

Review on Hadron Spectroscopy

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I review some of the lattice results on spectroscopy and resonances in the past years. For the conventional hadron spectrum computations, focus has been put on the isospin breaking effects, QED effects, and simulations near the physical pion mass point. I then go through several single-channel scattering studies within Lüscher formalism, a method that has matured over the past few years. The topics cover light mesons and also the charmed mesons, with the latter case intimately related to the recently discovered exotic XYZ particles. Other possible related formalisms that are available on the market are also discussed.

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1. Introduction

Spectroscopy computations have always been an important ingredient and a benchmark in lattice chromodynamics (lattice QCD) since its very early ages. In recent years, owing to the development in both theoretical understandings (physically and algorithmically) and computer hardware, lattice QCD computations have gradually made the crossover into the precision era, see e.g. the latest compilation of the FLAG Working Group [1, 2] for more details. In the light hadron spectrum sector, for example, lattice computations have been performed by various groups using full dynamical quarks with different number of dynamical flavors and different fermion realizations and the final results are very encouraging, see e.g. Ref. [3]. For hadrons involving heavy quarks, precise lattice computations also exist for the charmonia below the open charm threshold, see e.g. Ref [4], and for the single charmed mesons [5].

To accomplish these precise lattice computations, one has to control a number of systematic effects. These include finite lattice spacing errors (lattice artifacts), heavier than physical quark masses (the chiral limit, or more precisely, the physical quark mass limit), finite volume effects, isospin breaking errors and QED effects. Let us first briefly discuss these issues in the following.

To control the finite lattice spacing errors, improved actions and/or finer lattices have been utilized. As for the chiral limit, people have also been able to simulate right on the physical point or close enough to the physical quark mass values so as to make use of variants of Chiral Perturbation theory to access the true physical values for interested quantities. In fact, lattice QCD has provided an extra probe than true experiments in Nature in the sense that one can conveniently study the quark mass dependence of any physical quantity at any value of the quark mass. This particular dependence usually provides much more information of the theory than only at a particular value, even though it is the true physical value. We will see an example of this in subsection 3.2.

For the finite volume errors, since most hadrons are resonances instead of stable particles, scattering states are necessarily included together with the single-hadron states with the same quantum numbers. This is particularly important for unstable hadrons (that is, resonances) above some threshold under strong interaction. One of course then needs the connection between the discrete energy levels in a finite box and the scattering information parameterized by the S -matrix elements. The main theoretical framework, known as Lüscher's formalism in the community, has been established for more than two decades. However, only in recent years, the applications of this formalism in real lattice computations have matured. As we will see in this review, now we are able to reproduce e.g. the rho resonance with rather good precision, a task that could only be dreamed of in the 90's.

Another systematic effects comes from the isospin breaking and quantum electrodynamics (QED). In this problem, there is the famous mystery of neutron-proton mass splitting which is a subtle balance of the isospin breaking effects and the QED effects. In the past few years, two lattice groups, namely BMW and QCDSF/UKQCD, have studied this challenging problem and we will briefly discuss their results in subsection 3.1.

For the hadrons with heavy flavors, in recent years, partly due to the recent experimental progress in the so-called XYZ particles, lattice computations have also played an active role. Various groups have studied these newly discovered structures in both the charm and bottom sectors. Although these lattice studies are still not systematic enough to really nail down the true nature of

these exotic particles, these studies have certainly provided important non-perturbative information for these states.

This short review is organized as follows. In Sec. 2, I will first recapitulate the basic theoretical formalisms that have been utilized in typical lattice spectrum computations. Apart from the Lüscher's formalism that has been mentioned above, I will also discuss three other available formalisms on the market: the Hamiltonian Effective Field Theory method (HEFT), the HAL QCD method and the Optical Potential method. Then, starting in Sec. 3, I will go over the developments in spectroscopy in recent two years or so: starting from the light hadron spectrum, then move on to the single-channel scattering of light mesons and charmed mesons, the latter topic is heated up recently due to experimental discoveries of new near-threshold structures. Some related developments involving bottom quarks will be mentioned as well.

It should be noted though many topics that is related to spectroscopy is not discussed here. One important subject is the scattering involving baryons which is partly reviewed in Savage's talk on lattice nuclear physics [6]. I will also refer to Wilson's topical talk, which on the lattice calculation using coupled-channel Lüscher formalism [7], for multi-channel lattice computations in hadron scattering.

2. Theoretical methods utilized in spectrum computations

In a typical lattice QCD spectrum computation, one targets a specific channel with designated quantum numbers. Then a collection of interpolating operators $\{\mathcal{O}_\alpha, \alpha = 1, 2, \dots, N_{op}\}$ is chosen which carry the same quantum numbers of interest. Using Monte Carlo simulations, the following correlation matrix is estimated numerically,

$$\mathcal{C}_{\alpha\beta}(t) = \left\langle \mathcal{O}_\alpha(t) \mathcal{O}_\beta^\dagger(0) \right\rangle, \quad (2.1)$$

where $\langle \dots \rangle$ stands for the expectation value in the QCD vacuum which is usually achieved by using a sample of gauge field configurations. By solving the so-called generalized eigen-value problem (GEVP) for a judiciously chosen time-slice t_0 :

$$\mathcal{C}(t) \cdot u_\alpha = \lambda_\alpha(t, t_0) \mathcal{C}(t_0) \cdot u_\alpha, \quad \lambda_\alpha(t, t_0) \simeq e^{-E_\alpha(t-t_0)}, \quad (2.2)$$

one obtains the generalized eigenvalues $\lambda_\alpha(t, t_0)$. These eigenvalues are related to the exact energy eigenvalues, E_α , of the QCD Hamiltonian via $\lambda_\alpha(t, t_0) \simeq e^{-E_\alpha(t-t_0)}$. Therefore, through such a process, one obtains the exact energy eigenvalues of the system in a particular channel. It should be noted that, these operators share the same good quantum numbers respected by the QCD Hamiltonian, in other words, all operators that carry the same quantum numbers mix within QCD. This includes single hadron operators, two-hadron operators, etc. In fact, most resonances can be studied using this method as will be illustrated in subsection 3.3 and 3.4.

