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Wess-Zumino-Witten term in QCD-like theories

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The so-called chiral soliton lattice was recently found to describe the ground state of the dense QCD matter in strong magnetic fields. Such a state consists of a periodic array of topological solitons, spontaneously breaks the parity and the translational symmetry and is known to appear also in condensed-matter systems such as chiral magnets. Motivated by the fact that the QCD-like theories such as the two-color QCD are accessible to the lattice simulations even at finite densities, we continue this work by investigating the ground state of the two-color QCD in strong magnetic fields. The analytic approach of low-energy effective field theory is used, hence, as a first step the gauged Wess-Zumino term reproducing the chiral anomaly has to be found. The well-known shape of the WZ term relevant for the QCD symmetry breaking pattern was generalized in order to be applicable also to the QCD-like theories.

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1. Introduction

An inhomogeneous neutral pion condensate was recently found to form the ground state of dense QCD matter in a strong magnetic field [1, 2]. Similar phase appears for chiral magnets [3] and its name Chiral Soliton Lattice (CSL) was adopted, since also in the QCD case, the chiral symmetry is broken by this ground state formed by a periodic array of topological solitons. A crucial ingredient for appearance of this phase is the coupling of neutral pions to the electromagnetic field due to the chiral anomaly which then also implies that the neutral pion condensate carries non-zero baryon charge and magnetic moment. Consequently, already for chemical potentials in the range of $400-800\,\mathrm{MeV}$, the baryon density reaches values of few-times nuclear saturation density [2] and the CSL phase could be relevant for the physics of neutron stars if the magnetic fields of the order of $10^{18-19}\,\mathrm{G}$ were present.

Let us emphasize that the above result is fully analytic and, based on systematic low-energy effective theory, it is also model-independent. The comparison with the lattice calculations would be desirable, however, due to notorious sign problem present for QCD with non-zero baryon chemical potential, such lattice results can't be expected in near future.

Our aim is to find a similar phase in a theory which does not suffer from the sign problem and, hence, could be tested by lattice methods. A class of QCD-like theories with quarks in real or pseudo-real representations of the gauge group fits this requirement, the simplest example being the so-called two-color QCD based on SU(2) gauge group with quarks in the fundamental representation [4].

Remarkably, if N_f copies of quarks reside in (pseudo)real representation of the gauge group, the flavor symmetry is extended from usual $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ to $\mathrm{SU}(2N_f)$. The chiral condensate $\langle q\bar{q}\rangle$ then breaks this symmetry to $\mathrm{SO}(2N_f)$ or $\mathrm{Sp}(2N_f)$ in the real or pseudoreal case, respectively. The effective field theory based on $\mathrm{SU}(2N_f)/\mathrm{SO}(2N_f)$ or $\mathrm{SU}(2N_f)/\mathrm{Sp}(2N_f)$ coset spaces has to be used, accordingly.

As mentioned above, the coupling of neutral pions to electromagnetic field arises as a consequence of the chiral anomaly which is captured by the so-called Wess-Zumino-Witten (WZW) term [5, 6] in the chiral perturbation theory. This term is well known in case of the usual QCD $\mathrm{SU}(N_f)_\mathrm{L} \times \mathrm{SU}(N_f)_\mathrm{R} \to \mathrm{SU}(N_f)_\mathrm{V}$ symmetry breaking pattern, however, the explicit formulas for the gauged WZW terms for general coset spaces are missing. More precisely, the results are known for the chiral coset spaces of the type $G_\mathrm{L} \times G_\mathrm{R}/G_\mathrm{V}$ (see, e.g., [7]) and certain information on the WZW terms for general coset spaces is available in the mathematically oriented literature [8, 9], however, substantial mathematical background and lengthy calculations are needed in order to translate the latter results to explicit formulas usable for physics applications. Therefore, general formulas for the gauged WZW terms where derived as a part of this project. This could provide an input not only for our study of QCD-like theories in strong magnetic fields, but also for various other physics applications of these topological terms.

The text is organized as follows. In section 2 we describe the construction of WZW terms based on the theory of cohomology whereas the results obtained in this way are summarized in section 3. Particle physics applications of the gauged WZW terms are discussed in section 4 and we conclude in section 5 with some preliminary implications for the case of our interest, the QCD-like theories in strong magnetic fields.

