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Numerical study of ADE-type $\mathcal{N} = 2$ Landau–Ginzburg models

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At an extremely low-energy scale, it is believed that the two-dimensional $\mathcal{N} = 2$ Wess–Zumino model becomes an $\mathcal{N} = 2$ superconformal field theory (SCFT). We study this theoretical conjecture of the Landau–Ginzburg (LG) description by numerical simulations based on a supersymmetric-invariant momentum-cutoff regularization. First, from the two-point function of the energy-momentum tensor, we measure the central charge of the ADE minimal models. Second, we develop a method to take the continuum limit, and perform a precision measurement of the scaling dimension in the A-type minimal model. All our results show a coherence picture being consistent with the conjectured LG/SCFT correspondence.

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1. Introduction

It is believed that the two-dimensional (2D) massless $\mathcal{N}=2$ Wess–Zumino (WZ) model with a quasi-homogeneous superpotential provides a Lagrangian-level realization of the 2D $\mathcal{N}=2$ superconformal field theory (SCFT). Such a SCFT would be a scale-invariant theory on the nontrivial infrared (IR) fixed point of the WZ model, while all massive modes are decoupled. This conjecture of the Landau–Ginzburg (LG) description has been theoretically analyzed from various aspects; e.g., see [1]. To a solvable ADE-type minimal model, the corresponding superpotential of the WZ model is shown in table 1 [2]. It is, however, difficult to prove this theoretical conjecture directly, since the coupling constant becomes strong at the IR region and the perturbation theory possesses IR divergences. The LG description is remarkably a non-perturbative phenomenon.

Algebra	Superpotential W	Central charge c
A_n	Φ^{n+1}, $n \geqq 1$	$3-6/(n+1)$
D_n	$\Phi^{n-1}+\Phi\Phi'^2$, $n \geqq 3$	$3-6/2(n-1)$
E_6	$\Phi^3+\Phi'^4$	$3-6/12$
E_7	$\Phi^3+\Phi\Phi'^3$	$3-6/18$
E_8	$\Phi^3+\Phi'^5$	$3-6/30$

Table 1: ADE classification [2]

An alternative approach to this issue may be provided by a non-perturbative calculational method such as the lattice field theory. This kind of numerical method, when further developed, may enable us to compute directly scattering amplitudes in a superstring theory whose world sheet theory is given by an $\mathcal{N}=2$ SCFT; the theory possesses the superstring compactification to the Calabi–Yau quintic threefold. Such a theory is in general not a minimal model nor a product of minimal models. With regard to this point, the LG description realizes a specific strongly-interacting Lagrangian corresponding to the Calabi–Yau manifold [3, 4]. A numerical approach to the 2D $\mathcal{N}=2$ WZ model would be useful to investigate a superstring theory.

As is well recognized, however, the lattice regularization is generally incompatible with the supersymmetry (SUSY). The lattice parameters should be fine-tuned so that a lattice model yields the target SUSY continuum theory. To this issue, a possible solution is that we construct the 2D $\mathcal{N}=2$ WZ model on the lattice on the basis of the so-called Nicolai map [5, 6]. For example, in the lattice formulation from [7], one nilpotent SUSY is exactly preserved at finite lattice spacing, and the vacuum energy is canceled even on the lattice owing to the lattice Nicolai map. Moreover, it can be argued that, to all orders of perturbation theory, the full SUSY is automatically restored in the continuum limit without any fine tuning. By using this formulation [7], the scaling dimension of the scalar field in the A_2-type theory with the cubic superpotential was measured [8]. This numerical study achieved a triumph of the lattice field theory.

Some what later, the authors in [9] examined the same A_2-type WZ model by using the formulation from [10], and measured the scaling dimension and the central charge. The formulation [10] is based on the Nicolai mapping and the momentum cutoff regularization, and preserves the full set of SUSY as well as the translational invariance even with a finite cutoff. Then, the construction of the Noether currents associated with spacetime symmetries, e.g., the supercurrent and the energy-momentum tensor (EMT), is straightforward. This feature enables us to compute the central charge, which appears in two-point functions of such Noether currents.