The next step in a spectroscopy calculation relies on how these energy eigenvalues, the E_α 's, are treated. It is clear that, at least in principle, these eigenvalues are not hadron mass values themselves, although they will approximate the hadron masses if the hadron resonance being considered is narrow enough. For generic resonances, it is also known that they are related to the scattering matrix elements within the so-called Lüscher's formalism, see e.g. Ref. [8, 9]. The theoretical formalism brought forward by Lüscher, first illustrated for single-channel scattering of two identical

spinless bosons in center-of-mass frame, was later on generalized in many ways so as to deal with more complicated scattering processes of various situations of the two hadrons in single or even in multi-channel scattering [10–30]. Up to now the theoretical formalism capable of dealing with the most general two-particle to two-particle scattering (single or multi-channel) is available and one in principle can utilize this to relate S -matrix elements to the eigenvalues obtained from GEVP. For the single-channel case, the formalism is rather straightforward. The practical lattice computations have matured over the past few years and some examples will be discussed in the following sections. For the case of multi-channel scattering, the application of the formalism is more elaborate and complicated. One needs some concrete parameterizations of the S -matrix to proceed. I refer to the contribution of Hadron Spectrum Collaboration (HSC) in these proceedings [7] for concrete examples.

Let us briefly review the main ideas behind Lüscher's formalism. In its simplest case, one considers two identical spinless bosons with mass m interact via a short-ranged interaction with range parameterized by R . In the center of mass frame, the elastic scattering phase of the two particle is described by the scattering phases $\delta_l(k)$ where k is the scattering momentum of the particle in the center of mass frame while l designates various partial waves. If the two particles are put inside a box of size $L \gg R$, then the infinite-volume scattering phase $\delta_0(k)$ ¹ is related to the exact energy of the two-particle system inside the box, $E(L)$, via,

$$\tan \delta_0(\bar{k}) = \frac{\pi^{3/2} q}{\mathcal{Z}_{00}(1; q^2)}, \quad (2.3)$$

where q and \bar{k} are related by $q = \bar{k}L/(2\pi)$ and \bar{k} is further related to $E(L)$ via $2\sqrt{\bar{k}^2 + m^2} = E(L)$. Thus, by measuring the values of $E(L)$, which are nothing but those E_α 's obtained from the GEVP process in Eq. (2.2), one can extract the scattering phase shift $\delta_0(\bar{k})$ at those energies.

Although Lüscher's formalism has been established for almost two decades, the real lattice simulations using the formalism have only matured in recent years, particularly in the single-channel scenario. In the multichannel case, the usage of the formalism tends to be rather involved and complicated, see e.g. Ref. [7]. It is therefore desirable to search for other theoretical formalisms. Up to now, three methods have been put forward: the Hamiltonian Effective Field Theory (HEFT) approach [31, 32], the HAL QCD method [33, 34] and the Optical Potential (OP) method [35]. These methods will be briefly discussed below.

In the Hamiltonian Effective Field Theory approach, one constructs an effective Hamiltonian starting from the non-interacting bare Fock states of relevant hadrons and parameterize their interactions using phenomenologically known results. For example, one could parameterize the interacting part of the Hamiltonian in terms of form factors and compute the low-energy scattering phase shifts which are compared with the known experimental results. Demanding that these scattering phase shifts to agree with the experiments will constrain the undetermined parameters appearing in the effective Hamiltonian. After this procedure, the same effective Hamiltonian can be diagonalized numerically in a finite box with a particular chosen volume, yielding the eigenvalues E_α which can be compared directly with the relevant results from corresponding lattice simulations.

¹This assumes that s -wave scattering dominates, neglecting the contributions of higher partial waves. This is true in the case of near-threshold scattering to be discussed in the following.

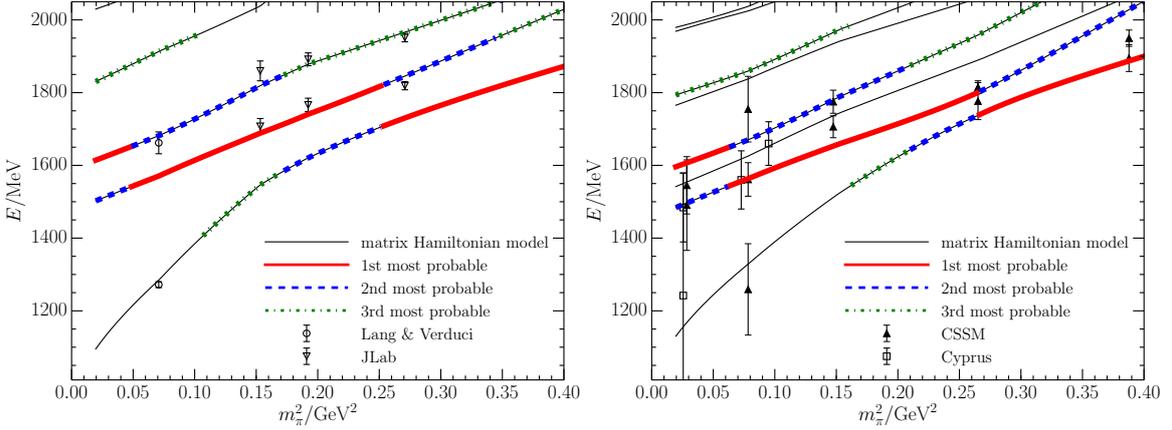


Figure 1: The energy eigenvalues in $J^P = (1/2)^-$ channel of the Hamiltonian for two different volumes with $L \simeq 2\text{fm}$ (left panel) and $L \simeq 3\text{fm}$ (right panel) at various pion masses, taken from Ref. [32]. The HEFT results are shown as different line types which are compared with the existing lattice data.