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2. WZW terms and the theory of cohomology

The so-called coset construction [10, 11] is a well-established method for finding G-invariant Lagrangian densities describing the interactions of the Goldstone bosons corresponding to the spontaneous symmetry breaking of a group G to its subgroup H. On the other hand, it is sufficient that the action is G-invariant and the Langrangian density can be invariant only up to a surface term. There are also phenomenological reasons why to believe that the terms included in the coset construction are not sufficient: the chiral anomaly (responsible, e.g., for the decay $\pi^0 \to \gamma\gamma$) is not captured by these terms.

The term which describes the chiral anomaly in case of QCD was first recognized by Wess and Zumino [5]. Its nice geometrical interpretation was revealed later by Witten [6]: it was shown that the Lagrangian density invariant up to a surface term in 4 spacetime dimensions can be written as an invariant Lagrangian in 5 dimensions. This statement was generalized to other coset spaces in [12], let us present here the argument for spacetimes with general dimension d.

First, let us assume that the boundary conditions in the infinity allow the compactification of the d-dimensional spacetime to a sphere S^d. If in addition the d-th homotopy group of the coset space G/H is trivial[1]: $\pi_d(G/H) = 0$, the Goldstone boson field $U(x)$ can be extended to a mapping $\tilde{U}(x,t)$, $t \in [0,1]$ by putting $\tilde{U}(x,1) = U(x)$, $\tilde{U}(x,0) = 0$. This mapping then attains values in a $d+1$-dimensional disc D^{d+1} in G/H with the boundary $U(S^d)$. Well-defined G-invariant actions which do not depend on the extension $\tilde{U}(x,t)$ correspond to closed G-invariant $d+1$-forms ω_{d+1} on G/H. Such closed forms can be (at least locally) written as a differential of a d-form $\tilde{\omega}_d$ and by the Stokes theorem we find

$$S_{\mathrm{WZW}} \propto \int_{D^{d+1}} \omega_{d+1} = \int_{D^{d+1}} \mathrm{d}\tilde{\omega}_d = \int_{U(S^d)} \tilde{\omega}_d = \int_{S^d} U^* \tilde{\omega}_d. \tag{2.1}$$

If the d-form $U^*\tilde{\omega}_d$ representing the Lagrangian density is G-invariant, such a term would be covered already in the coset construction. Hence, the case when the forms $U^*\tilde{\omega}_d$ are *not* G-invariant is of most interest. The corresponding forms ω_{d+1} are then closed but not exact, i.e., they are generators of the $d+1$st de Rham cohomology group of G/H. Indeed, it was shown in [12] that these generators are in one to one correspondence with the terms in the Lagrangian density invariant up to a total derivative (under the conditions on homotopy of G/H mentioned above).

In [9] the cohomology generators of degrees 2, 3, 4 and 5 were explicitly constructed for any coset space G/H and compact, connected group G. On the other hand, the gauged versions of these generators were given only implicitly, in terms of lower-degree differential forms satisfying certain hierarchy of equations. In [14] we follow the work [9] by finding the explicit formulas for the gauged cohomology generators and the corresponding Lagrangian densities. For simplicity we restrict ourselves to semi-simple and simply-connected G and connected H which ensures that G/H is simply connected (the way, in which this condition can be relaxed is described in [9]). Selection of our results is given in the next section.

Let us note, that the proportionality sign in (2.1) was used since the normalization of the differential forms describing the WZW term is not fixed by the symmetry-based differential geometry methods used here. The normalization factor has to be found eventually by the matching to the

[1] For a classification of WZW terms avoiding the assumption on homotopy see [13].

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underlying microscopic theory. This factor is required to be quantized in order to obtain an action independent of the extension $\tilde{U}(x,t)$ if the homotopy group $\pi_{d+1}(G/H)$ is nontrivial [6].

3. Results: Gauged WZW terms for a general coset space

For sake of brevity, we include here only the results for the case $d=4$ which is of most relevance for particle physics. All the results are given in terms of differential forms, the dictionary for translating these to ordinary Lagrangian densities can be found in [14].

