpha_4)\beta_3 + C'(\alpha_4)\right)\frac{\alpha_3}{\lambda_1}\frac{\alpha_4 d\alpha_4}{\eta}\,, \tag{4.9}$$

and,

$$d\beta_4 = d\left(\frac{\lambda_2}{\beta_3}\right) = -\lambda_2 \frac{1}{\beta_3^2} d\beta_3 = -\frac{\beta_4^2}{\lambda_2} d\beta_3\,, \quad \lambda_2 = \frac{m_3^2 t(s+t)}{s^3} \tag{4.10}$$

Given a polynomial function of $\{\alpha_i, \beta_i\}$, we can use the chain rule to generate the on-shell IBPs.

Again, we also consider Levi-Civita insertions. In total, we generate 25 on-shell IBPs and 6 Levi-Civita insertions identities. Hence there are $38-25-6=7$ MIs for the non-planar crossed box diagram with three internal mass scales. Define that $X = (l_1+p_4)^2/2$ and $Y = (l_2+p_1)^2/2$, and the MIs can be chosen as:

$$\left\{I_{\text{xbox}}[X^3], I_{\text{xbox}}[Y^2], I_{\text{xbox}}[XY], I_{\text{xbox}}[X^2], I_{\text{xbox}}[X], I_{\text{xbox}}[Y], I_{\text{xbox}}[1]\right\}\,. \tag{4.11}$$

For this non-planar diagram, the analytic integral reduction relations are significantly more complicated. The integral reduction at the level of crossed box takes about 22 minutes with our Mathematica code, and the relations obtained by our method have been numerically verified by FIRE [3, 4].

5. Outlooks

Beyond the cases discussed, it would be interesting to study the ε-dependent part of the integral reduction, based on our method. One apparent difficulty is that the spacetime dimension is not a constant, so the cut solution's dimension or geometric structure is not fixed. This problem can be solved by using the dimension-dependent measure and dimension-independent propagators in the integrand. The research on two-loop maximal unitarity in dimensional regularization scheme [41], also based on algebraic geometry tools, help us to understand IBPs with dimensional regularization from a geometric viewpoint.

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