In Fig. 1, we have shown the results from HEFT [32] of the $N^*(1535)$ (the negative parity excitation of the nucleon) spectrum as a function of the pion mass value squared. Different line types are the HEFT results which are compared with relevant lattice data (points with error-bars) at two different volumes one with $L \sim 2\text{fm}$ (left panel) and one with $L \sim 3\text{fm}$ (right panel). Different line types and colors will illustrate the composition of the energy eigenstate in terms of the original bare Fock states. Interested readers can refer to Ref. [32] for further explanations of their color coding. In general, the HEFT can describe the lattice data rather well.

Some authors of Ref. [32] also believe that, using this approach, they could also resolve the Roper resonance [36], the first excited nucleon state in $J^P = (1/2)^+$ channel. There has been a puzzle in this channel for some years, namely the Roper turns out to be much higher than its physical value for most of the lattice computations except for those using overlap fermions. It is known that the overlap fermions, though quite expensive in terms of simulation, has the best chiral behavior which is considered to be the major reason for this discrepancy [37]. The authors of Ref. [36] thus claim that they have clarified the puzzle using HEFT, since the E_α 's that are measured on a particular lattice are not exactly the mass values themselves, but rather have to be converted to the values quoted in the experiments. However, people using overlap fermions in their simulations do not agree, see e.g. K.F. Liu's review on this subject [37, 38]. But it is fair to say that we are getting closer to the final solution of this puzzle than a few years ago.

Let us now come to the so-called HAL QCD method. The HAL QCD method utilizes the so-called Nambu-Bethe-Salpeter wavefunction that can be directly measured on the lattice. The HAL QCD collaboration has utilized this method over the years in the study of nuclear force and also in the process of baryon-baryon scattering. I refer to Savage's review [6] for specific discussions on these issues. Recently, HAL QCD also utilized their method to analyze the nature of the $Z_c(3900)$ structure [39] which will be discussed in subsection 3.4.

The Optical Potential (OP) method [35] is relatively new which attempts to measure the optical potential directly on the lattice. It is a very appealing approach that could evade some of the complications that will be present in Lüscher approach. Using synthetic data, the authors of Ref. [35]

have shown the successful application of this method. However, it remains to be seen how this method is implemented in real lattice simulations. On the theoretical side, it is also tempting to further understand the relation of this method with the other methods, say the HAL QCD method mentioned above.

3. Recent progress in lattice spectroscopy

In this section, I will go over recent progress in the field of lattice spectroscopy computations. I will start from the more conventional ones and then move on to single-channel scattering processes for the light mesons and the charmed mesons, the latter is intimately related to the newly discovered XYZ particles. Focus will be put on the charmed sector, though other exotic structures will also be mentioned. Scattering processes involving baryons will be covered by Savage's review [6].

3.1 Proton Neutron mass splitting

Lattice QCD have come to a stage that can address the subtle and difficult problem of proton-neutron mass splitting, which requires proper treatment of both QCD and QED on the lattice. This mass splitting is tiny, roughly 0.14% of its average mass value, and has far-reaching phenomenological consequences which concerns the very existence and stability of the usual baryonic matter. As we will see, it is rather subtly fine-tuned in terms of basic parameters of QCD and QED. Two groups, namely Budapest-Marseille-Wuppertal Collaboration (BMW) and the QCDSF-UKQCD Collaboration have made attempts towards this goal in the past few years which will be recapitulated in the following.

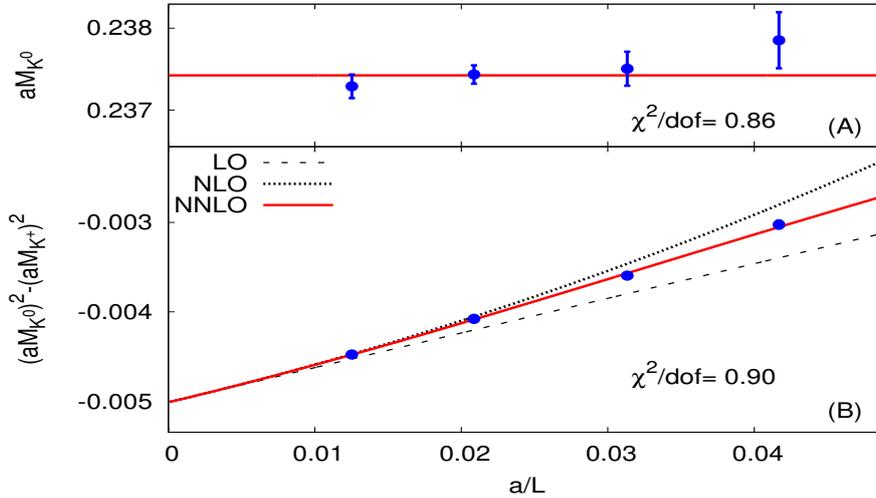


Figure 2: The finite volume dependence of the neutral and the charged kaon masses from Ref. [40].