3.1 Notation

Let us summarize the notation largely adopted from [9] which will be used for expressing our results. First of all, we choose to work with the anti-hermitian generators of the group G satisfying

$$[T_A, T_B] = f_{AB}{}^C T_C. \tag{3.1}$$

We index by capital Latin letters the generators of G in general, specifically, the generators of the unbroken subgroup H will be denoted as $T_{\alpha,\beta,\dots}$, whereas the broken generators as $T_{a,b,\dots}$. As in the coset construction [10, 11] we introduce the matrix Goldstone field U transforming as

$$U \xrightarrow{g} U' = gUh^{-1}, \tag{3.2}$$

for $g \in G$ and $h = h(g,U) \in H$. We assume from the beginning that the symmetry is gauged, i.e., g is local. The gauged Maurer-Cartan (MC) Lie-algebra valued 1-form is defined as

$$\bar{\theta} \equiv \bar{\theta}^A T_A \equiv U^{-1}(\mathrm{d}+A)U = \theta + \bar{A}, \tag{3.3}$$

where $\theta \equiv \bar{\theta}|_{A=0}$. Further, $A \equiv A^B T_B$ is the gauge connection of G transforming as $A \xrightarrow{g} gAg^{-1} + g\mathrm{d}g^{-1}$. Let us note that if one wants to compare our results with other works where the generators of G are chosen to be Hermitian, the replacement $A \to -\mathrm{i}A$ is needed.

The transformation rule for the MC form reads $\bar{\theta} \xrightarrow{h} h\bar{\theta}h^{-1} + h\mathrm{d}h^{-1}$. It is then handy to divide this form to broken and unbroken parts:

$$\bar{\phi} \equiv \bar{\theta}^a T_a \xrightarrow{h} h\bar{\phi}h^{-1}, \qquad \bar{V} \equiv \bar{\theta}^\alpha T_\alpha \xrightarrow{h} h\bar{V}h^{-1} + h\mathrm{d}h^{-1} \tag{3.4}$$

where $\bar{V}$ behaves as a gauge connection of H. The corresponding field strength 2-form can be defined as[2]

$$\bar{W} \equiv \mathrm{d}\bar{V} + \bar{V}^2 \xrightarrow{h} h\bar{W}h^{-1}. \tag{3.5}$$

Analogously, the gauge connection of G gives rise to another field-strength 2-form

$$F \equiv \mathrm{d}A + A^2 \xrightarrow{g} gFg^{-1}. \tag{3.6}$$

Finally, let us recall that the gauged MC form satisfies the MC structure equation

$$\mathrm{d}\bar{\theta} + \bar{\theta}^2 = U^{-1}FU \equiv \bar{F}, \qquad \mathrm{d}\bar{\theta}^A + \tfrac{1}{2} f^A{}_{BC}\bar{\theta}^B\bar{\theta}^C = \bar{F}^A. \tag{3.7}$$

[2]Let us note that we will be omitting the wedge symbol in the products of differential forms. Recalling also the matrix structure of $\bar{V}$, this means, e.g., $\bar{V}^2 = T_\alpha T_\beta \bar{\theta}^\alpha \wedge \bar{\theta}^\beta = \frac{1}{2}[T_\alpha, T_\beta]\bar{\theta}^\alpha \wedge \bar{\theta}^\beta = \frac{1}{2} f_{\alpha\beta}{}^\gamma T_\gamma \bar{\theta}^\alpha \wedge \bar{\theta}^\beta$.

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All the forms $\bar{\phi}$, $\bar{W}$ and $\bar{F}$ then transform linearly under the adjoint action of H and can be used as covariant building blocks when constructing invariant Lagrangians. In addition, their covariant derivatives can be formed using the gauge connection $\bar{V}$. On the other hand, differential forms which are *not* G-invariant will be also constructed in this work, consequently, it is useful to introduce ungauged versions of the forms above:

$$\phi = \bar{\phi}|_{A=0}, \quad V = \bar{V}|_{A=0}, \quad W = \mathrm{d}V + V^2. \tag{3.8}$$

3.2 Results for $d = 4$

As discussed in [9], the generators of the 5th de Rham cohomology group are classified by constant fully symmetric G-invariant tensors d_{ABC} which vanish on the unbroken subgroup H, i.e., $d_{\alpha\beta\gamma} = 0$. Up to an overall scale, such a tensor is unique for any simple compact Lie group, and is non-vanishing only in case of $\mathrm{SU}(N)$ groups with $N \geq 3$. Considering semi-simple groups, the most general fully symmetric invariant tensor can be expressed as

$$d_{ABC} = \frac{1}{2}\sum_j d_j \,\mathrm{tr}_j(T_A\{T_B, T_C\}) \tag{3.9}$$

where j runs over all simple components of G, the trace is done over the j-th simple component and the set of coefficients d_j is constrained only by the fact that $d_{\alpha\beta\gamma} = 0$. It is then possible to introduce a shorthand matrix notation for the differential forms constructed below if one defines

