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Aspects of the Bosonic Spectral Action: Successes and Challenges

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A short introduction on elements of noncommutative geometry, which offers a purely geometric interpretation of the Standard Model and implies a higher derivative gravitational theory, is presented. Physical consequences of almost commutative manifolds are briefly discussed and cosmological consequences of the gravitational sector, which is shown not to be plagued by linear instability, are highlighted. Successes and challenges are discussed. A novel spectral action proposal based on zeta function regularisation is briefly presented.

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1. Introduction

Classical cosmology, tested by a variety of precise astrophysical measurements, is built upon Einstein's theory of General Relativity and the Cosmological Principle. General Relativity is however a classical theory, hence its validity breaks down at very high energy scales. The Cosmological Principle, namely the assumption of a continuous space-time characterised by homogeneity and isotropy on large scales, is valid only once we consider late-eras of our universe, characterised by energies far below the Planck scale. At very early times, very close to the Big Bang and the so-called Planck era, quantum corrections can no longer be neglected and geometry may altogether lose the meaning we are familiar with. To describe the physics near the Big Bang, a Quantum Gravity theory and the associated appropriate space-time geometry is hence required.

The available Quantum Gravity proposals can be divided into two classes. On the one hand, there is String Theory/M-theory, according which matter consists of one-dimensional objects, strings, which can be either closed or open (without ends). Different string vibrations would represent different particles; splitting and joining of strings would then correspond to different particle interactions. On the other hand, there are non-perturbative approaches to quantum gravity; some examples are Loop Quantum Gravity, a Euclidean approach to quantum gravity like Causal Dynamical Triangulations, and Group Field Theory. The latter class of models adopts the hypothesis that space is not infinitely divisible, instead it has a granular structure, hence it is made out of quanta of space. In the former class of models, matter is the important ingredient; in the latter one, matter is, so far, (rather artificially) added. These two classes of models can be considered as following a *top-down approach*, whilst they both inspire cosmological models leading to several observational consequences.

In the following, I will consider a *bottom-up approach*, in the sense that I will focus on a proposal attempting to guess the small-scale structure of space-time near the Planck era, using our knowledge of well-tested particle physics at the electroweak scale. More precisely, I will focus on Noncommutative Spectral Geometry (NCSG). One may argue that at the Planck energy scale, quantum gravity implies that space-time is a wildly noncommutative manifold. However, at an intermediate scale, one may assume that the algebra of coordinates is only a mildly noncommutative algebra of matrix valued functions, which if appropriately chosen, may lead to the Standard Model of particles physics coupled to gravity. It is important to note that according to the NCSG proposal, to construct a quantum theory of gravity coupled to matter, the gravity-matter interaction is the most important ingredient to determine the dynamics; this consideration is not the case for either of the two classes of proposals mentioned previously.

I will first briefly introduce elements of NCSG and then discuss some of its phenomenological consequences, focusing on its successes and some open questions [1, 2, 3].

2. NCSG in a nutshell

Noncommutative spectral geometry [4, 5, 6] postulates that the Standard Model (SM) of particle physics is a phenomenological model which dictates the space-time geometry in order to get the SM action. In its simplest approach, one considers that at each point of a four-dimensional Riemannian manifold there is an internal zero-dimensionality discrete space. Such a Kaluza-Klein type mildly noncommutative manifold, given by the product $\mathcal{M} \times \mathcal{F}$ of a compact four-dimensional smooth Riemannian spin manifold $\mathcal{M}$ and a discrete noncommutative zero-dimensionality space $\mathcal{F}$, is called an *almost commutative* manifold.