The BMW group simulated $1 + 1 + 1 + 1$ flavor QCD+QED and study the finite volume corrections that arise due to the long-range nature of the electromagnetic interaction. They compared the so-called QED_L [41, 42] and the QED_{TL} prescription [40]. Due to its long range feature, treatment of the electromagnetic field in a finite volume needs special care. In the literature, there

have been the QED_{TL} and QED_L prescriptions that can be applied to the electromagnetic fields. The QED_{TL} prescription basically fixed the zero-momentum mode of the field $A_\mu(x)$ to zero which violates the reflective positivity, since this constraint on $\tilde{A}_\mu(0)$ involves time slices that are far apart in real space. It also introduces potential dangers since the charged particle propagators are ill-defined. The QED_L prescription only do so for the spatial-momentum zero mode but for every individual time-slice. Therefore it does not involve far separated time slices and thereby preserve reflective positivity. It however violates cubic symmetry which can be viewed as a finite volume effect. The BMW group compares the shift in the pole mass of a point particle in both QED perturbation theory and numerical simulations and show that the QED_L prescription looks normal. Fig. 2 illustrates the finite volume dependence of the neutral and the charged kaon masses. It is seen that the neutral kaon shows little (exponentially small) volume dependence while the charge one shows considerable finite-volume corrections that are well described by theoretical expectations.

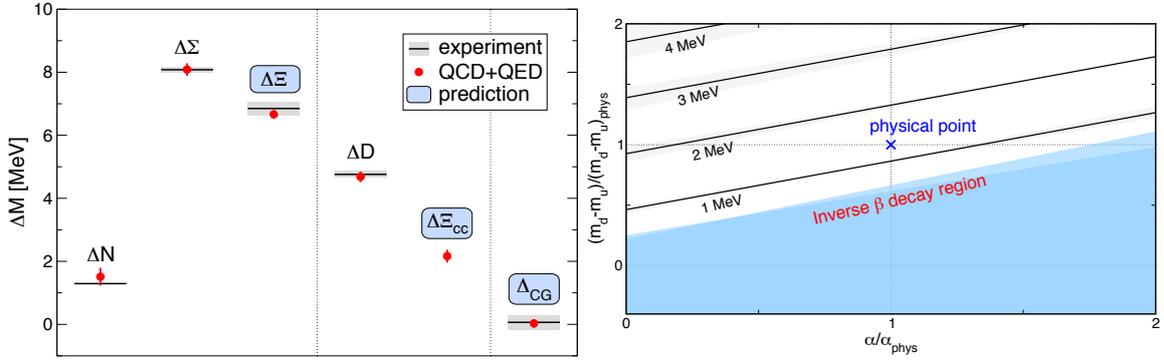


Figure 3: The mass splittings of various hadrons (left panel) and the corresponding allowed regions (right panel) in terms of quark masses and fine structure constant [40].

After treating the finite volume corrections carefully, BMW collaboration proceeds to obtain the mass splittings for various hadrons: the nucleon ΔN , the Σ baryon, the Ξ baryon, the D meson, the doubly charmed Ξ baryon and also the so-called Coleman-Glashow mass difference Δ_{CG} , which is supposed to vanish if the Coleman-Glashow relation is valid [43]. These results are shown in the left panel of Fig. 3. It is seen that the agreement with the existing experimental data is excellent. For example, the tiny mass splitting between the proton and the neutron is obtained. Their results even predict other mass splittings that have not yet been measured experimentally.

It is useful to separate the QCD and QED contributions to various mass splittings: $\Delta M_X = \Delta_{QCD}M_X + \Delta_{QED}M_X$, where $\Delta_{QCD}M_X$ is proportional to $\delta m = m_d - m_u$ while $\Delta_{QED}M_X$ is proportional to α_{EM} . This separation of course is ambiguous at the order of $\mathcal{O}(\alpha\delta m)$. BMW argued that, to a good approximation, the mass splitting of the Σ baryon comes solely from QCD. After fitting their data sets, the mass splitting between the neutron and proton can be separated into QCD and QED contributions. This is illustrated in the right panel of Fig. 3. The horizontal and vertical axis are essentially $\alpha \equiv \alpha_{EM}$ (the fine-structure constant due to electromagnetism) and $\delta m = m_d - m_u$ (the difference of the down and up quark masses) measured by their corresponding physical values. The color shaded region is ruled out due to the inverse β decay. The contour lines indicate constant $m_N - m_p$ values and the true physical point is indicated by a cross. This figure illustrates the subtle balance between the QCD and the QED contributions to neutron-proton mass splitting.

The QCDSF-UKQCD collaboration made a similar attempt [44–46]. They choose to tackle the problem from the so-called $SU(3)$ symmetric point of pure QCD simulations where all three quarks have the same masses. The so-called Dashen scheme was adopted and the conversion to other schemes was also discussed. They studied the octet baryons, the octet mesons, quark masses as well as vacuum structures. Using Dashen scheme, QCDSF is able to separate the QCD and QED contributions to the neutron-proton mass difference. They have also checked the Coleman-Glashow relation [43]. Fig. 4 shows the summary of their results. In the left panel, the mass splittings for the pion (π), kaon (K), nucleon (N), Σ baryon and Ξ baryon are compared with the experimental values which are indicated by the horizontal bars. The overall agreement is impressive. In the right panel, the equivalent of right panel of Fig. 3 is shown but plotted in a different way. The neutron-proton mass difference is plotted by separating the contribution from QCD (indicated by the parameter m_u/m_d) and QED (indicated by the parameter α_{EM}). The two color shaded regions are actually forbidden from cosmological point of view since there is either no fusion or no regular star formation.

