PROCEEDINGS OF SCIENCE

Quantum Gravity and Dimensional Transmutation

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I summarise the salient features of Dimensional Transmutation and describe its recent application to renormalisable Quantum Gravity

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1. Introduction

Our goal is ambitious: to construct a theory of gravitation, and gravitational interactions, which is both scale invariant (in the classical approximation), and Ultra-Violet (UV) complete by virtue of the fact that all the coupling constants in the theory are asymptotically free. The tool for this is Dimensional Transmutation (DT); which refers to the fact that there are field theories with no explicit mass parameter which nevertheless have a massive spectrum (which may be complicated) generated by radiative corrections. There are two types of DT:

- Non-perturbative: for example Massless QCD
  - Chiral symmetry breaking and confinement are triggered by the large value of the strong coupling α_s.
- Perturbative
  - Scalar QED (Coleman-Weinberg) [1]
  - R^2 Gravity [2]-[11]

2. Dimensional Transmutation in Perturbation Theory

The effective potential $V(\phi)$ is the leading term in a derivative expansion of the 1PI effective action, $\Gamma(\phi)$:

$$\Gamma(\phi) = -V(\phi) + f(\phi)\partial^\mu\phi\partial_\mu\phi + \cdots \tag{2.1}$$

Let us begin by considering the simplest possible theory: massless $\lambda\phi^4$. Including one-loop corrections the potential is given by

$$V(\phi) = \frac{\lambda}{4!}\phi^4 + \frac{\kappa}{4}\left(\frac{1}{2}\lambda\phi^2\right)^2 \ln\frac{\phi^2}{\mu^2} \tag{2.2}$$

where $\kappa = (16\pi^2)^{-1}$ and μ is the renormalisation scale. Notice that V is not analytic in ϕ.

2.1 The False Minimum

$$V(\phi) = \frac{\lambda}{4!}\phi^4 + B\kappa\phi^4\ln\frac{\phi^2}{\mu^2}, \quad \text{where} \quad B = \frac{\lambda^2}{16}. \tag{2.3}$$

Consider the dependence of V on ϕ for fixed μ, $\lambda > 0$. Clearly $V \to 0$ as $\phi \to 0$, and $V \to \infty$ as $\phi \to \infty$. Moreover V clearly has a minimum for nonzero ϕ, where

$$\frac{\kappa\lambda^2}{4}\ln\frac{\phi^2}{\mu^2} = -\frac{\lambda}{6} - \frac{\kappa\lambda^2}{8}. \tag{2.4}$$

Normally we are happy with perturbation theory if the coupling constant is "sufficiently" small. But here even if λ is small we see that the one loop correction to the potential is comparable to the tree term, a classic sign that we cannot trust the result.

2.2 RG Evolution

Of course what we forgot above is that λ depends on μ:

$$\mu\frac{\partial\lambda}{\partial\mu} = \beta_\lambda(\lambda) = 3\kappa\lambda^2 \quad \text{with solution} \quad \lambda(t) = \frac{\lambda_0}{1-3\kappa\lambda_0 t}, \tag{2.5}$$

where $t = \ln(\mu/\mu_0)$. Moreover $V(\phi)$ satisfies a RG equation:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta_\lambda(\lambda)\frac{\partial}{\partial\lambda} - \gamma(\lambda)\phi\frac{\partial}{\partial\phi}\right]V(\phi,\lambda,\mu) = 0, \tag{2.6}$$

whence we have $\beta_\lambda^{(1)} = 48\kappa B = 3\kappa\lambda^2$. *So we could have used* $\beta_\lambda^{(1)}$ *to calculate B.*

We can solve the RG equation, but the essentials of the result are given by simply deciding that we will evaluate V choosing $\mu = \phi$. Then, as long as $\lambda(t)$ is small, we have to a good approximation

$$V = \frac{1}{4!}\lambda(t)\phi^4 = \frac{1}{4!}\frac{\lambda_0}{1-3\kappa\lambda_0 t}\phi^4 \quad \text{with now} \quad t = \ln\frac{\phi}{\mu_0}. \tag{2.7}$$

We still have $V \to 0$ as $\phi \to 0$, but now there is no sign of the (spurious) minimum; V increases monotonically with ϕ. As ϕ increases above μ_0, eventually λ reaches a Landau pole but of course we cannot evaluate V confidently once $\lambda(t) >> 1$.

2.3 Dimensional Transmutation

We can reach the same conclusion in a different way.

$$V(\phi) = \frac{\lambda}{4!}\phi^4 + B\kappa\phi^4\ln\frac{\phi^2}{\mu^2}. \tag{2.8}$$

Now instead of $\mu = \phi$ let's pick $\mu = \langle\phi\rangle$ where $\langle\phi\rangle$ is defined as a function of λ and μ by the equation $\frac{\partial V}{\partial\phi} = 0$. Plugging this into Eq. (7) we at once get the relation

$$\langle\phi\rangle^3\left(\frac{\lambda}{6} + 2B\kappa\right) = \langle\phi\rangle^3\left(\frac{\lambda}{6} + \frac{\kappa\lambda^2}{8}\right) = 0. \tag{2.9}$$

Notice that we can have a $\langle\phi\rangle \neq 0$ solution if $\lambda = -4/(3\kappa)$. Unfortunately this is is (a) negative and (b) large. It is clearly unphysical; although *not* because it is negative, as we shall see when we consider SQED.