The main idea we will follow is to characterise Riemannian manifolds by *spectral data*, and then apply the same procedure in the case of almost commutative manifolds. Hence, let us first consider a compact four-dimensional Riemannian spin manifold $\mathcal{M}$. The set $C^\infty(\mathcal{M})$ of smooth infinitely differentiable functions forms an algebra $\mathcal{A} = C^\infty(\mathcal{M})$ under point-like multiplication. Then consider the Hilbert space $\mathcal{H} = L^2(\mathcal{M}, S)$ of square-integrable spinors S on the spin manifold $\mathcal{M}$. Note that the algebra $\mathcal{A} = C^\infty(\mathcal{M})$ acts on the Hilbert space $\mathcal{H} = L^2(\mathcal{M}, S)$ as multiplication operators. Finally, consider the Dirac operator $\partial\!\!\!/ = -i\gamma^\mu \nabla^S_\mu$, acting as a first order differential operator on the spinors S. The canonical triple $(C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), \partial\!\!\!/)$ encodes the space-time structure. In addition, we introduce the γ_5 operator, which is just a $\mathbb{Z}_2$-grading, with $\gamma_5^2 = 1, \gamma_5^\star = \gamma_5$. It plays the rôle of a chirality operator, in the sense that it decomposes $\mathcal{H}$ into a positive and negative eigen-space: $L^2(\mathcal{M}, S) = L^2(\mathcal{M}, S)^+ \otimes L^2(\mathcal{M}, S)^-$. Let us also introduce an antilinear isomorphism $J_\mathcal{M}$, playing the rôle of a charge conjugation operator on spinors, with $J^2_\mathcal{M} = -1$, $J_\mathcal{M} \partial\!\!\!/ = \partial\!\!\!/ J_\mathcal{M}$, $J_\mathcal{M}\gamma_5 = \gamma_5 J_\mathcal{M}$.

In a similar way, the noncommutative space $\mathcal{F}$, which encodes the internal degrees of freedom at each point in space-time $\mathcal{M}$, can be described by the real spectral triple $(\mathcal{A}_\mathcal{F},\ \mathcal{H}_\mathcal{F},\ \mathcal{D}_\mathcal{F})$. It lead to a gauge theory on the spin manifold $\mathcal{M}$. Here $(\mathcal{A}_\mathcal{F}$ is an involution of operators on the finite-dimensional Hilbert space $\mathcal{H}_\mathcal{F}$ of Euclidean fermions. The matrix algebra $\mathcal{A}_\mathcal{F}$ contains all information usually carried by the metric. The axioms of the spectral triples imply that the Dirac operator of the internal space is the fermionic mass matrix. Hence, $\mathcal{D}_\mathcal{F}$ is a 96×96 matrix in terms of the 3×3 Yukawa mixing matrices and a real constant necessary to obtain neutrino mass terms. Consider also a grading $\gamma_\mathcal{F}$, with $\gamma_\mathcal{F} = +1$ for left-handed and $\gamma_\mathcal{F} = -1$ for right-handed fermions, and a conjugation operator $J_\mathcal{F}$:

$$J_\mathcal{F} = \begin{pmatrix} & 1_{48} \\ 1_{48} & \end{pmatrix} .$$

The almost commutative manifold $\mathcal{M} \times \mathcal{F}$ is thus given by the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, with

$$\begin{aligned} \mathcal{A} &= C^\infty(\mathcal{M}) \otimes \mathcal{A}_\mathcal{F} = C^\infty(\mathcal{M}, \mathcal{A}_\mathcal{F}) , \\ \mathcal{H} &= \mathcal{L}^2(\mathcal{M}, S) \otimes \mathcal{H}_\mathcal{F} = \mathcal{L}^2(\mathcal{M}, S \otimes \mathcal{H}_\mathcal{F}) , \\ \mathcal{D} &= \partial\!\!\!/ \otimes \mathbb{I} + \gamma_5 \otimes \mathcal{D}_\mathcal{F} . \end{aligned}$$

The finite dimensional algebra $\mathcal{A}_\mathcal{F}$, which is the main input, must be in agreement with noncommutative geometry properties, while it must be chosen such that it can lead to the SM. The appropriate choice is [7]

$$\mathcal{A}_\mathcal{F} = M_a(\mathbb{H}) \oplus M_k(\mathbb{C}) ,$$

with $M_a(\mathbb{H})$ the algebra of quaternions and $M_k(\mathbb{C})$ the algebra of complex $k \times k$ matrices with $k = 2a$. The lowest allowed value of k in order to obtain 16 fermions in each of the three generations, where the number of generations is an (external) physical input, is $k = 4$.