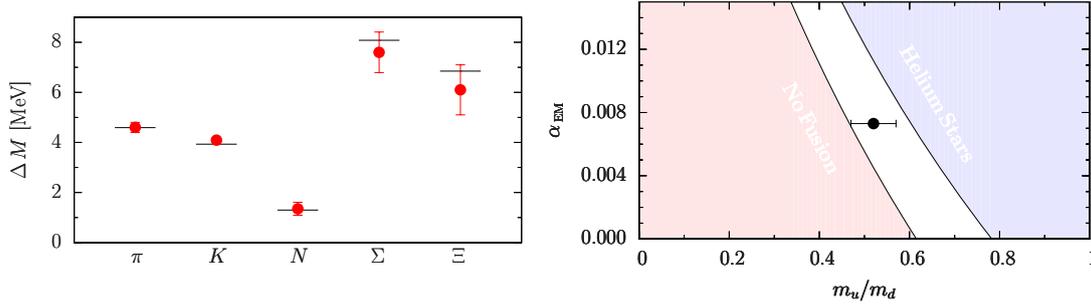


Figure 4: The final mass splittings for various hadrons (left panel) and the QCD (in terms of m_u/m_d) and QED contributions to the neutron-proton mass splittings [46].

3.2 Simulation at the physical point

Apart from the above mentioned lattice computations, the ETM collaboration performed a simulation at the physical pion mass [47]. The simulation was done at $\beta = 2.10$ with a clover term added. Various quantities have been computed and compared with the experimental results and other lattice results. These includes, meson and baryon masses, decay constants and their ratios, quark masses, etc. Chiral behavior is inspected with care. Fig. 5 shows a typical chiral behavior of $r_0 m_\pi^2 / f_\pi$ (left panel) and m_N / m_π (right panel) vs. $(r_0 m_\pi)^2$ where r_0 being the Sommer scale. It is seen that the direct simulated results, the purple triangle in the left panel and the blue square in the right, agree very well with the expected result obtained from chiral extrapolation (line in the left panel) and the corresponding experimental results (the green hexagrams in both panels). In the right panel, some of the $2 + 1 + 1$ results from ETMC (the red points) are also shown for reference.

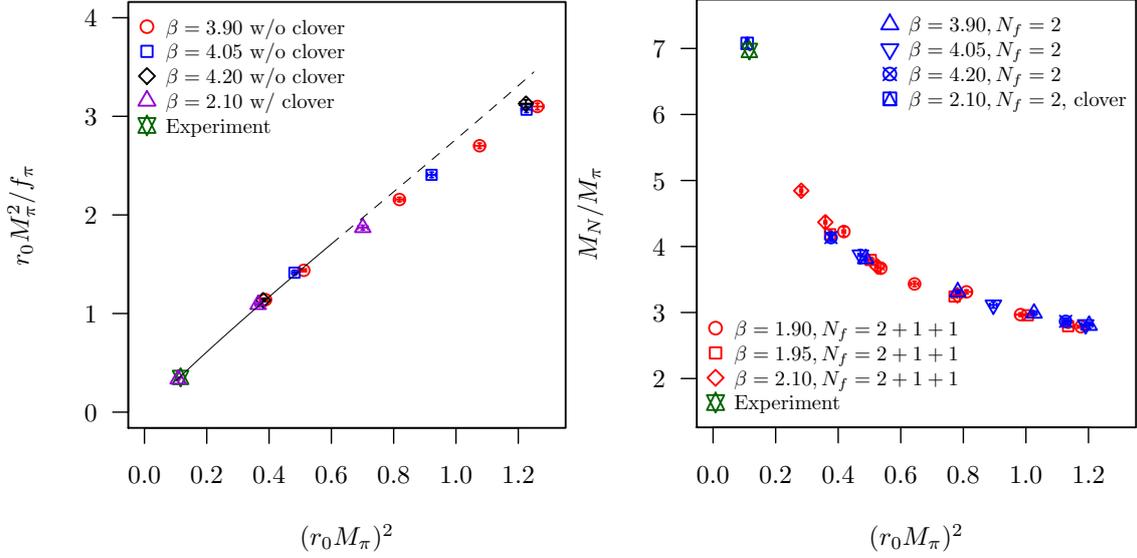


Figure 5: Typical chiral and continuum extrapolation in physical pion mass simulation of ETMC [47]. Left panel shows the quantity $r_0 m_\pi^2 / f_\pi$ while the right panel shows the mass ratio of the nucleon to the pion. The physical simulation point corresponds to the open purple triangle in the left and the blue square in the right panel, both of which having the smallest value of $(r_0 m_\pi)^2$.

In Ref. [47], ETMC also studied other important hadronic quantities: various hadron masses and decay constants (and their ratios), quark masses and the hadronic contributions to the lepton anomalous magnetic moments. They have compared their lattice results with the experimental ones and good agreements are found.

3.3 Single-channel scattering of light mesons

As mentioned in the previous section, lattice computations utilizing Lüscher's formalism have matured in recent years and there have been a number of such calculations in the past year. I will go through the light meson scattering computations first.

The ETM Collaboration has been studying the low-energy pion-pion scattering processes using their $2+1+1$ flavor ensembles [48, 49]. In Ref. [48], the $\pi\pi$ scattering length in $I=2$ channel, denoted by $a_0^{I=2}$, is obtained in the continuum and physical pion mass limit. As we all know now, scattering length is a very important low-energy quantity that will enter many effective field theory analysis. For example, the πN scattering length might be the crucial point to resolve the discrepancy for the sigma term between the existing lattice computations and the dispersion relations [50]. Therefore, pion-pion scattering length serves as a good benchmark quantity for lattice computations.