2.4 Massless SQED

The one-loop V for massless scalar QED in the Landau gauge is:

$$V(\phi) = \frac{\lambda}{4!}\phi^4 + \kappa(\frac{5}{72}\lambda^2 + \frac{3}{4}e^4)\phi^4\ln\frac{\phi^2}{\mu^2}. \tag{2.10}$$

If we once again choose $\mu = \langle\phi\rangle$ then we find the condition

$$\frac{\lambda}{6} + \kappa(\frac{5}{36}\lambda^2 + \frac{3}{2}e^4) = 0. \tag{2.11}$$

This differs crucially from the ungauged case. We can have $|\lambda|, e \ll 1$ with $\lambda \sim -\kappa e^4$ so that Eq. (11) is perturbatively believable: we can easily see that higher loop effects will be smaller. But what about $\lambda < 0$? Is V unbounded?

The crucial point is that it only makes sense to set $\mu = \langle\phi\rangle$ for values of ϕ *near* $\langle\phi\rangle$. To look at V for $\phi >> \langle\phi\rangle$ we must revert to our previous method of choosing $\mu \sim \phi$. Then, as in the ungauged case, we see that (as long as $\lambda(\mu)$ and $e(\mu)$ are both small, the potential is given to a good approximation by

$$V(\phi) = \frac{\lambda(\phi)}{4!}\phi^4, \tag{2.12}$$

but now the RG evolution of λ is quite different from the ungauged case. Crucially, λ can be negative at small ϕ and positive at large ϕ. So we can have $\lambda < 0$ at the extremum but also a bounded V.

We have the RG equations:

$$\frac{d\lambda}{dt} = \kappa\left(\frac{10}{3}\lambda^2 - 12\lambda e^2 + 36e^4\right), \tag{2.13}$$

$$\frac{de}{dt} = \frac{1}{3}\kappa e^3. \tag{2.14}$$

If we write $Y = \lambda/e^2$ then

$$\frac{dY}{dt} = \kappa e^2(\frac{10}{3}Y^2 - \frac{38}{3}Y + 36). \tag{2.15}$$

The quadratic on the RHS is always positive; so as $t \to \pm\infty, Y \to \pm\infty$. Hence as $t \to \infty$, $\lambda \to \infty$; for negative t we need to solve the equation for Y, which is not too difficult.

The solution is

$$e^2 = \frac{e_0^2}{1 - \frac{2}{3}\kappa t e_0^2}, \tag{2.16}$$

$$\lambda = \frac{e^2}{10}\left[\sqrt{719}\tan\left(\frac{1}{2}\sqrt{719}\ln(e^2) + \theta\right) + 19\right]. \tag{2.17}$$

Here θ is a constant chosen so that $\lambda = \lambda_0$ when $t = 0$. We see that as $t \to \infty$, $\lambda \to \infty$ (at a finite t value) so that $V(\phi) \to \infty$ as λ approaches a Landau pole. As $t \to -\infty$, $\lambda \to -\infty$, but $V \to 0$ since as $t \to -\infty$, $\phi \to 0$. Perturbation theory breaks down at both $t \to \pm\infty$, but there is an intermediate region where it is possible to have $\lambda \approx -9\kappa e^4$ (so as to satisfy Eq. (2.11)) and hence a perturbatively credible extremum, which as we shall now see is a minimum.

2.5 The Higgs mass

We had (for $\kappa\lambda << 1$)

$$V(\phi) = \frac{\lambda}{4!}\phi^4 + B\kappa\phi^4\ln\frac{\phi^2}{\mu^2}, \tag{2.18}$$

and setting $\frac{\partial V}{\partial \phi} = 0$ with $\mu = \langle\phi\rangle$ we found $\lambda + 12\kappa B = 0$. It is easy to show that

$$m_H^2 = \frac{\partial^2 V}{\partial \phi^2} = 8\kappa B\langle\phi\rangle^2 \approx 6\kappa e^4\langle\phi\rangle^2 = \frac{3\alpha}{2\pi}m_W^2, \tag{2.19}$$

where m_W is the gauge boson mass.

In the Standard Model, this fails since $B < 0$ due to the top quark Yukawa coupling contribution, as we now demonstrate.

2.6 The Standard Model Case

As in the previous examples we have a potential of the form

$$V = \phi^4\left[\frac{\lambda}{4!} + B\kappa\ln\frac{\phi^2}{\mu^2}\right], \tag{2.20}$$

where if we choose $\mu = \langle\phi\rangle$ then we get

$$\lambda + 12\kappa B = 0 \quad \text{and} \quad m_H^2 = 8\kappa B\langle\phi\rangle^2. \tag{2.21}$$

Now in the SM,

$$B = \frac{1}{12}\lambda^2 + \frac{3}{64}\left(3g^4 + 2g^2g'^2 + g'^4\right) - \frac{3}{2}h^4, \tag{2.22}$$

where h is the top quark Yukawa coupling.

