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Scattering amplitudes in TMD factorisation via BCFW recursion

Mirko SERINO[*]
Institute of Nuclear Physics, Polish Academy of Sciences
E-mail: mirko.serino@ifj.edu.pl

An extension of the BCFW on-shell recursion relation suitable to compute gauge invariant scattering amplitudes with off-shell particles is presented for Yang-Mills theories with fermions. It allows to compute amplitudes needed for the study of multi-parton scattering at hadron colliders in the framework of Transverse-Momemtum-Dependent (TMD) factorisation.

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[*]Speaker.

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1. Introduction

Scattering amplitudes are the very core of the predictions for observables at particle accelerators: cross sections are obtained by convoluting partonic scattering amplitudes, describing the interaction of the elementary constituents of the colliding hadrons on a smaller time scale, with universal functions (PDF's) describing the distributions of such partons inside the protons, which account for evolution phenomena taking place on a time scale longer than the parton scattering itself.

In the *Transverse-Momentum-Dependent (TMD) factorisation* or *k_T-factorisation* approach [1, 2], the amplitudes entering the calculation of cross sections feature particles with off-shell momenta, due to a non vanishing transverse component, which is additional w.r.t. the hadron (longitudinal) momentum fraction carried by the parton.

In order for results to be physical, amplitudes need to be gauge invariant, a property whose definition is far form trivial in case there are off-shell legs. A dramatic improvement in the calculation of on-shell scattering amplitudes has been achieved ever since 2005, when the BCFW recursion procedure was first introduced, originally for pure Yang-Mills theories [3, 4] and later extended to include amplitudes with fermions [5]. The question whether this recursion can be generalised to amplitudes with off-shell partons was solved in [6] in the case of gluons and extended to amplitudes with a fermion pair in [7].

2. Definitions

We always consider scattering amplitudes with all particles outgoing. The momentum k^μ can be decomposed in terms of its light-like direction p^μ, satisfying $p\cdot k = 0$ and, if the particle is off-shell, of a transversal part, following

$$k^\mu = x(q)p^\mu - \frac{\kappa}{2}\frac{\langle p|\gamma^\mu|q]}{[pq]} - \frac{\kappa^*}{2}\frac{\langle q|\gamma^\mu|p]}{\langle qp\rangle}\,, \tag{2.1}$$

with q^μ an auxiliary lightlike 4-momentum

$$x(q) = \frac{q\cdot k}{q\cdot p} \quad , \quad \kappa = \frac{\langle q|\not{k}|p]}{\langle qp\rangle} \quad , \quad \kappa^* = \frac{\langle p|\not{k}|q]}{[pq]}\,. \tag{2.2}$$

The coefficients κ and κ^* can be shown to be independent of the auxiliary momentum q^μ, in the sense that any other lightlike vector q' can be used in its place, provided $k\cdot q' \neq 0$ and

$$k^2 = -\kappa\kappa^*\,. \tag{2.3}$$

We consider *to-ordered* or *dual* amplitudes, which contain only planar Feynman graphs and are constructed with color-stripped Feynman rules. Every scattering amplitude, including the basic 3-point functions with off-shell particles, can be found via the recursion itself, provided one knows 3-point on-shell amplitudes, which can be built from symmetry principles anyway. No use of Feynman rules is necessary at any step. However, the knowledge of the Feynman rules necessary to compute off-shell gauge-invariant scattering amplitudes is necessary to properly identify the poles in the scattering amplitudes (see below) when applying the recursion: they can be found in [8, 9].

3. BCFW recursion

For every particle with momentum k_i^μ, an orthogonal direction p_i^μ is given,

$$\begin{aligned}
&k_1^\mu + k_2^\mu + \cdots + k_n^\mu = 0 \qquad \text{momentum conservation}\\
&p_1^2 = p_2^2 = \cdots = p_n^2 = 0 \qquad \text{light-likeness}\\
&p_1\cdot k_1 = p_2\cdot k_2 = \cdots = p_n\cdot k_n = 0\,. \qquad \text{eikonal condition}
\end{aligned}$$

In the case of an on-shell particle, direction and momentum are the same vector.

The polarisation vectors for gluons can be expressed as

$$\varepsilon_+^\mu = \frac{\langle q|\gamma^\mu|g]}{\sqrt{2}\langle qg\rangle}\,, \quad \varepsilon_-^\mu = \frac{\langle g|\gamma^\mu|q]}{\sqrt{2}[gq]}\,, \tag{3.1}$$

where q is the auxiliary lightlike vector and g is a short-hand notation for the gluon momentum. We denote gluon spinors with the numbers of the corresponding particles, whereas quarks and antiquarks are always indicated by q and $\bar{q}$ respectively.