The result of $M_{\pi^\pm} a_0^{I=2}$ from Ref. [48] is plotted in Fig. 6 (marked as ETM (2015)) together with those from chiral perturbation theory and other previous lattice computations. The two points to the left are from continuum determinations, either using leading order chiral perturbation theory (marked as LO χPT) or together with dispersion relations (marked as CGL (2001)). All the rest are from various lattice computations performed with the chiral and continuum extrapolations. The error-bars stand for the statistical errors while the red shaded bands on the points indicate the

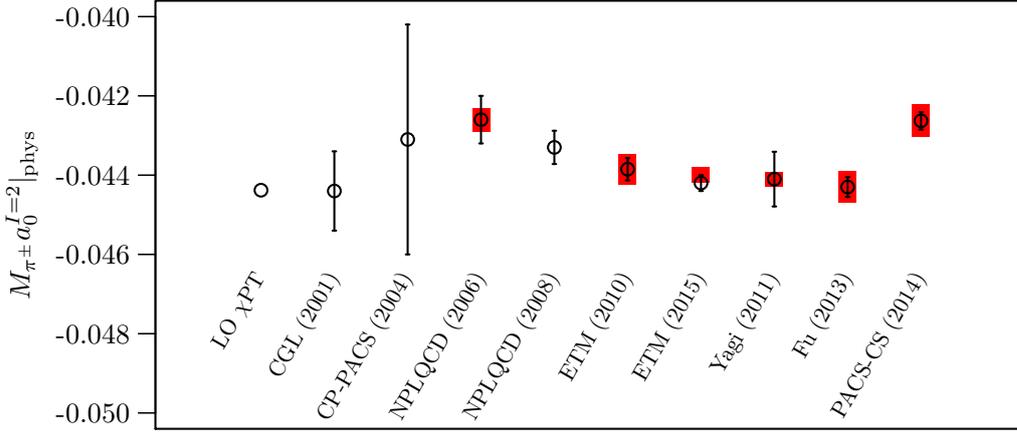


Figure 6: Comparison of ETMC [48]’s result, marked as ETM (2015), for $a_0^{I=2}$ with existing results in the literature. The two points to the far left correspond to leading order chiral perturbation w/o dispersion analysis from the continuum. The rest are all lattice results. Red bands indicate the systematic errors where available.

systematic errors where available. Careful analysis of the statistical and systematic errors have been performed in the study of Ref. [48] and similar studies in other channels of pion-pion and pion-kaon scattering are under way.

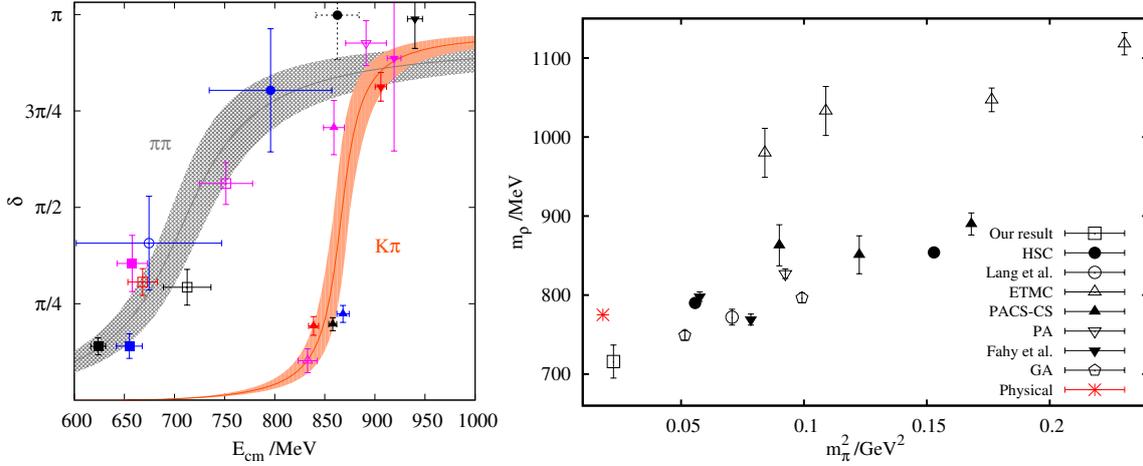


Figure 7: Left panel shows the phase shifts of $\pi\pi$ and $K\pi$ in the corresponding channel form RQCD computation [51]. The right panel illustrates the summary of various lattice determinations for m_ρ as a function of pion mass.

In recent years, the computation of the rho resonance becomes not only feasible but also quite fashionable, illustrating our understanding of the resonance nature of ρ meson. In the past year or so, there have been a number of computations of this using different techniques and with various fermion realizations [51–54]. Some of these computations have already been quite systematic in the sense that a series of lattice ensembles have been utilized in the computation, enabling one to estimate various systematics in a reliable fashion. As an example, in Fig. 7 we show the phase shifts obtained by the RQCD Collaboration using almost physical pion mass ($m_\pi \simeq 150\text{MeV}$) with

$N_f = 2$ Wilson fermions. They studied both $\pi\pi$ and $K\pi$ scattering in the p -wave and the scattering phase shifts are shown in the left panel of Fig. 7, depicting the resonance structure of the ρ (the $\pi\pi$ channel) and the K^* (the $K\pi$ channel), respectively. In the right panel, they summarize the Breit-Wigner mass of the ρ from various lattice groups. It is seen that most $N_f = 2$ results (the open symbols) undershoot the corresponding physical value (the red star) and this is likely due to the quenching of the strange quark, as the authors of Ref. [54] have argued. The filled symbols are from $N_f = 2 + 1$ simulations and they seem to converge to the physical result rather well.

To summarize, concerning the single-channel scattering of light mesons, physical properties have been studied with good accuracy and control.

3.4 Single-channel scattering of charmed mesons and the XYZ particles

In recent years, a handful of near-threshold structures have been observed in the experiments. This happened in both the bottom and the charm sector. These structures, though the nature of them remains to be clarified, have been called the XYZ particles. A wealth of phenomenological explanations have been put forward including: conventional quarkonium, molecular states, tetra-quark states, etc., see e.g. Ref. [55] and references therein. Lattice studies can also shed some light on these possible explanations. Below, I will focus on lattice studies on some of the Z_c states.