Now $h \sim 1$, $g \sim 0.64$, $g' \sim 0.36$ so

$$B = \frac{1}{12}\lambda^2 - 1.4. \tag{2.23}$$

The extremum condition $\lambda + 12\kappa B = 0$ has solutions, $\lambda \sim 0.05, -158$. In the first, $B < 0$ and the extremum is a maximum. In the second, the extremum is a minimum and $m_H \sim 2.5$TeV, but manifestly this is not believable; in fact we are back in the same situation as the pure $\lambda\phi^4$ case. (Occasional claims to the contrary in the literature notwithstanding: they are incorrect).

In their original paper[1], Coleman and Weinberg (CW) had $h = 0$, whereupon $B = \frac{1}{12}\lambda^2 + 0.0295$. The extremum condition then had the sound solution $\lambda \sim -12\kappa \times 0.0295$, so $B > 0$, predicting $m_H \sim 9.5$GeV.

3. Quantum Gravity

The obvious (naïve) treatment of Einstein-Hilbert gravity is to expand the metric about a flat (Euclidean or Minkowski) background

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_P}, \tag{3.1}$$

so that

$$S = \int d^4x\sqrt{g}M_P^2 R \sim \int d^4x\left(M_P\Box h + h\Box h + \frac{1}{M_P}h^2\Box h + \cdots\right). \tag{3.2}$$

The number of terms increases very rapidly, and more so if we expand about a more general background, $\eta_{\mu\nu} \to g^B_{\mu\nu}$.

The theory is very non-renormalisable because there are h^n vertices for all n and each has two derivatives. But obviously from an effective field theory point of view we are in good shape as long as $\Box \ll M_P^2$

3.1 R^2 Gravity

In the Effective field theory spirit we might decide to write

$$S = \int d^4x\sqrt{g}\left(M_P^2 R + \alpha R^2_{\mu\nu\rho\sigma} + \beta R^2_{\mu\nu} + \gamma R^2 + O(R^3/M_P^2)\right), \tag{3.3}$$

where α, β, γ are dimensionless. Clearly for momenta $\ll M_P$, the R^2 terms are small corrections.

Or we might entertain the possibility that the theory

$$S = \int d^4x\sqrt{g}\left(M_P^2 R + \alpha R^2_{\mu\nu\rho\sigma} + \beta R^2_{\mu\nu} + \gamma R^2\right) + S_{\text{matter}}, \tag{3.4}$$

(or even a *scale invariant* theory without the R term) is UV complete.

Both the $R+R^2$ and the R^2 theories are renormalisable, but have unitarity "issues":

$$M_P^2 R + R^2 \sim h\Box h + h\Box^2 h/M_P^2 + \cdots \tag{3.5}$$

so the propagator is like

$$\frac{M_P^2}{M_P^2 k^2 + k^4} \sim \frac{1}{k^2} - \frac{1}{k^2 + M_P^2} \tag{3.6}$$

and we see a “wrong-sign” propagator, as in the Lee-Wick model[12].

3.2 The basic model

The theory we shall study is

$$S = \int d^4x\sqrt{g}\left(\frac{1}{2a}C^2_{\mu\nu\rho\sigma} + \frac{1}{3b}R^2 + cG\right) + S_{\text{matter}}, \tag{3.7}$$

where C is the Weyl tensor and G is the Gauss-Bonnet term which is a total derivative:

$$C^2_{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2, \tag{3.8}$$

$$G = \frac{1}{4}\varepsilon^{\kappa\lambda\alpha\beta}\varepsilon_{\mu\nu\gamma\delta}R^{\mu\nu}{}_{\alpha\beta}R^{\gamma\delta}{}_{\kappa\lambda} = R^2_{\kappa\lambda\mu\nu} - 4R^2_{\mu\nu} + R^2. \tag{3.9}$$

We introduce a single scalar field ϕ and

$$S_{\text{matter}} = \int d^4x\sqrt{g}\left[\frac{1}{2}(\nabla\phi)^2 + \frac{\lambda}{4}\phi^4 - \frac{\xi\phi^2}{2}R\right]. \tag{3.10}$$

S is invariant under the scale transformation $\phi(x) \to e^{\alpha}\phi(x), \quad g_{\mu\nu}(x) \to e^{-2\alpha}g_{\mu\nu}(x)$.

3.3 Remark on the A-theorem

In fact, the β-function β_c represents a generalisation to the quantised R^2-gravity case of the Euler anomaly coefficient, and thus a candidate for an A-function as proposed by Cardy[13], manifesting a 4-dimensional c-theorem. (For a recent discussion see Ref. [10].) Results for this anomaly coefficient (without quantising gravity) at 3-5 loops have been calculated [14], [15]. At one loop in our case we have

$$\beta_c = -\frac{196}{45} - \frac{1}{360}\left[N_0 + 11N_{1/2} + 62N_1\right] \tag{3.11}$$

where N_0 etc represent the number of scalar, fermion and vector fields.