$$\langle X \rangle \equiv \sum_j d_j \,\mathrm{tr}_j X. \tag{3.10}$$

Our main result concerning the gauged generator of the 5th de Rham cohomology group for general G/H can be then expressed as

$$\omega_5 = \big\langle \tfrac{1}{10}\bar{\phi}^5 - \tfrac{1}{2}(\bar{W} + \bar{F})\bar{\phi}^3 + (\bar{W}^2 + \bar{F}^2)\bar{\phi} + \tfrac{1}{2}(\bar{W}\bar{F} + \bar{F}\bar{W})\bar{\phi} \big\rangle. \tag{3.11}$$

This form is G-invariant but in general not closed,

$$\mathrm{d}\omega_5 = d_{ABC}F^A F^B F^C = \langle F^3 \rangle. \tag{3.12}$$

Since $d_{\alpha\beta\gamma} = 0$, this expression is vanishing when only unbroken generators are gauged. There is, however, an obstruction to gauging the broken generators which is related to the appearance of the chiral anomaly in the underlying microscopic theory as will be shown below. Let us first observe that the right-hand side of (3.12) is the third Chern character which can be written as a differential of the Chern-Simons 5-form:

$$\omega_5^{\mathrm{CS}} = \big\langle F^2 A - \tfrac{1}{2}FA^3 + \tfrac{1}{10}A^5 \big\rangle, \qquad \mathrm{d}\omega_5^{\mathrm{CS}} = \langle F^3 \rangle. \tag{3.13}$$

Consequently, the form $\omega_5 - \omega_5^{\mathrm{CS}}$ is closed and gives rise to a well-defined action in four spacetime dimensions. However, this form is not gauge-invariant:

$$\delta\omega_5^{\mathrm{CS}} = -\mathrm{d}\mathscr{A}_4, \qquad \mathscr{A}_4 = \big\langle \varepsilon\,\mathrm{d}\big(A\mathrm{d}A + \tfrac{1}{2}A^3\big) \big\rangle. \tag{3.14}$$

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The last expression corresponds to the well-known formula for the so-called consistent anomaly[3] [15].

Using transgression methods [8], a 4-form corresponding to the gauge-field-dependent part of the WZW term can be found:

$$
\begin{aligned}
\omega_{5A} \equiv\ & \omega_5(A) - \omega_5(0) - \omega_5^{\mathrm{CS}} = \mathrm{d}\tilde{\omega}_{4A} \\
\tilde{\omega}_{4A} =\ & \big\langle \tfrac{1}{2}\phi^3(\bar{A}+\bar{A}_\parallel) + \tfrac{1}{4}\phi\bar{A}_\perp\phi(\bar{A}+\bar{A}_\parallel) + \tfrac{1}{2}\phi^2[\bar{A}_\perp,\bar{A}_\parallel] \\
& + \phi\big(\tfrac{1}{2}\bar{A}_\perp^3 + \tfrac{3}{4}\bar{A}_\perp^2\bar{A}_\parallel + \tfrac{3}{4}\bar{A}_\parallel\bar{A}_\perp^2 + \tfrac{1}{2}\bar{A}_\parallel^2\bar{A}_\perp + \tfrac{1}{2}\bar{A}_\parallel\bar{A}_\perp\bar{A}_\parallel + \tfrac{1}{2}\bar{A}_\perp\bar{A}_\parallel^2 + \bar{A}_\parallel^3\big) \\
& + \tfrac{1}{2}\bar{A}_\perp\bar{A}_\parallel^3 - \tfrac{1}{2}\bar{A}_\parallel\bar{A}_\perp^3 - \tfrac{1}{4}\bar{A}_\parallel\bar{A}_\perp\bar{A}_\parallel\bar{A}_\perp + \tfrac{1}{2}\bar{F}[\bar{A}+\tfrac{1}{2}\bar{A}_\parallel,\phi] \\
& + \tfrac{1}{2}(\bar{W}+W)[\tfrac{1}{2}\bar{A}+\bar{A}_\parallel,\phi] + (\tfrac{1}{2}\bar{F}+\tfrac{1}{2}\bar{W}+\tfrac{1}{4}W)[\bar{A}_\parallel,\bar{A}_\perp]\big\rangle. \qquad (3.15)
\end{aligned}
$$

As anticipated, the forms which are not G-invariant do not have a simple canonical expression in terms of covariant building blocks mentioned in section 3.1. The symbols $\parallel$ and $\perp$ are used to indicate the unbroken and broken part of the 1-form $\bar{A}$ defined in (3.3), respectively.