It is important to note that the choice of an almost commutative manifold has deep physical implications. The spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, J, \gamma)$ defining the almost commutative manifold $\mathcal{M} \times$

$\mathscr{F}$ can be written as

$$(\mathscr{A},\mathscr{H},\mathscr{D},J,\gamma) = (\mathscr{A}_1,\mathscr{H}_1,\mathscr{D}_1,J_1,\gamma_1)\otimes(\mathscr{A}_2,\mathscr{H}_2,\mathscr{D}_2,J_2,\gamma_2)\ , \tag{2.1}$$

with

$$\mathscr{A}=\mathscr{A}_1\otimes\mathscr{A}_2\ ,\ \mathscr{H}=\mathscr{H}_1\otimes\mathscr{H}_2\ ,\ \mathscr{D}=\mathscr{D}_1\otimes 1+\gamma_1\otimes\mathscr{D}_2\ ,\gamma=\gamma_1\otimes\gamma_2\ ,\ J=J_1\otimes J_2\ , \tag{2.2}$$

where $J^2=-1,[J,\mathscr{D}]=0,[J_1,\gamma_1]=0$ and $\{J,\gamma\}=0$.
The algebra doubling is strongly related to dissipation, to the the gauge structure of the SM, whilst it offers a way to generate the seeds of quantisation [8]. Moreover, the doubling of the algebra offers a natural explanation for neutrino mixing, since by linking the algebra doubling to the deformed Hopf algebra, one can build Bogogliubov transformations and argue the emergence of neutrino mixing [9].

In order to extract physical consequences of the NCSG construction, one needs to obtain a Lagrangian. To do so we will apply the spectral action principle, according which the bosonic part of the action is of the form

$$\mathrm{Tr}(f(\mathscr{D}_A^2/\Lambda^2))\ ,$$

where $\mathscr{D}_A$ is the fluctuated Dirac operator, f is a cutoff function (a positive function that goes to zero for large values of its argument) and Λ a cutoff scale, denoting the energy scale at which the Lagrangian is valid. Hence, the bosonic part of the action sums up eigen-values of the fluctuated Dirac operator, smaller than the cutoff energy scale. Since this action depends on the cutoff energy scale and (mildly) on the cutoff function, we will call it *the cutoff bosonic spectral action*. One then evaluates the trace with heat kernel techniques, and writes the bosonic cutoff spectral action in terms of Seeley-de Witt coefficients. The asymptotic expansion of the trace thus reads

$$\mathrm{Tr}(f(\mathscr{D}_A^2/\Lambda^2))\sim 2f_4\Lambda^4 a_0(\mathscr{D}_A^2)+2f_2\Lambda^2 a_2(\mathscr{D}_A^2)+f(0)a_4(\mathscr{D}_A^2)+\mathscr{O}(\Lambda^{-2})\ , \tag{2.3}$$

in terms of only three of the momenta of the cutoff function f, given by

$$f_4=\int_0^\infty f(u)u^3du\ ,\ f_2=\int_0^\infty f(u)udu\ ,f_0=f(0)\ . \tag{2.4}$$

related respectively to the cosmological constant, the gravitational constant and the coupling constants at unification. Performing a straightforward but long calculation, one finally writes the cutoff bosonic spectral action, modulo gravitational terms, as