An aside: it is easy to show that an A- function exists in $d=3$ renormalisable Chern-Simons theories [16].

4. Dimensional Transmutation

Our goal now is to show that the gravitational couplings can play the same role as the electromagnetic coupling in SQED, so that ϕ develops a vev. Then $\xi\langle\phi\rangle^2 R$ becomes the Einstein term.

We will also show that there is a region of parameter space such that all the couplings are asymptotically free, so that the model can be UV complete.

4.1 Equations of Motion

The equations of motion for ϕ and the trace of the one for the metric are

$$-\xi\phi R - \Box\phi + \lambda\phi^3 = 0, \tag{4.1}$$

$$-\xi\phi^2 R + (\nabla\phi)^2 + \lambda\phi^4 = \Box\left(\frac{4}{b}R - 3\xi\phi^2\right). \tag{4.2}$$

It is easy to see that these are compatible for ϕ constant and, defining $r = \phi^2/R$,

$$r = r_0 = \xi/\lambda. \tag{4.3}$$

We will assume a maximally symmetric space, such that

$$R_{\mu\nu\rho\sigma} = \frac{1}{12}R\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right). \tag{4.4}$$

The classical action then becomes, with $\int d^4x\sqrt{g} \equiv \frac{V_4}{R^2}$, V_4 being a dimensionless volume element:

$$S/V_4 = \frac{b}{3} + \frac{c}{6} + \frac{\lambda r^2}{4} - \frac{\xi r}{2} = \frac{b}{3} + \frac{c}{6} - \frac{\xi^2}{4\lambda}. \tag{4.5}$$

Of course

$$\left.\frac{\partial S}{\partial r}\right|_{r=r_0} = 0. \tag{4.6}$$

We shall refer to imposition of Eq. (4.6) as “going on-shell”.

4.2 Radiative corrections

We will assume that the metric remains maximally symmetric and consider radiative corrections to the action for constant (R,ϕ):

$$\Gamma(\lambda_i, r, \rho/\mu) = S(\lambda_i, r) + B(\lambda_i, r)\log(\rho/\mu) + \frac{C(\lambda_i, r)}{2}\log^2(\rho/\mu) + \ldots, \tag{4.7}$$

where $\rho = \sqrt{R}$ and λ_i stand for the dimensionless couplings. We seek the extremum of Γ with respect to r,ρ choosing $\mu = \langle\rho\rangle = v$ at the extremum. Then

$$\left.\frac{\partial}{\partial r}\Gamma(\lambda_i, r, \rho/\mu)\right|_{r_0,v} = \left.\frac{\partial}{\partial r}S(\lambda_i, r)\right|_{r_0,v} = \left.S'\right|_{r_0,v} = 0, \tag{4.8a}$$

$$\left.\rho\frac{\partial}{\partial\rho}\Gamma(\lambda_i, r, \rho/\mu)\right|_{r_0,v} = \left.B(\lambda_i, r)\right|_{r_0,v} = 0. \tag{4.8b}$$

These results are exact to all orders in the loop expansion! Thus the conditions for an extremum involves a relation among the dimensionless couplings, valid at a *specific* renormalisation scale $\mu = v$. The relation Eq. (4.8b) is analagous to Eq. (2.11) in the SQED case, except that there is no *tree* contribution in Eq. (4.8b). It must be emphasised (because of misleading claims to the contrary in the literature) that this sort of relation only holds at a SPECIFIC rnormalisation scale, and the consequences of assuming that it is RG invariant are entirely spurious.

4.3 Stability

The matrix of second derivatives of the action is determined by

$$\frac{\partial^2}{\partial r^2}\Gamma(\lambda_i,r,\rho/\mu)\Big|_{r_0,v} = \frac{\partial^2}{\partial r^2}S(\lambda_i,r)\Big|_{r_0}, \tag{4.9a}$$

$$\rho\frac{\partial^2}{\partial r\partial\rho}\Gamma(\lambda_i,r,\rho/\mu)\Big|_{r_0,v} = \frac{\partial}{\partial r}B(\lambda_i,r)\Big|_{r_0}, \tag{4.9b}$$

$$\rho^2\frac{\partial^2}{\partial\rho^2}\Gamma(\lambda_i,r,\rho/\mu)\Big|_{r_0,v} = C(\lambda_i,r_0). \tag{4.9c}$$

This matrix has two eigenvalues ϖ_i that may be approximated as

$$\varpi_1(r_0,v) = \frac{S''}{2} + O(\hbar^2), \qquad \varpi_2(r_0,v) = \frac{1}{2}\left[C_2 - \frac{(B_1')^2}{S''}\right] + O(\hbar^3), \tag{4.10}$$

where B_1, C_2 are the leading contributions to B, C respectively. So $\varpi_1 = \lambda(v)/2$, and ϖ_2, although of order $\hbar^2$, can in fact be determined by one-loop results, using the Renormalisation Group.