3.3 Simplification in case of symmetric coset spaces

For the symmetric coset spaces G/H one can introduce an automorphism $\mathscr{R}$ of the Lie algebra under which the generators of H do not transform: $\mathscr{R}(T_\alpha) = T_\alpha$ whereas the broken generators change the sign $\mathscr{R}(T_a) = -T_a$. If one chooses a parametrization of U where this matrix is inverted by the automorphism $\mathscr{R}$ (which is the case, e.g., for the exponential parameterization $U(x) = e^{\mathrm{i}\pi^a(x)T_a}$ with $\pi^a(x)$ denoting the Goldstone fields), it is possible to define a field variable that transforms linearly under the whole group G [10, 11],

$$
\Sigma(x) \equiv U(x)^2, \qquad \Sigma \xrightarrow{g} g\Sigma\mathscr{R}(g)^{-1}. \qquad (3.16)
$$

Projecting the broken and unbroken parts of the relevant differential forms using the automorphism $\mathscr{R}$, one obtains

$$
\begin{aligned}
\tilde{\omega}_{4A} = \big\langle & -\tfrac{11}{32}\mathrm{d}\Sigma\mathrm{d}\Sigma^{-1}\mathrm{d}\Sigma\Sigma^{-1}A + \tfrac{5}{32}\mathrm{d}\Sigma^{-1}\mathrm{d}\Sigma\mathrm{d}\Sigma^{-1}\Sigma A_{\mathscr{R}} + \tfrac{3}{32}\mathrm{d}\Sigma A_{\mathscr{R}}\mathrm{d}\Sigma^{-1}A + \tfrac{11}{64}\mathrm{d}\Sigma\Sigma^{-1}A\mathrm{d}\Sigma\Sigma^{-1}A \\
& - \tfrac{5}{64}\mathrm{d}\Sigma^{-1}\Sigma A_{\mathscr{R}}\mathrm{d}\Sigma^{-1}\Sigma A_{\mathscr{R}} + \tfrac{1}{4}\mathrm{d}\Sigma\mathrm{d}\Sigma^{-1}\Sigma A_{\mathscr{R}}\Sigma^{-1}A - \tfrac{1}{4}\mathrm{d}\Sigma^{-1}\mathrm{d}\Sigma\Sigma^{-1}A\Sigma A_{\mathscr{R}} \qquad (3.17) \\
& - \tfrac{9}{32}\mathrm{d}\Sigma\Sigma^{-1}A\Sigma A_{\mathscr{R}}\Sigma^{-1}A + \tfrac{7}{32}\mathrm{d}\Sigma^{-1}\Sigma A_{\mathscr{R}}\Sigma^{-1}A\Sigma A_{\mathscr{R}} + \tfrac{3}{32}\mathrm{d}\Sigma A_{\mathscr{R}}\Sigma^{-1}A^2 - \tfrac{5}{32}\mathrm{d}\Sigma^{-1}A\Sigma A_{\mathscr{R}}^2 \\
& + \tfrac{5}{32}\mathrm{d}\Sigma A_{\mathscr{R}}^2\Sigma^{-1}A - \tfrac{3}{32}\mathrm{d}\Sigma^{-1}A^2\Sigma A_{\mathscr{R}} + \tfrac{13}{32}\mathrm{d}\Sigma\Sigma^{-1}A^3 - \tfrac{3}{32}\mathrm{d}\Sigma^{-1}\Sigma A_{\mathscr{R}}^3 + \tfrac{1}{8}\Sigma A_{\mathscr{R}}\Sigma^{-1}A\Sigma A_{\mathscr{R}}\Sigma^{-1}A \\
& - \tfrac{5}{16}\Sigma A_{\mathscr{R}}\Sigma^{-1}A^3 + \tfrac{3}{16}\Sigma^{-1}A\Sigma A_{\mathscr{R}}^3 + \tfrac{1}{8}(-\mathrm{d}\Sigma F_{\mathscr{R}}\Sigma^{-1}A - \mathrm{d}\Sigma A_{\mathscr{R}}\Sigma^{-1}F + \mathrm{d}\Sigma^{-1}F\Sigma A_{\mathscr{R}} + \mathrm{d}\Sigma^{-1}A\Sigma F_{\mathscr{R}}) \\
& - \tfrac{7}{16}\mathrm{d}\Sigma\Sigma^{-1}\{F,A\} + \tfrac{1}{16}\mathrm{d}\Sigma^{-1}\Sigma\{F_{\mathscr{R}},A_{\mathscr{R}}\} + \tfrac{3}{8}\Sigma A_{\mathscr{R}}\Sigma^{-1}\{F,A\} - \tfrac{1}{8}\Sigma^{-1}A\Sigma\{F_{\mathscr{R}},A_{\mathscr{R}}\}\big\rangle,
\end{aligned}
$$