The starting point of the on-shell BCFW recursion relation is the residue theorem

$$\lim_{z\to\infty} f(z) = 0 \Rightarrow \oint \frac{dz}{2\pi\mathrm{i}}\,\frac{f(z)}{z} = 0\,, \tag{3.2}$$

where the integration contour encloses all the poles of the rational function $f(z)$ and extends to infinity, implying that the function at the origin $f(0)$ is given by the sum over the residues at the single poles in the complex plane,

$$f(0) = -\sum_i \frac{\lim_{z\to z_i} f(z)\,(z-z_i)}{z_i}\,. \tag{3.3}$$

Now, if $f(z) = \mathscr{A}(z)$, where $\mathscr{A}(z)$ is a scattering amplitude which is turned into a function of a complex variable without spoiling momentum conservation and on-shellness, it is enough to identify the single poles in z appearing in some of the propagators in order to reconstruct the amplitude in terms of simpler building blocks. These are found to be products of on-shell lower-point amplitudes times an intermediate propagator, on the ground of general unitarity requirements [3,4].

In order to make a scattering amplitude a rational function of a complex variable z in a way that suits the off-shel case, two particles are picked up, say i and j, and each particle's direction is chosen to be the reference vector for the other, so that their momenta with transverse component are

$$\begin{aligned}
k_i^\mu &= x_i(p_j)\,p_i^\mu - \frac{\kappa_i}{2}\,\frac{\langle i|\gamma^\mu|j]}{[ij]} - \frac{\kappa_i^*}{2}\,\frac{\langle j|\gamma^\mu|i]}{\langle ji\rangle}\\
k_j^\mu &= x_j(p_i)\,p_j^\mu - \frac{\kappa_j}{2}\,\frac{\langle j|\gamma^\mu|i]}{[ji]} - \frac{\kappa_j^*}{2}\,\frac{\langle i|\gamma^\mu|j]}{\langle ij\rangle}\,.
\end{aligned} \tag{3.4}$$

Let the shift vector be

$$e^\mu = \frac{1}{2}\,\langle i|\gamma^\mu|j]\,, \quad p_i\cdot e = p_j\cdot e = e\cdot e = 0\,. \tag{3.5}$$

The shifted momenta are thus

$$\begin{aligned}\hat{k}_i^\mu &= k_i + ze^\mu \;=\; x_i(p_j)\,p_i^\mu - \frac{\kappa_i - z[ij]}{2}\,\frac{\langle i|\gamma^\mu|j]}{[ij]} - \frac{\kappa_i^*}{2}\,\frac{\langle j|\gamma^\mu|i]}{\langle ji\rangle} \\ \hat{k}_j^\mu &= k_j - ze^\mu \;=\; x_j(p_i)\,p_j^\mu - \frac{\kappa_j}{2}\,\frac{\langle j|\gamma^\mu|i]}{[ji]} - \frac{\kappa_j^* + z\langle ij\rangle}{2}\,\frac{\langle i|\gamma^\mu|j]}{\langle ij\rangle}\end{aligned} \tag{3.6}$$

Momentum conservation and either on-shellness or the eikonal conditions $p_i \cdot \hat{k}_i = 0$ and $p_j \cdot \hat{k}_j = 0$ are preserved by the shift (3.6). It is also possible to choose $e^\mu = 1/2\,\langle j|\gamma^\mu|i]$ as shift vector, implying that the shifted quantities are $\hat{\kappa}_i^*$ and κ_j. The changes induced in the momenta or in the directions by the first shift vector are:

$$e^\mu \;=\; \frac{1}{2}\langle i|\gamma^\mu|j] \Leftrightarrow \begin{cases} i \text{ off-shell:} & \hat{\kappa}_i = \kappa_i - z[ij] \\ i \text{ on-shell:} & |\hat{i}] = |i] + z|j] \\ j \text{ off-shell:} & \hat{\kappa}_i^* = \kappa_j^* + z\langle ij\rangle \\ j \text{ on-shell:} & |\hat{j}\rangle = |j\rangle - z|i\rangle \end{cases} \tag{3.7}$$

It is basic to the BCFW argument that (3.7) implies that the large z behaviours of the polarisation vectors of shifted gluons are

$$e^\mu = \frac{1}{2}\langle i|\gamma^\mu|j] \Rightarrow \varepsilon^\mu_{i-} \sim \frac{1}{z} \quad \text{and} \quad \varepsilon^\mu_{j+} \sim \frac{1}{z}\,, \tag{3.8}$$

whereas the opposite helicity polarisation vectors of shifted gluons stay constant. It is important for us, in order for our argument to work in general, to include in our amplitudes the propagators of the external off-shell particles, who play the same role as the gluon polarisation vectors in the on-shell case.