4.4 The Renormalisation Group

Both B_1 and C_2 can be found via the Renormalisation Group in terms of the one loop β-functions. In a "Landau"-type gauge we have

$$\left[\gamma_\rho\frac{\partial}{\partial\rho} - \mu\frac{\partial}{\partial\mu}\right]\Gamma(\lambda_i,r,\rho/\mu) = (1+\gamma_\rho)\rho\frac{\partial\Gamma}{\partial\rho} = \left[\beta_{\lambda_i}\frac{\partial}{\partial\lambda_i} - \gamma_r r\frac{\partial}{\partial r}\right]\Gamma(\lambda_i,r,\rho/\mu). \tag{4.11}$$

On shell we have

$$\beta_{\lambda_i}\frac{\partial}{\partial\lambda_i}\Gamma(\lambda_i,r,\rho/\mu)\Big|_{r_0,v} = 0, \tag{4.12}$$

But to determine B_1 and C_2 we need to apply the RG off-shell, giving

$$B_1(\lambda_i,r) = \beta^{(1)}_{\lambda_i}\frac{\partial}{\partial\lambda_i}[S(\lambda_i,r)] - \gamma^{(1)}_r rS'(\lambda_i,r), \tag{4.13a}$$

$$B_1'(\lambda_i,r) = \beta^{(1)}_{\lambda_i}\frac{\partial}{\partial\lambda_i}S'(\lambda_i,r) - \gamma^{(1)}_r\frac{\partial}{\partial r}\big(rS'(\lambda_i,r)\big), \tag{4.13b}$$

$$C_2(\lambda_i,r) = \left[\beta^{(1)}_{\lambda_i}\frac{\partial}{\partial\lambda_i} - \gamma^{(1)}_r r\frac{\partial}{\partial r}\right]B_1(\lambda_i,r). \tag{4.13c}$$

4.5 The RG solution

On-shell (that is, imposing Eq. (4.6)), it turns out that (unlike B_1' and C_2 individually) both B_1 and ϖ_2 are independent of γ_r[1]:

$$B_1(\lambda_i,r_0) = \beta^{(1)}_{\lambda_i}\frac{\partial}{\partial\lambda_i}\big[S(\lambda_i,r)\big]\big|_{r_0,v}, \tag{4.14}$$

$$\varpi_2 = \frac{1}{2}\left[\left(\beta^{(1)}_{\lambda_i}\frac{\partial}{\partial\lambda_i}\right)^2\big[S(\lambda_i,r)\big] - \frac{1}{S''}\left(\beta^{(1)}_{\lambda_i}\frac{\partial}{\partial\lambda_i}S'(\lambda_i,r)\right)^2\right]\Big|_{r_0,v} \tag{4.15}$$

recall that

$$\varpi_1 = \lambda(v)/2 \tag{4.16}$$

[1]This was inevitable since on-shell they represent physical quantities, while γ_r is gauge dependent

5. The β-functions and the Fixed Points

It is useful to introduce variables $x = b/a$, $y = \lambda/a$ and $du = \kappa a dt = \kappa a d(\ln\mu)$, where $1/\kappa \equiv 16\pi^2$. Then at one-loop, as well as

$$\frac{da}{dt} = -\frac{799}{60}\kappa a^2, \quad \frac{dc}{dt} = -\frac{523}{120}\kappa; \tag{5.1}$$

we have [8]:

$$\frac{dx}{du} \equiv \overline{\beta}_x = -\frac{10}{3}\left[1 - \frac{1099}{200}x + \frac{1}{8}x^2 + \frac{1}{80}(1+6\xi)^2 x^2\right]; \tag{5.2a}$$

$$\frac{d\xi}{du} \equiv \overline{\beta}_\xi = (6\xi+1)y + \frac{\xi}{6}\left(\frac{20}{x} - x(6\xi+1)(3\xi+2)\right); \tag{5.2b}$$

$$\frac{dy}{du} \equiv \overline{\beta}_y = 18y^2 + y\left(\frac{1099}{60} - \frac{1}{2}x(1+6\xi)^2\right) + \frac{\xi^2}{8}(20 + (6\xi+1)^2 x^2). \tag{5.2c}$$

The system of β-functions has some fixed points, shown in Table 1.

	x	ξ	y	Nature
1.	**39.78082**	**0**	**0**	**UV stable**
2.	0.18282	0.	0.	IR stable
3.	0.18292	.083150	-1.005218	saddle point
4.	36.9666	.058999	.787391	saddle point
5.	43.7762	$-.16404$	-1.01350	saddle point
6.	43.7770	$-.165507$	$-.00377560$	saddle point

Table 1: Fixed Points

Remarkably, one of the fixed points with $y = \xi = 0$ is UV stable (it is easy to see that a FP with $y = 0$ must have $\xi = 0$). Since a is AF, this FP corresponds to AF for all the couplings (a, b, c, ξ, λ). With regard to the IR stable FP, note that in approaching it from any starting values of the couplings, one would eventually lose perturbative believability since in the IR the coupling "a" approaches a Landau pole.