$$\begin{aligned}S_\Lambda &= \frac{-2af_2\Lambda^2+ef_0}{\pi^2}\int|\phi|^2\sqrt{g}d^4x+\frac{f_0}{2\pi^2}\int a|D_\mu\phi|^2\sqrt{g}d^4x-\frac{f_0}{12\pi^2}\int aR|\phi|^2\sqrt{g}d^4x\\ &-\frac{f_0}{2\pi^2}\int\left(g_3^2G^i_\mu G^{\mu i}+g_2^2F^a_\mu F^{\mu\nu a}+\frac{5}{3}g_1^2B_\mu B^\mu\right)\sqrt{g}d^4x+\frac{f_0}{2\pi^2}\int b|\phi|^4\sqrt{g}d^4x+\mathscr{O}(\Lambda^{-2})\ ,\end{aligned} \tag{2.5}$$

with a,b,c,d,e constants depending on the Yukawa parameters. Adding to the above bosonic action S_Λ, the fermionic part

$$(1/2)\langle J\Psi,\mathscr{D}_A\Psi\rangle\ ;\ \Psi\in\mathscr{H}^+\ , \tag{2.6}$$

one obtains the full SM Lagrangian. The cutoff scale is set at the Grand Unified Theories (GUT) scale, since among the relations between the coefficients in the spectral action one obtains the relation $g_2^2 = g_3^2 = (5/3)g_1^2$ for the three couplings, valid in the context of several GUTs groups. Following a renormalisation group analysis [10] one then obtains predictions for the SM, which turn out to be in agreement with the most current experimental data. In particular, one obtains a Higgs doublet with a negative mass term and a positive quartic term, which implies that the electroweak symmetry is spontaneously broken. Note, that current developments [11, 12, 13, 14] of the noncommutative spectral geometry proposal are in agreement with the experimentally found Higgs mass.

3. The gravitational sector

Noncommutative spectral geometry leads to an *extended* gravitational theory, in the sense that the gravitational sector includes additional terms beyond the ones of the Einstein-Hilbert action. The gravitational part of the cutoff bosonic spectral action, in Euclidean signature, reads [10]

$$\begin{aligned} S_{\rm gr}^{\rm E} = \int \Big(&\frac{1}{2\kappa_0^2}R + \alpha_0 C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \gamma_0 + \tau_0 R^\star R^\star + \frac{1}{4}G_{\mu\nu}^i G^{\mu\nu i} + \frac{1}{4}F_{\mu\nu}^\alpha F^{\mu\nu\alpha} \\ &+ \frac{1}{4}B^{\mu\nu}B_{\mu\nu} + \frac{1}{2}|D_\mu \mathbf{H}|^2 - \mu_0^2|\mathbf{H}|^2 - \xi_0 R|\mathbf{H}|^2 + \lambda_0|\mathbf{H}|^4 \Big) \sqrt{g}\, d^4x \,, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \kappa_0^2 &= \frac{12\pi^2}{96 f_2\Lambda^2 - f_0\mathfrak{c}} \;\; , \;\; \alpha_0 = -\frac{3f_0}{10\pi^2} \,, \\ \gamma_0 &= \frac{1}{\pi^2}\left(48 f_4\Lambda^4 - f_2\Lambda^2\mathfrak{c} + \frac{f_0}{4}\mathfrak{d}\right) \;\; , \;\; \tau_0 = \frac{11 f_0}{60\pi^2} \,, \\ \mu_0^2 &= 2\Lambda^2\frac{f_2}{f_0} - \frac{\mathfrak{e}}{\mathfrak{a}} \;\; , \;\; \xi_0 = \frac{1}{12} \,, \\ \lambda_0 &= \frac{\pi^2\mathfrak{b}}{2f_0\mathfrak{a}^2} \;\; , \;\; \mathbf{H} = (\sqrt{a f_0}/\pi)\phi \,, \end{aligned} \tag{3.2}$$

with $\mathbf{H}$ a rescaling of the Higgs field ϕ to normalize the kinetic energy, and $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ parameters related to the particle physics model. Let me remark that this action has to be seen *à la Wilson*. We have hence obtained the Einstein-Hilbert action with a cosmological term, a conformal Weyl term and a conformal coupling of the Higgs field to the background geometry. Note that the fourth term is a topological term related to the Euler characteristic of the space-time manifold; thus a nondynamical term. In addition to the gravitational sector written above, one obtains also the SM action.