One of these Z_c state is $Z_c(3900)$, observed by BESIII, Belle and CLEO-c collaborations [56–58] whose quantum numbers are: $I^G(J^{PC}) = 1^+(1^{+-})$. The state is just around the $\bar{D}D^*$ threshold and interacts strongly with the $\bar{D}D^*$ final states. Therefore, it was naturally conjectured to be a loosely bound state of the two relevant charmed mesons. However, things might be more complicated than this. There are in fact quite a number of other thresholds below that of $\bar{D}D^*$, e.g. $\eta_c\rho$, $\pi J/\psi$ etc., therefore multi-channel effects might be relevant here. This is particularly complicated for lattice computations since it relies on the operator building process as we described in section 2. More importantly, if one would like to pursue the usual Lüscher’s approach, one in principle has to deal with a multi-channel situation, much more complicated than the single-channel version as in the $\pi\pi$ sector which we described in subsection 3.3. This has been realized for quite some time, see e.g. Ref. [59] and references therein.

Prelovsek *et al* studied this problem using one set of $16^3 \times 32$ $N_f = 2$ Wilson fermion gauge field ensemble that corresponds to $m_\pi \sim 266\text{MeV}$. The lattice spacing ($a \sim 0.124\text{fm}$) and the volume ($L \sim 2\text{fm}$) are all fixed. However, they used a rather elaborate multi-channel operator basis with distillation, a novel smearing technique [61, 62]. Their operator basis includes the so-called tetra-quark operators as well.² They obtained the finite volume spectra, namely those E_α ’s, and compare the spectrum with the free two-meson spectra and see if an extra state should emerge [60], a strategy that had been successfully utilized in their previous search for $X(3872)$ [63]. The summary plot of their spectra is shown in Fig. 8 in which no new exotic state could be identified. Therefore, their conclusion is negative, namely no new exotic state is found below 4.2GeV from their lattice computation.

To simplify the multi-channel nature of the problem, China Lattice QCD (CLQCD) attempts to single out the most important channel of the problem and proceeds with the single-channel Lüscher

²One should keep in mind that these tetra-quark operators are in fact related to the two meson operators via Fierz rearrangement. Therefore, the two sets of operators are in fact not linearly independent.

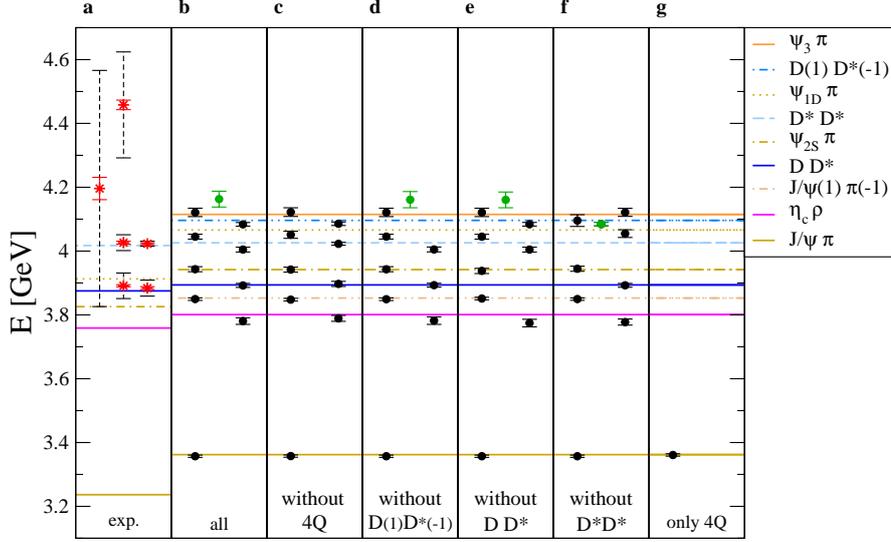


Figure 8: The spectra obtained by Prelovsek *et al*, taken from Ref. [60].

approach. Normally, from phenomenological arguments and experimental facts, one knows that the newly discovered structure strongly couples to one particular channel. For example, for the case of $Z_c(3900)$, the most important channel is \bar{D} and a D^* while for $Z_c(4025)$ the major channel is $D^*\bar{D}^*$. Then, building two-meson operators in these corresponding channels with the right quantum numbers will allow us to explore the problem within this single-channel approximation. CLQCD had estimated the effects of other operators/channels, making sure that they do not ruin the major channel correlators [64, 65]. Partially twisted boundary conditions have been utilized to fully explore the near threshold region. Then, using single-channel Lüscher formalism combined with effective range expansion,

$$k \cot \delta(k) = \frac{1}{a_0} + \frac{1}{2}r_0 k^2 + \dots, \quad (3.1)$$

they were able to determine the elastic scattering length a_0 and the effective range r_0 of the two charmed mesons in the near-threshold energy region, where k stands for the magnitude of the scattering momentum of the two mesons in the center of mass frame.

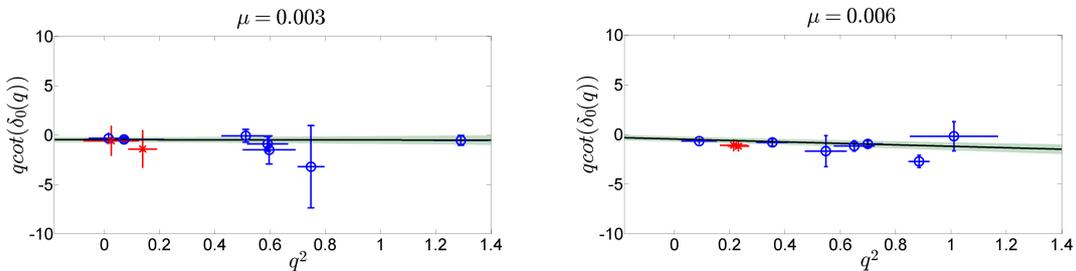


Figure 9: The quantity $q \cot(\delta_0(q^2))$ is plotted vs. the dimensionless scattering momentum squared, q^2 , at two different pion mass values, taken from Ref. [65].