5.1 The solution for $B_1 = 0$

Recall we showed that a CW extremum of the potential corresponds to $B = 0$. We find for $B_1^{(os)}$ (that is, B_1 on-shell) the following expression:

$$\begin{aligned} B_1^{(os)} \quad &= \tfrac{1}{240x^2y^2}\big(1620x^4\xi^6 + 540x^4\xi^5 + 45x^4\xi^4 - 4320x^3\xi^4 y \\ &-360x^3\xi^3 y + 60x^3\xi^2 y + 900x^2\xi^4 + 2880x^2\xi^2 y^2 + 1800x^2\xi^2 y \\ &-480x^2\xi y^2 - 826x^2y^2 - 2400x\xi^2 y - 2400xy^2 + 1600y^2\big), \end{aligned} \tag{5.3}$$

which can (remarkably) be rewritten as follows:

$$B_1^{(os)} = 12(X + \frac{1}{4})^2 + \frac{20}{3}(Y + \frac{3}{4})^2 - \frac{291}{40}, \tag{5.4}$$

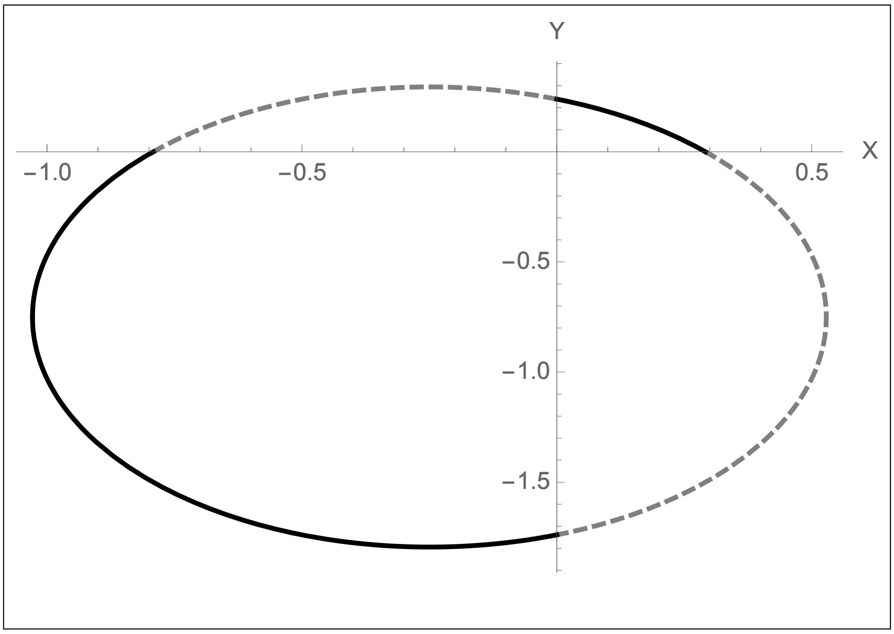


Figure 1: The $B_1 = 0$ ellipse.

in terms of new variables $X = \xi' z'$ and $Y = z'/x$, where $z' = z - 1$, $\xi' = \xi + 1/6$, and $z \equiv 3x\xi^2/(4y)$. Thus points with $B_1 = 0$ lie on an ellipse. The space of solutions of $B_1 = 0$ corresponding to a stable minimum is quite large.

In Fig. 1 we show the ellipse defined by Eq. (5.4). The two quadrants $Y, X > 0$ and $Y, X < 0$ are distinguished because they correspond to $X/Y > 0$ which is necessary to obtain $\xi > 0$ at the DT extremum; necessary in turn to obtain the correct sign for the induced Einstein-Hilbert term. In Fig. 2(a,b) we show the ranges of (x, X, Y) with $\xi > 0$ and then with $\varpi_2 > 0$ as well. Finally in Fig. 3(a,b) we give examples of how the couplings run, starting from near the UVFP and then running *down* in the hope of entering the region where DT occurs. In fact, in Fig. 3(a) the couplings run towards the IRFP, whereas in Fig. 3(b) y becomes negative and approaches a singularity.

The bad news is that although there is a substantial region of parameter space corresponding to a local minimum, the basin of attraction of the only UV stable fixed point does not include this region in which DT minima occur.