To use the gravitational part of the bosonic spectral action, one must assume that it is also valid for a Lorentzian manifold. The next question one has to address is whether the gravitational sector of the bosonic spectral action, leading to a fourth-order gravitational theory, is not plagued by linear instability. Considering the spectral action within a four-dimensional manifold with torsion, one can show that in the vacuum case, the equations of motion reduce to the second order Einstein's equations, implying linear stability [15]. To address the nonvacuum case, one can then consider the spectral action of an almost commutative geometry and show that the Hamiltonian is bounded from below, securing also in this case, which is of interest to us here, the linear stability of the theory [15].

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The equations of motion that one obtains from gravitational part of the asymptotic expression of the cutoff bosonic spectral action are [16]

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \frac{1}{\beta^2}\delta_{\rm cc}\left[2C^{\mu\lambda\nu\kappa}_{;\lambda;\kappa} + C^{\mu\lambda\nu\kappa}R_{\lambda\kappa}\right] = \kappa_0^2\delta_{\rm cc}T^{\mu\nu}_{\rm matter}\ , \tag{3.3}$$

with

$$\beta^2 \equiv -\frac{1}{4\kappa_0^2\alpha_0} \quad \text{and} \quad \delta_{\rm cc} \equiv [1 - 2\kappa_0^2\xi_0\mathbf{H}^2]^{-1}\ , \tag{3.4}$$

where β^2 (or equivalently, α_0, or f_0) is related to the coupling constants at unification, and $\delta_{\rm cc}$ encodes the conformal coupling between the Higgs field and the Ricci scalar. Neglecting the non-minimal coupling (namely, setting $\delta_{\rm cc} = 1$) one can show that since the Weyl tensor vanishes for a Friedmann-Lemaître-Robertson-Walker space-time, then in such a case there are no corrections to Einstein's equations [16]. Any modifications at leading order will only arise for anisotropic and inhomogeneous geometries [16].

As energies however increase, one may no longer set $\delta_{\rm cc} = 1$, and the corresponding background equations are [16]

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \kappa_0^2\left[\frac{1}{1-\kappa_0^2|\mathbf{H}|^2/6}\right]T^{\mu\nu}_{\rm matter}\ , \tag{3.5}$$

where we have set $\beta = 0$, just for simplicity. Thus one observes that the nonminimal coupling between the Higgs field and the Ricci scalar, leads to an effective gravitational constant, or equivalently, or equivalently, to an enhancement of the self-interaction of the Higgs field.

Given that the model has no freedom to introduce extra scalar fields, one may wonder whether the Higgs field, through its nonminimal coupling to the background geometry, could play the rôle of the inflaton[17, 18]. To address this question one looks for a flat region of the Higgs potential. Considering the renormalisation of the Higgs self-coupling up to two-loops, one finds that for each value of the top quark mass, there is a value of the Higgs mass where the Higgs potential is locally flattened [18]. However, the flat region is very narrow and to achieve a sufficiently long inflationary era, the slow-roll must be very slow, leading to an amplitude of density perturbations incompatible with Cosmic Microwave Background data (CMB) [18].