CLQCD utilized $N_f = 2$ twisted mass configurations with three different pion masses ranging from 300MeV to 485MeV. However, all configurations are at a fixed lattice spacing ($a \sim 0.067\text{fm}$) and a fixed physical volume ($L \sim 2.1\text{fm}$). In Fig. 9 the effective range fitting from CLQCD's study on $Z_c(4025)$ is illustrated [65]. The horizontal axis is proportional to the scattering momentum squared, $q^2 = (kL/(2\pi))^2$ while the vertical axis shows the quantity $q^2 \cot \delta(q^2)$. The straight lines are linear fits according to Eq. 3.1 near the threshold $q^2 = 0$ and the intercepts of the straight lines basically yield $1/a_0$. Slightly negative scattering lengths have been obtained. This is quite analogous to the case of $\pi\pi$ scattering in the $I = 2$ channel. Their results thus indicate that the two charmed mesons seem to have weak repulsive interactions and no bound states are found for all three pion mass values in their lattice computation. Similar situation has been witnessed in the study of $Z_c(3900)$ [64]. However, this weak repulsion scenario is not universal for all cases of charmed mesons. In a similar lattice study on $Z(4430)$, a structure close to the threshold of a D_1 and \bar{D}^* which was first observed by Belle [66] back in 2008 and later on verified by LHCb [67], CLQCD did find attractive interaction between the two charmed mesons in both quenched lattice QCD and in the unquenched case [68].

The two different approaches mentioned above are in fact quite complimentary to each other. The main conclusion they reach is also similar: No indication from the lattice computation has been observed for the state $Z_c(3900)$ and $Z_c(4025)$. A similar preliminary study using HISQ lattices by the Fermilab and MILC collaboration also fails to identify any indications for these exotic states [69].

On the other hand, HAL QCD collaboration tackled the problem using their HAL QCD approach. They used improved Wilson gauge field configurations PAC-CS with $2 + 1$ dynamical flavors at one lattice spacing ($a \sim 0.09\text{fm}$) and one volume ($L \sim 2.9\text{fm}$). They did check the pion mass dependence by simulating at three different pion masses ranging from 410MeV to 700MeV. Their conclusion was that, $Z_c(3900)$ is a threshold cusp that is due to multi-channel interaction effects, see Ref. [39]. In particular, the $\eta_c\rho$ channel and the $J/\psi\pi$ channel all interact strongly with the DD^* channel. It would really be nice to check this result using a different approach. For example, in the particular channel of $Z_c(3900)$, one could carry out a coupled channel study using Lüscher formalism, which is feasible if one pre-selects say only two or three most important channels, and see if a similar conclusion could be reached.

One should keep in mind that these studies discussed above are still quite preliminary to draw any definite conclusions. In particular, usually only one volume at one lattice spacing have been utilized in these lattice searches and more systematic studies are very much welcome here. Needless to say that the nature of these exotic structures, whether it is a resonance or a bound states or even just multichannel effects, remains a challenging problem and hopefully lattice will provide us with more information in the future.

3.5 Other exotic structures from the lattice

Recently, Lang *et al* have also studied the counterparts of the above mentioned XYZ particles in the bottom sector [70]. Near threshold exotics also show up in the charm-strange and bottom-strange mesons. For example, there have been lattice studies on the D_s and B_s mesons [71, 72]. At this conference, there have also been a few reports on multi-heavy tetraquark states, see e.g.

Ref. [73]. HAL QCD have also reported their new results in baryon-baryon scattering, please refer to the review of Savage [6] and references therein for further details.

4. Summary and outlook

Generally speaking, lattice studies of spectroscopy has entered the precision stage. To be more specific, one has to focus on different subfields in lattice spectroscopy. In this short review, I have gone over a number of these developments in the past year or so.

In hadron spectroscopy involving only the light quarks, especially for those stable ones under strong interaction, it is clear that the field has already entered the precision era. The lattice studies in this field are rather systematic and precise, not only to percent level, but in some cases to per mil level. People have also started to consider not only QCD but also QED simultaneously. We could also perform studies close enough to the physical pion mass point so that comparisons with effective field theories can be carried out. In fact, the scope of precise lattice computations goes beyond just the light quark sector. For charmonium below the open charm threshold, things are also rather precise, see e.g HPQCD's results [4, 5] on the hyperfine splittings. Generally speaking, for hadrons that are stable under strong interaction, rather good accuracy have been obtained.

However, for hadrons that can decay under strong interactions, one in principle needs to study the scattering process of the decay products and this is where Lüscher formalism has come into play. In recent years, a lot of progress has been gained in this direction, both theoretically and in practical simulations. As we see in subsection 3.3, rather good accuracy has been obtained for light meson scattering. Therefore, for single-channel scattering of light mesons, also rather precise and systematic results can be obtained. As I showed you in this review, numerous computations have been performed on the rho resonance. We have seen from Wilson's talk that people have also been able to tackle multi-channel scattering problems within Lüscher formalism.

For particles involving heavy quarks, especially those beyond the threshold, one has to deal with the scattering of the relevant hadrons. The complication here is that usually this is typically a multi-channel situation and a brut-force treatment using the conventional Lüscher method is complicated. However, within certain approximations, the progress in this field is also steady, but more studies are definitely required. Although in this review, I only focused on the charmed meson case, lattice computations in this direction will definitely have very important impact on the experiments that have been fast developing in recent years. It is also desirable to search for other equivalent or complementary methods that can handle the multi-channel scattering of multi-hadron systems.

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