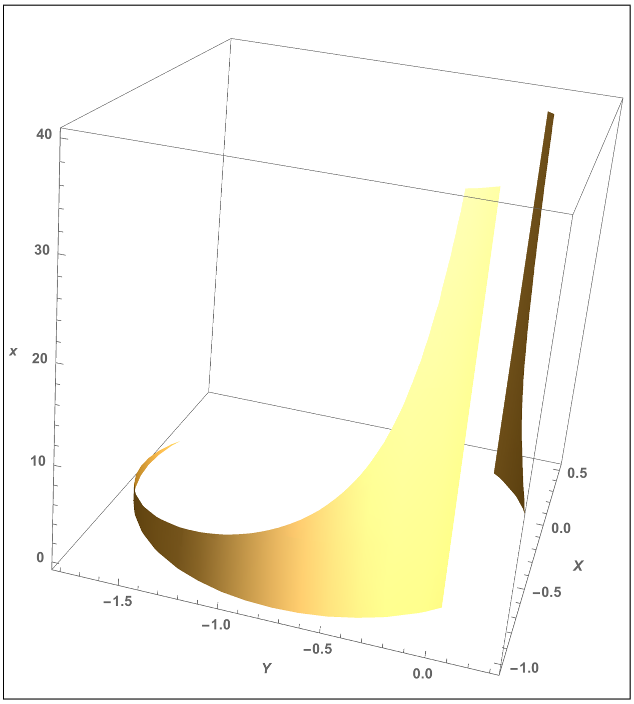
6. Towards a realistic theory

Consider $SO(10)$ with an adjoint scalar representation and a set of n_f 10-dimensional two-component (or Majorana) fermion representations. The scalar potential is

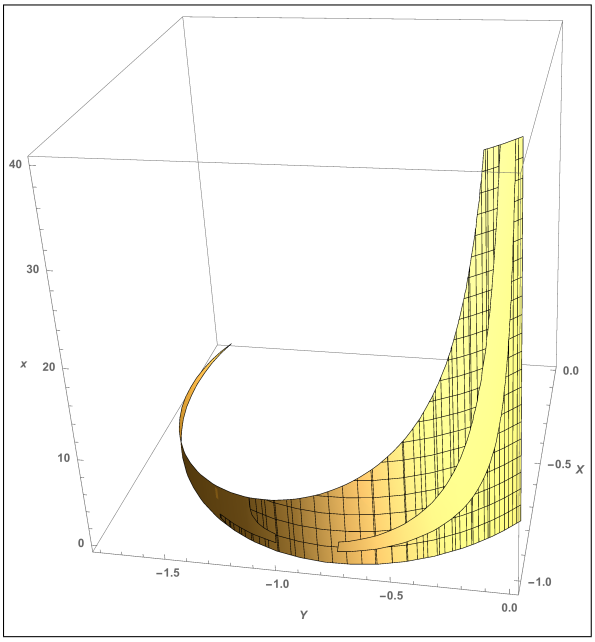
$$V = \frac{\lambda_1}{24}(\phi^a\phi^a)^2 + \frac{\lambda_2}{24}\mathrm{Tr}\Phi^4, \quad \Phi = \phi^a R^a \tag{6.1}$$

where we choose R^a to be the matrix representation of the group generators in the fundamental representation, with

$$\mathrm{Tr}\left[R^a R^b\right] = \frac{1}{2}. \tag{6.2}$$

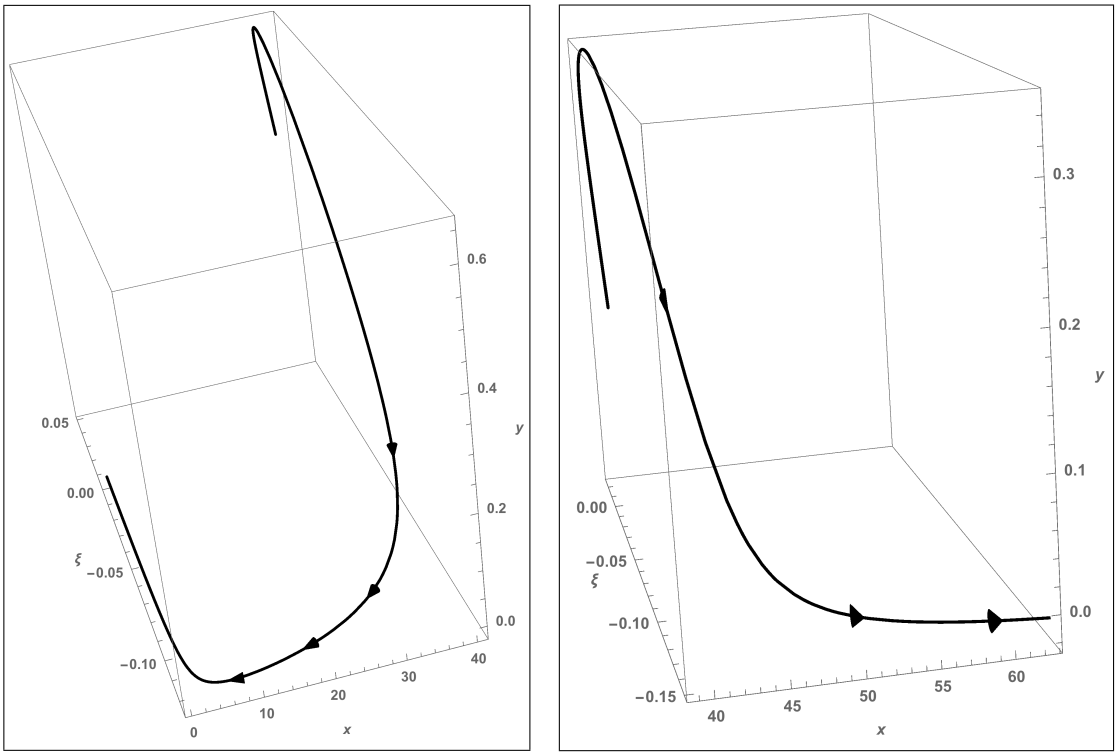


(a) $\xi > 0$.