Not all of the shift vector choices are suitable to apply the BCFW recursion, because some of them lead to a violation of the basic hypothesis of the residue theorem

$$\lim_{z\to\infty} \mathscr{A}(z) = 0\,. \tag{3.9}$$

In [4] it was found that with the shift vector $e^\mu = \frac{1}{2}\langle i|\gamma^\mu|j]\;\mathscr{A}(z) \xrightarrow{z\to\infty} 0$ for three possible helicity choices of the shifted particles, namely $(h_i,h_j) = (-,+),(-,-),(+,+)$. It is easy to obtain a diagrammatic proof for the first case, that we dub the *original BCF prescription*, and all of our results for amplitudes with 1 off-shell particle refer to shifts which reduce to this case or, if $e^\mu = \frac{1}{2}\langle j|\gamma^\mu|i]$, to $(h_i,h_j) = (+,-)$.

As eikonal quark vertices only depend on the direction and eikonal quark propagators can only contribute powers of z to the denominator, the BCFW argument extends to our case immediately, if we shift two on-shell gluons. This also works for amplitudes with a fermion pair [5]. Then, if one of the shifted gluons is off-shell and the other one is on-shell, we require the helicity of the latter to agree with the original BCF prescription [6]. Finally, if both shifted gluons are off-shell, they both will contribute a factor $1/z$ with either shift vector choice. It is quite general that if both shifted particles are off-shell, BCFW recursion works for both shift vectors. A thorough discussion of all possible cases can be found in [7].

	$e^\mu = \frac{1}{2}\langle i\vert\gamma^\mu\vert j]$	$e^\mu = \frac{1}{2}\langle j\vert\gamma^\mu\vert i]$
C^g	$(h_i,h_j)=(+,+)$ $\vert\hat{k}_i] = \sqrt{x_i}\,\vert i]\,,\ \vert\hat{k}_i\rangle = \frac{\not{k}_i\vert j]}{\sqrt{x_i}[ij]}$ $\hat{\kappa}_j^* = \frac{\langle j\vert\not{k}_i+\not{k}_j\vert i]}{[ji]}$ or $\vert\hat{j}\rangle = \frac{(\not{k}_i+\not{p}_j)\vert i]}{[ji]}$	$(h_i,h_j)=(-,-)$ $\vert\hat{k}_i\rangle = \sqrt{x_i}\,\vert i\rangle\,,\ \vert\hat{k}_i] = \frac{\not{k}_i\vert j\rangle}{\sqrt{x_i}\langle ij\rangle}$ $\hat{\kappa}_j = \frac{\langle i\vert\not{k}_i+\not{k}_j\vert j]}{\langle ji\rangle}$ or $\vert\hat{j}] = \frac{(\not{k}_i+\not{p}_j)\vert i\rangle}{\langle ji\rangle}$
D^g	$(h_i,h_j)=(-,-)$ $\vert\hat{k}_j\rangle = \sqrt{x_j}\,\vert j\rangle\,,\ \vert\hat{k}_j] = \frac{\not{k}_j\vert i\rangle}{\sqrt{x_j}\langle ji\rangle}$ $\hat{\kappa}_i = \frac{\langle j\vert\not{k}_j+\not{k}_i\vert i]}{\langle ij\rangle}$ or $\vert\hat{i}] = \frac{(\not{k}_j+\not{p}_i)\vert i\rangle}{\langle ij\rangle}$	$(h_i,h_j)=(+,+)$ $\vert\hat{k}_j] = \sqrt{x_j}\,\vert j]\,,\ \vert\hat{k}_j\rangle = \frac{\not{k}_j\vert i]}{\sqrt{x_j}[ji]}$ $\hat{\kappa}_i^* = \frac{\langle i\vert\not{k}_j+\not{k}_i\vert j]}{[ij]}$ or $\vert\hat{i}\rangle = \frac{(\not{k}_j+\not{p}_i)\vert i]}{[ij]}$

Table 1: Shifted quantities needed for the evaluation of C^g and D^g residues. The reported helicity of the on-shell gluon is meant in the case it is on-shell.