where the short-hand notation $A_{\mathscr{R}} \equiv \mathscr{R}(A)$, $F_{\mathscr{R}} \equiv \mathscr{R}(F)$ was introduced.

[3] Strictly speaking, the Chern-Simons 5-form is defined by the second relation in (3.13), hence, the expression for ω_5^{CS} can be changed by adding a differential of any form depending on the gauge fields only. The form $\tilde{\omega}_{4A}$ below and the expression for the anomaly can be then changed accordingly, the choice for ω_5^{CS} (3.13) is distinguished by the fact that the corresponding shape of the anomaly satisfies the so-called Wess-Zumino consistency conditions [15]. On the other hand, in [9] a closed 5-form is defined as $\omega_5(U,A) - \omega_5(\mathbb{1},A)$ leading to different representation of the anomaly.

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3.3.1 Chiral coset spaces $G_{\mathbf{L}} \times G_{\mathbf{R}} \to G_{\mathbf{V}}$

Further simplification of (3.17) is possible for the well-explored case of $G_{\mathrm{L}} \times G_{\mathrm{R}} \to G_{\mathrm{V}}$ coset spaces. Here, $d_{\alpha\beta\gamma} = 0$ must be ensured by a proper choice of d_i in (3.9), in particular, we have used $d_L = 1$ and $d_R = -1$ in (3.10) for deriving the formula below. The generators of $G_{\mathrm{L}} \times G_{\mathrm{R}}$ can be written as $(T_{\mathrm{L},A}, T_{\mathrm{R},B})$, with $T_{\mathrm{L},A} = \pm T_{\mathrm{R},B}$ corresponding to unbroken and broken generators respectively. The automorphism $\mathscr{R}$ then swaps the two entries in the parenthesis. In order to comply with the notation usually used in QCD, we write the linearly transforming variable (3.16) as (Σ, Σ^{-1}) with $\Sigma \in G$. Further, the gauge fields are denoted as (A_L, A_R) and the corresponding field-strength 2-form is denoted as (F_L, F_R). Using this notation, one obtains

$$\begin{aligned}\tilde{\omega}_{4A} = \mathrm{tr}\big(&-\tfrac{1}{2}\mathrm{d}\Sigma\mathrm{d}\Sigma^{-1}\mathrm{d}\Sigma\Sigma^{-1}A_{\mathrm{L}} + \tfrac{1}{4}\mathrm{d}\Sigma\Sigma^{-1}A_{\mathrm{L}}\mathrm{d}\Sigma\Sigma^{-1}A_{\mathrm{L}} + \tfrac{1}{2}\mathrm{d}\Sigma\mathrm{d}\Sigma^{-1}\Sigma A_{\mathrm{R}}\Sigma^{-1}A_{\mathrm{L}} \\ &-\tfrac{1}{2}\mathrm{d}\Sigma\Sigma^{-1}A_{\mathrm{L}}\Sigma A_{\mathrm{R}}\Sigma^{-1}A_{\mathrm{L}} + \tfrac{1}{4}\mathrm{d}\Sigma A_{\mathrm{R}}\Sigma^{-1}A_{\mathrm{L}}^2 + \tfrac{1}{4}\mathrm{d}\Sigma A_{\mathrm{R}}^2\Sigma^{-1}A_{\mathrm{L}} + \tfrac{1}{2}\mathrm{d}\Sigma\Sigma^{-1}A_{\mathrm{L}}^3 \\ &+\tfrac{1}{8}\Sigma A_{\mathrm{R}}\Sigma^{-1}A_{\mathrm{L}}\Sigma A_{\mathrm{R}}\Sigma^{-1}A_{\mathrm{L}} - \tfrac{1}{2}\Sigma A_{\mathrm{R}}\Sigma^{-1}A_{\mathrm{L}}^3 - \tfrac{1}{4}\mathrm{d}\Sigma F_{\mathrm{R}}\Sigma^{-1}A_{\mathrm{L}} - \tfrac{1}{4}\mathrm{d}\Sigma A_{\mathrm{R}}\Sigma^{-1}F_{\mathrm{L}} \\ &-\tfrac{1}{2}\mathrm{d}\Sigma\Sigma^{-1}\{F_{\mathrm{L}}, A_{\mathrm{L}}\} + \tfrac{1}{2}\Sigma A_{\mathrm{R}}\Sigma^{-1}\{F_{\mathrm{L}}, A_{\mathrm{L}}\}\big) - (\Sigma \leftrightarrow \Sigma^{-1}, \mathrm{L} \leftrightarrow \mathrm{R}).\end{aligned} \tag{3.18}$$