Finally, one can study the effects of NCSG in a perturbed background. Let us consider linear perturbations $g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}$ around a Minkowski background $\eta_{\mu\nu}$. The linearised equation of motion then reads [19]

$$\left(1 - \frac{1}{\beta^2}\Box_\eta\right)\Box_\eta\bar{h}^{\mu\nu} = -2\kappa^2T^{\mu\nu}_{\rm matter}\,, \tag{3.6}$$

where $\kappa^2 \equiv 8\pi G$ and $\beta^2 = 5\pi^2/(6\kappa^2 f_0)$. Note that $T^{\mu\nu}_{\rm matter}$ is taken to lowest order in $\gamma^{\mu\nu}$. To write the above equation, we have defined the tensor

$$\bar{h}_{\mu\nu} = \bar{\gamma}_{\mu\nu} - \frac{1}{3\beta^2}Q^{-1}\left(\eta_{\mu\nu}\Box_\eta - \partial_\mu\partial_\nu\right)\gamma, \tag{3.7}$$

with

$$Q \equiv 1 - \frac{1}{\beta^2}\Box_\eta \quad \text{and} \quad \bar{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\gamma\,. \tag{3.8}$$

One can then impose constraints on β from astrophysical data. More precisely, one may restrict β (and therefore f_0) by requiring that the energy lost to gravitational radiation by binary pulsar systems agrees with the (standard) General Relativity prediction within observational uncertainties. Such consideration implied a (rather weak) limit, namely $\beta \gtrsim 7.55 \times 10^{-13}\mathrm{m}^{-1}$ [20], which however can be improved if in the future data from rapidly orbiting nearby binaries become available.

Considering data from the Gravity Probe B and the LARES experiments, the limit on β has been improved, namely $\beta \gtrsim 7.1 \times 10^{-5}\mathrm{m}^{-1}$ and $\beta \gtrsim 1.2 \times 10^{-6}\mathrm{m}^{-1}$, respectively [21, 22].

The tighter constraint on the parameter β can be set using torsion balance experiments. It turns out that the modifications to the Newton potentials induced by the spectral action are similar to those due to a fifth-force potential. One thus finds $\beta \gtrsim 10^4\mathrm{m}^{-1}$, which is by far the strongest limit [21].

4. The zeta function regularisation

The cutoff bosonic spectral action, defined and explored previously, has certainly several merits. It leads to a description of geometry in terms of spectral properties of operators and can provide an explanation of the most successful particle physics we have, namely the Standard Model. Since not all gauge groups can fit into the framework, one may conclude that absence of large groups (like SO(10)) prevents proton decay; hence an encouraging outcome of the proposal. However, the meaning of the cutoff scale remains unclear, the dimensional parameters appear with incorrect values (a *hierarchy problem*), and there is also a (mild) dependence on the cutoff function. Moreover, and maybe this is the most important concern, the asymptotic expansion, valid only in the weak-field approximation, invalidates the theory in the ultraviolet regime, when one expects noncommutative geometry to play an important rôle.

To address such issues, the *zeta bosonic spectral action*

$$S_\zeta \equiv \lim_{s\to 0} \mathrm{Tr}\mathscr{D}^{-2s} \equiv \zeta(0,\mathscr{D}^2) \ , \tag{4.1}$$

has been proposed [23]. It is just the a_4 heat kernel coefficient associated with the Laplace type operator $\mathscr{D}^2$:

$$S_\zeta = a_4\left[\mathscr{D}^2\right] = \int d^4x\sqrt{g}L \ \text{ with } \ L(x) = a_4(\mathscr{D}^2, x) \ , \tag{4.2}$$

and leads to the Lagrangian density

$$\begin{aligned} L(x) \ =& \ \alpha_1 M^4 + \alpha_2 M^2 R + \alpha_3 M^2 H^2 + \alpha_4 B_{\mu\nu}B^{\mu\nu} + \alpha_5 W^\alpha_{\mu\nu}W^{\mu\nu\alpha} + \alpha_6 G^a_{\mu\nu}G^{\mu\nu a} \\ & + \alpha_7 H\left(-\nabla^2 - \frac{R}{6}\right)H + \alpha_8 H^4 + \alpha_9 C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \alpha_{10} R^*R^* \ , \end{aligned} \tag{4.3}$$

where $B_{\mu\nu}$, $W_{\mu\nu}$ and $G_{\mu\nu}$ are respectively the field strength tensors of the U(1), SU(2) and SU(3) gauge fields; the coefficients $\alpha_1, .., \alpha_{10}$ are dimensionless constants determined by the Dirac operator, the term R^*R^* is the Gauss-Bonnet density, and C denotes the Weyl tensor. Note that the $\alpha_1, \alpha_2, \alpha_3$ coefficients cannot be taken by the spectral action, i.e. the lower-dimensional operators must be normalised by hand.