3.1 The residues

The single poles in z appear due to the denominators of the gluon or fermion propagators. Our scattering amplitude $\mathscr{A}(0)$ is given by a sum over 4 possible kinds of residues

$$\mathscr{A}(0) = \sum_{s=g,f}\left(\sum_p \sum_{h=+,-} \mathrm{A}^s_{p,h} + \sum_i \mathrm{B}^s_i + \mathrm{C}^s + \mathrm{D}^s\right). \tag{3.10}$$

The index s refers to the particle species, namely gluons or fermions; h is the helicity; K^μ denotes the momentum flowing through the propagators exhibiting poles. The residues in the case of gluon poles are the following: $\mathrm{A}^g_{p,h}$ are due to the poles which appear in the original BCFW recursion. The index p stands for the cyclically ordered distributions of the particles into two subsets; the shifted particles are never on the same sub-amplitude. The pole is due to an intermediate virtual gluon, whose shifted momentum squared, $K^2(z)$, is on-shell for

$$z = -\frac{K^2}{2\,e\cdot K}\,.$$

B^g_i residues are due to poles appearing in the auxiliary eikonal quarks propagators whose denominator vanishes. This means $p_i\cdot\hat{K}(z)=0$, where $\hat{K}$ is the momentum flowing through the propagator. The location of these poles is

$$z = -\frac{2\,p_i\cdot K}{2\,p_i\cdot e}\,.$$

If the i-th particle is on-shell, these terms are not present.
C^g and D^g denote the same kind of residues: they appear respectively when the shifted i-th or j-th gluons are off-shell. They are due to the vanishing of the shifted momentum in the propagator: $k_i^2(z)=0$or $\quad k_j^2(z)=0$. Table 1 summarises the results for C^g and D^g terms which are needed to carry on computations.

Now fermions: the terms $\mathrm{A}^f_{p,h}$ and B^f_i are almost exactly the same as in the gluon case, except that in the B^f_i terms one has an eikonal (anti-)quark and a zero-momentum photon needed to account for the off shell (anti-)quark If the i-th particle is an on-shell fermion, these terms are not present.

h

	Admitted on-shell	Not admitted on-shell
C^f $e^\mu = \frac{\langle\bar{q}\lvert\gamma^\mu\rvert g]}{2}$	$(h_{\bar{q}}, h_g) = (-,+)$ $\lvert\hat{k}_{\bar{q}}] = \sqrt{x_{\bar{q}}}\,\lvert\bar{q}]\,,\ \lvert\hat{k}_{\bar{q}}\rangle = \frac{\not{k}_{\bar{q}}\lvert g]}{\sqrt{x_{\bar{q}}}[\bar{q}g]}$ $\hat{\kappa}^*_g = \frac{\langle g\lvert\not{k}_{\bar{q}}+\not{k}_g\rvert\bar{q}]}{[g\bar{q}]}$ or $\lvert\hat{g}\rangle = \frac{(\not{k}_{\bar{q}}+\not{p}_g)\lvert\bar{q}]}{[g\bar{q}]}$	$(h_{\bar{q}}, h_g) = (+,+)$ $\lvert\hat{k}_{\bar{q}}] = \sqrt{x_{\bar{q}}}\,\lvert\bar{q}]\,,\ \lvert\hat{k}_{\bar{q}}\rangle = \frac{\not{k}_{\bar{q}}\lvert g]}{\sqrt{x_{\bar{q}}}[\bar{q}g]}$ $\hat{\kappa}^*_g = \frac{\langle g\lvert\not{k}_{\bar{q}}+\not{k}_g\rvert\bar{q}]}{[g\bar{q}]}$ or $\lvert\hat{g}\rangle = \frac{(\not{k}_{\bar{q}}+\not{p}_g)\lvert\bar{q}]}{[g\bar{q}]}$
D^f $e^\mu = \frac{\langle g\lvert\gamma^\mu\rvert\bar{q}]}{2}$	$(h_g, h_{\bar{q}}) = (-,+)$ $\lvert\hat{k}_{\bar{q}}\rangle = \sqrt{x_{\bar{q}}}\,\lvert\bar{q}\rangle\,,\ \lvert\hat{k}_{\bar{q}}] = \frac{\not{k}_{\bar{q}}\lvert g\rangle}{\sqrt{x_j}\langle\bar{q}g\rangle}$ $\hat{\kappa}_g = \frac{\langle\bar{q}\lvert\not{k}_{\bar{q}}+\not{k}_g\rvert g]}{\langle g\bar{q}\rangle}$ or $\lvert\hat{g}] = \frac{(\not{k}_{\bar{q}}+\not{p}_g)\lvert g\rangle}{\langle g\bar{q}\rangle}$	$(h_g, h_{\bar{q}}) = (-,-)$ $\lvert\hat{k}_{\bar{q}}\rangle = \sqrt{x_{\bar{q}}}\,\lvert\bar{q}\rangle\,,\ \lvert\hat{k}_{\bar{q}}] = \frac{\not{k}_{\bar{q}}\lvert g\rangle}{\sqrt{x_j}\langle\bar{q}g\rangle}$ $\hat{\kappa}_g = \frac{\langle\bar{q}\lvert\not{k}_{\bar{q}}+\not{k}_g\rvert g]}{\langle g\bar{q}\rangle}$ or $\lvert\hat{g}] = \frac{(\not{k}_{\bar{q}}+\not{p}_g)\lvert g\rangle}{\langle g\bar{q}\rangle}$