This result agrees with [7] if one takes into account that the action remains unchanged if a differential of a 3-form is added to this 4-form.

3.3.2 $\mathbf{SU}(2N_f)/\mathbf{SO}(2N_f)$ and $\mathbf{SU}(2N_f)/\mathbf{Sp}(2N_f)$ coset spaces

Most importantly, the WZW term relevant for the case of quarks in (pseudo)real representations of the gauge group was derived. Both the relevant coset spaces can be defined by following relations for unbroken and broken generators, respectively:

$$T_\alpha^T \Sigma_0 + \Sigma_0 T_\alpha = 0, \qquad T_a^T \Sigma_0 - \Sigma_0 T_a = 0. \tag{3.19}$$

Here Σ_0 is a fixed real unitary matrix describing the ground state of the system which is symmetric in case of the $\mathrm{SU}(2N_f)/\mathrm{SO}(2N_f)$ coset space, whereas in case of $\mathrm{SU}(2N_f)/\mathrm{Sp}(2N_f)$, it is antisymmetric. It can be chosen for instance in the block form

$$\Sigma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad \text{(real)}, \qquad \Sigma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \quad \text{(pseudoreal)}. \tag{3.20}$$

Both these coset spaces are symmetric which can be easily seen by introducing the automorphism $\mathscr{R}(T_A) = -\Sigma_0 T_A^T \Sigma_0^{-1}$. The linearly transforming $2N \times 2N$ matrix variable $\Sigma = U^2$ has the conjugation property $\Sigma^T = \Sigma_0 \Sigma \Sigma_0^{-1}$ which allows to relate pairwise the terms in the general formula (3.17). In the literature, it is usually introduced $\tilde{\Sigma}(x) \equiv \Sigma(x)\Sigma_0$, with simple transformation properties $\tilde{\Sigma} \xrightarrow{g} g\tilde{\Sigma}g^T$. In terms of this variable the WZW 4-form can be written as

$$\begin{aligned}\tilde{\omega}_{4A} = \mathrm{tr}\big(&-\tfrac{1}{2}\mathrm{d}\tilde{\Sigma}\mathrm{d}\tilde{\Sigma}^{-1}\mathrm{d}\tilde{\Sigma}\tilde{\Sigma}^{-1}A + \tfrac{1}{4}\mathrm{d}\tilde{\Sigma}\tilde{\Sigma}^{-1}A\mathrm{d}\tilde{\Sigma}\tilde{\Sigma}^{-1}A - \tfrac{1}{4}\mathrm{d}\tilde{\Sigma}\mathrm{d}\tilde{\Sigma}^{-1}\tilde{\Sigma}A^T\tilde{\Sigma}^{-1}A + \tfrac{1}{4}\mathrm{d}\tilde{\Sigma}^{-1}\mathrm{d}\tilde{\Sigma}\tilde{\Sigma}^{-1}A\tilde{\Sigma}A^T \\ &+\tfrac{1}{2}\mathrm{d}\tilde{\Sigma}\tilde{\Sigma}^{-1}A\tilde{\Sigma}A^T\tilde{\Sigma}^{-1}A - \tfrac{1}{2}\mathrm{d}\tilde{\Sigma}\tilde{\Sigma}^{-1}A^3 + \tfrac{1}{8}\tilde{\Sigma}A^T\tilde{\Sigma}^{-1}A\tilde{\Sigma}A^T\tilde{\Sigma}^{-1}A - \tfrac{1}{2}\tilde{\Sigma}A^T\tilde{\Sigma}^{-1}A^3 \\ &+\tfrac{1}{4}\mathrm{d}\tilde{\Sigma}A^T\tilde{\Sigma}^{-1}\mathrm{d}A - \tfrac{1}{4}\mathrm{d}\tilde{\Sigma}^{-1}A\tilde{\Sigma}\mathrm{d}A^T - \tfrac{1}{2}\mathrm{d}\tilde{\Sigma}\tilde{\Sigma}^{-1}\{\mathrm{d}A, A\} - \tfrac{1}{2}\tilde{\Sigma}A^T\tilde{\Sigma}^{-1}\{\mathrm{d}A, A\}\big).\end{aligned} \tag{3.21}$$