It is worth noting that the dimensionful quantity M corresponding to the Majorana mass of the right-handed neutrino in the Dirac operator, is considered here as being constant. Since for $M = 0$

there are no dimensionful constants in the bare Lagrangian, one concludes that the cosmological constant, the Higgs mass parameter and the gravitational constant would not arise from renormalisation. A nonzero element in the Dirac operator corresponding to a neutrino Majorana mass is also essential in order to get the experimentally found Higgs mass. Such a term in the Dirac operator can be either $a_i \psi^c \sigma(x) \psi$ (with i a generation index, $i = 1,2,3$), or of the more general form $\psi^c(a_i\sigma(x) + M_1)\psi$ with a_i, M_i constants for right- and left-handed neutrinos in the three generations. In the former case, no dimension zero and two operators appear in the classical action; thus one will have to achieve dynamical generation of the three scales upon quantisation. In the latter case, one chose to use constant terms in order to introduce the M^4, M^2H^2, M^2R terms in the action, and consequently get the respective counter terms via ultraviolet renormalisation.

Following the zeta spectral action, one has no higher (than 4) dimensional operators, hence the theory is renormalisable and local. In addition, there are no issues with asymptotic expansion and convergence. The zeta bosonic spectral action is purely spectral with no dependence on a cutoff function. Moreover, while in the context of the zeta spectral action, the gravitational spectral dimension is equal to 2, implying that the gravitational propagation decreases faster at infinity due to the presence of fourth-derivatives, within the cutoff spectral action, the spectral dimension vanishes for all spins [24].

5. Conclusions

Noncommutative spectral geometry on the one hand addresses conceptual issues of the SM, while on the other hand it offers a geometrical framework to study physics at the quantum gravity regime. In the noncommutative spectral geometry framework, gravity and the Standard Model fields are put together into geometry and matter on a Kaluza-Klein noncommutative space. Then making use of known experimental data at the well-tested electroweak scale, one tries to understand the small-scale space-time structure. Applying the spectral action within an almost commutative manifold, one gets gravity combined with Yang-Mills and Higgs. Hence, this approach offers a purely geometric interpretation of the SM coupled to gravity, and in addition it offers a natural framework to address early universe cosmology.

To address physical consequences of the spectral action one admits its validity in the Lorentzian signature; an important issue that deserves further investigation. Within the context of the cutoff bosonic spectral action, an important remaining open issue is the weak-field approximation, in the sense that the expansion in reverse orders of the cutoff scale is only valid when fields and their derivatives are smaller than the cutoff scale. In the context of the zeta spectral action, one may have to find a dynamical generation of the three dimensionful fundamental constants, namely the cosmological constant, the Higgs vacuum expectation value and the gravitational constant.

It remains an open question of whether inflation, if at all needed within a wildly nocnommutative manifold, can be naturally incorporated. The known scalar fields, appearing in the NCSG action, could provide through their nonminimal coupling to the background geometry an era of accelerated expansion but fail to match the cosmic microwave background temperature anisotropies data. Unfortunately, the successful R^2-type inflation [25], favoured by the Planck CMB [26] data, cannot be applied in the higher derivative gravitational theory obtained by noncommutative spectral geometry. It is not clear yet whether one can accommodate a dilaton-type inflation [27] or use a scalar field, in a beyond the Standard Model scenario like the Pati-Salam model [28], as a successful inflaton candidate.

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