Table 2: Shifted quantities needed for the evaluation of C^f and D^f residues in the case in which the antiquark is shifted. A completely similar pattern holds when q is shifted. The reported helicity of the gluon is meant in the case it is on-shell.

C^f and D^f denote exactly the same kind of residues as in the case of gluons. We summarize the results for C^f and D^f terms assuming we shift the antiquark. Shifting the quark is exactly the same, with obvious changes of labels.
All the results needed to work out these C^f and D^f-residues are listed in Table 2, where the distinction between choices which are suitable both in the on-shell and off-shell case and in the off-shell case only is stressed.

4. Some explicit results

We have focuses on amplitudes which always have one fermion pair and only one off-shell particle. We do not repeat here the explicit derivations of scattering amplitudes, but rather stress a feature of this off-shell case which is quite different with respect to the on-shell case. In the latter, all the the mostly-plus MHV amplitudes for Yang-Mills theories with fermions are given by

$$\begin{aligned} \mathscr{A}(g_1^+,\ldots,g_i^-,\ldots,g_j^-,\ldots,g_n^+) &= \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}\,, \\ \mathscr{A}(\bar{q}^+,\ldots,g_i^-,\ldots,g_n^+,q^-) &= \frac{\langle iq\rangle^3\langle i\bar{q}\rangle}{\langle\bar{q}1\rangle\langle 12\rangle\ldots\langle nq\rangle\langle q1\rangle}\,. \end{aligned} \tag{4.1}$$

The mostly-minus MHV amplitudes are just obtained by $\langle ab\rangle \leftrightarrow [ba]$.
In the off-shell case, what stands out with respect to this pattern is that amplitudes with all gluons having the same helicity, one fermion particle with opposite helicity and one off-shell fermion particle do not vanish, despite the do vanish in on-shell limit. For example $\mathscr{A}(\bar{q}^*,q^-,g_1^+,\ldots,g_n^+) \neq 0$. We call these amplitudes *subleading* because their analysis and comparison to the proper MHV amplitudes shows that they carry an extra factor whose absolute value is $\propto \sqrt{|k^2|}$, which is 0 when the particle is on-shell and significantly suppresses the amplitude for small transverse momenta [9].
We have chosen to call MHV the non-vanishing amplitudes featuring the maximum difference between the numbers of positive and negative helicity particles and, at the same time, not vanishing in the on-shell case. This classification singles out our subleading amplitudes, which would be the ones with the highest value of this difference, but it is specifically meant to be an intuitive extension

of the on-shell case. The structure of a subleading amplitude with an off-shell antiquark is given by

$$\mathscr{A}(g_1^+, g_2^+, \dots, g_{n-1}^+, \bar{q}, q, g_n^+) = \frac{-\langle \bar{q}q\rangle^3}{\langle 12\rangle\langle 23\rangle \dots \langle \bar{q}q\rangle\langle qn\rangle\langle n1\rangle}\,. \tag{4.2}$$

The case with an off-shell quark is completely analogous, of course.

5. Summary

TMD factorisation requires the computation of gauge-invariant scattering amplitudes with non vanishing transverse component of the momenta of one or two of the incoming particles. It was shown that an efficiently tree level evaluation of these amplitudes can be achieved via a generalisation of the BCFW recursion relation. This was first done in the pure Yang-Mills case [6], then fermions were recently included [7]. In particular, 5-point amplitudes with one off-shell parton are now completely known at tree-level [7].

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