Ironically, after going through the procedure of deriving gauged WZW terms for a general coset space, we have found out that the WZW term for the $\mathrm{SU}(2N_f)/\mathrm{Sp}(2N_f)$ coset space (which is the case of most interest for us since it corresponds to the two-color QCD), was derived previously in the literature on the composite Higgs models [16, 17]. These results agree with ours except for the sign of the $\Sigma A_{\mathscr{R}}\Sigma^{-1}A\Sigma A_{\mathscr{R}}\Sigma^{-1}A$ term.

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4. Other applications of WZW terms

Quantum field theories where the symmetry based on a group G is spontaneously broken to $H \subset G$ and the Goldstone bosons appear have wide applications in different branches of physics, consequently, the same holds for the WZW terms. Numerous examples from condensed matter physics were mentioned in [14], let us concentrate here on selected particle physics applications where the gauging of the WZW term is relevant.

Apart from the obvious application in QCD, there are beyond-standard-model theories where another QCD-like confining UV sector is assumed giving rise to extra pseudo-Goldstone bosons. In the composite Higgs models, one of these pseudo-Goldstone bosons replaces the fundamental scalar Higgs field of the Standard Model (SM), whereas other scenarios suggest that such pion-like fields could form the dark matter (DM). There are in fact only three possibilities for the chiral symmetry breaking pattern in the UV theory corresponding to the extra quark-like fields in complex, real and pseudoreal representations of the gauge group [18]. These are described respectively by the $G_{\mathrm{L}} \times G_{\mathrm{R}} \to G_{\mathrm{V}}$, $\mathrm{SU}(2N_f)/\mathrm{SO}(2N_f)$ and $\mathrm{SU}(2N_f)/\mathrm{Sp}(2N_f)$ coset spaces which are exactly the examples covered in sections 3.3.1 and 3.3.2.

For the composite Higgs models, the decay rates for the processes analogous to $\pi^0 \to \gamma\gamma$ from QCD are proportional to number of "colors" in the UV gauge sector, hence, the gauged WZW term could play a crucial role in probing the content of such theories on colliders [19, 17].

In case of the composite DM models, the role of the WZW model is two-fold. First, the $3 \to 2$ self-interactions of the DM particles determining the relic abundance are induced by the WZW term similarly as the $K^+K^- \to \pi^+\pi^0\pi^-$ interaction in QCD [20]. Second, in [21] the coupling of the DM sector to SM particles is ensured by gauging of a $U(1)$ symmetry and mixing of the corresponding vector boson $\mathscr{A}_\mu$ with the SM hypercharge vector boson. The gauged WZW term then gives rise to interactions of the type $\pi\pi \to \pi V$ with π being the DM particles and V being the dark photon (mixture of $\mathscr{A}_\mu$ and the SM Z_μ). Such interactions could contribute to the determination of the relic abundance and have to be suppressed so that the $3 \to 2$ self-interactions are dominant.

5. Conclusions and outlook

Formula for the gauged WZW term for a general coset space in four spacetime dimensions was found and its simplified version for selected coset spaces was given (see [14] for more examples and the WZW terms in case of $d \neq 4$). In particular, the coset spaces relevant for the QCD-like theories were investigated which is a starting point for our analysis of these theories in strong magnetic fields using the effective field theory methods. The result for the WZW term suggests that, indeed, a phase analogous to CSL phase in QCD will be present, e.g., for the simplest case of the two-color QCD with two quark flavours transforming in the fundamental representation which corresponds to the $\mathrm{SU}(4)/\mathrm{Sp}(4)$ coset space. We explicitely checked that there is a region in $\mu_B - B$ plane where the CSL phase is energetically more favourable than both the vaccuum and the diquark condensate phase which is also expected in the case of two-color QCD [4]. The detailed shape of the phase diagram is under investigation and these results will be published elsewhere [22].

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