(b) $\xi, \varpi > 0$.

Figure 2: $B_1 = 0$ with constraints.



(a) $x = 38.03, \xi = 0.0362, y = 0.458$ (b) $x = 39.27, \xi = 0.0106, y = 0.126$

Figure 3: Running couplings down from near the UVFP.

In the gauge theory case it is better to rescale the couplings with g^2 rather than a so we define $x_i = \lambda_i/g^2, \bar{a} = a/g^2$ and $du = g^2 dt$. Then the $\overline{\beta}$-functions are

$$\overline{\beta}_{x_1} = \frac{53}{3}x_1^2 + \frac{19}{12}x_1x_2 + \frac{1}{32}x_2^2 + (b_g - 48)x_1 + 27 + 3\Delta\beta_1 + x_1\Delta\beta_2 \tag{6.3a}$$

$$\overline{\beta}_{x_2} = 4x_1x_2 + \frac{19}{24}x_2^2 + (b_g - 48)x_2 + 72 + x_2\Delta\beta_2 \tag{6.3b}$$

$$\overline{\beta}_{\xi} = (\xi + \frac{1}{6})(\frac{47}{3}x_1 + \frac{19}{24}x_2 - 24) + \Delta\beta_\xi \tag{6.3c}$$

where the gravitational contributions are:

$$\overline{\beta}_a = \bar{a}(b_g - b_2\bar{a}) \tag{6.4a}$$

$$\overline{\beta}_w = \bar{a}(\frac{10}{3}w^2 - \frac{183}{10}w + \frac{5}{12} - \frac{1}{60}w(585 + 30n_f) + \frac{15}{8}(6\xi + 1)^2) \tag{6.4b}$$

$$\Delta\beta_1 = \xi^2(5 + \frac{1}{4w^2}(6\xi + 1)^2)\bar{a}^2 \tag{6.4c}$$

$$\Delta\beta_2 = (5 - (1 + 12\xi + 36\xi^2)/2/w)\bar{a}; \tag{6.4d}$$

$$\Delta\beta_\xi = \xi(\frac{10}{3}w - (9\xi^2 + 15\xi/2 + 1)/3/w)\bar{a}. \tag{6.4e}$$

Here $b_g = 28 - \frac{2}{3}n_f$ is (minus) the one loop gauge β-function coefficient and $b_2 = \frac{461}{20} + \frac{1}{2}n_f$ is (minus) the β_a-function coefficient. Obviously in the UV the coupling $\bar{a}$ approaches the fixed point $\bar{a} = b_g/b_2$. We have included n_f sets of 10-dimensional representations of two-component fermions, which we assume have no Yukawa couplings to Φ.

6.1 Fixed Points

In this model we find, (with $n_f = 41$ so as to produce a small value of b_g, $b_g = 2/3$) four fixed points. (Here we define $x = 1/w$). Fixed point No. 1 is UV attractive; the other three are saddle

	x	ξ'	x_1	x_2
1.	**116.45**	$\mathbf{-.11955 \times 10^{-4}}$	**1.02645**	**1.723027**
2.	116.45	$.817525 \times 10^{-4}$	1.4694	1.80214
3.	$.68769 \times 10^{-1}$	.103897	1.469455	1.8021408
4.	$.68701 \times 10^{-1}$;	$-.21299667 \times 10^{-1}$	1.0264541	1.7230271

Table 2: $SO(N)$ Fixed Points

points. So our first criterion for a successful model is fulfilled.

The next step: we have indeed shown that, as in the basic model, there is a region of parameter space such that Φ acquires a vev via DT. In this case, the vev breaks the $SO(10)$ symmetry so as to leave unbroken the maximal subgroup $SU(5) \otimes U(1)$. However, we have not yet established whether for this (or any other) model there is DT in the catchment basin of a UV Fixed Point.

7. Summary and Outlook

- Dimensionless transmutation can give a non zero $\langle\phi\rangle$ in a theory with scalar fields coupled to R^2 gravity, and hence generate an Einstein term in the "low energy" theory.

- In the simplest model the basin of attraction of the only UV stable fixed point does not include the region in which DT minima occur, so in this region the theory becomes strongly coupled or must be modified at high scales.

- More complicated models might remedy this, and also the nonzero $\langle\phi\rangle$ might break a Grand Unified symmetry. $\Lambda_{\text{DT}} \sim \sqrt{\lambda} M_P/\xi$, so we would require $\sqrt{\lambda}/\xi \sim 10^{-3}$ which is reasonable. The scenario may be compatible with a form of Higgs inflation.

- Problems: Unitarity, generating the electroweak scale ......

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