In this paper, we numerically study the 2D $\mathcal{N}=2$ WZ model, based on the momentum-cutoff regularization [10]. First, we focus on the A_2, A_3, D_3, D_4, E_6 ($\cong A_2 \otimes A_3$), and E_7 models. The method in [9] is generalized to the WZ model with multiple superfields and more complicated superpotentials. From the IR behavior of the EMT correlator, we numerically determine the central charge of these models [11, 12]. Second, we develop an extrapolation method to take the continuum and infinite-volume limit [13], while any extrapolation has been not done in the preceding numerical studies. Then, on the basis of the formulation [10], we perform a precision measurement of the scaling dimension in the A_2-type theory. Our results below show a coherence picture being consistent with the conjectured LG description of the ADE minimal models.

2. SUSY-preserving formulation using the Nicolai map

First of all, we briefly review the SUSY-preserving formulation in [10]. In what follows, the system is defined in a 2D Euclidean physical box $L_0 \times L_1$; let us work in the momentum space with a momentum cutoff, $p_\mu = 2\pi n_\mu / L_\mu$ ($n_\mu = 0, \pm 1, \dots, \pm L_\mu/2a$), where the Greek index μ runs over 0 and 1, and repeated indices are not summed over. Here, a is a unit of dimensionful quantities; the *continuum limit* $a \to 0$ removes the UV cutoff. For simplicity, we take $L/a = L_0/a = L_1/a$ as even integers.

Let us consider the 2D $\mathcal{N}=2$ WZ model with N_Φ supermultiplets, $\{\Phi_I\}_{I=1,\dots,N_\Phi}$, which consist of complex scalar fields $\{A_I\}$, and left- and right-handed spinors $\{(\psi_\alpha, \bar{\psi}_{\dot{\alpha}})_I\}$ ($\alpha = 1, 2$). Then, the action of the 2D $\mathcal{N}=2$ WZ model with a quasi-homogeneous superpotential $W(\{A\})$ is given by

$$S = \frac{1}{L_0 L_1} \sum_p \sum_I \Bigg[4 p_z A_I^*(-p) p_{\bar{z}} A_I(p) + \frac{\partial W(\{A\})}{\partial A_I}(-p) \frac{\partial W(\{A\})^*}{\partial A_I^*}(p) + (\bar{\psi}_1, \psi_2)_I(-p) \sum_J \begin{pmatrix} 2\delta_{IJ} p_z & \frac{\partial^2 W(\{A\})^*}{\partial A_I^* \partial A_J^*} * \\ \frac{\partial^2 W(\{A\})}{\partial A_I \partial A_J} * & 2\delta_{IJ} p_{\bar{z}} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \bar{\psi}_2 \end{pmatrix}_J (p) \Bigg], \tag{2.1}$$

where $p_z = (p_0 - i p_1)/2$, $p_{\bar{z}} = (p_0 + i p_1)/2$, and $*$ denotes the convolution

$$(\varphi_1 * \varphi_2)(p) \equiv \frac{1}{L_0 L_1} \sum_q \varphi_1(q) \varphi_2(p-q). \tag{2.2}$$

The field products in $\partial W(\{A\})/\partial A_I$ and $\partial W(\{A\})/\partial A_I \partial A_J$ are understood as the convolution.

A remarkable property of the system is the existence of the so-called Nicolai map [5, 6]. This mapping simplifies the path integral drastically; the formulation makes essential use of it. Now, we introduce new variables $\{N\}$ as

$$N_I(p) = 2 i p_z A_I(p) + \frac{\partial W(\{A\})^*}{\partial A_I^*}(p), \tag{2.3}$$

which specify the Nicolai map from $\{A\}$ to $\{N\}$. Note that the fermion determinant coincides with the Jacobian associated with this mapping up to the sign. After eliminating $\{(\psi,\bar{\psi})\}$, the partition function is given by

$$\mathscr{Z}=\int\prod_{|p_\mu|\leq\pi}\prod_{I}[dN_I(p)dN_I^*(p)]\,e^{-S_B}\sum_k \operatorname{sign}\det\frac{\partial(\{N\},\{N^*\})}{\partial(\{A\},\{A^*\})}\Bigg|_{\{A\}=\{A\}_k},\tag{2.4}$$

where S_B is the bosonic part of the action, $S_B=(1/L_0L_1)\sum_p\sum_I N_I^*(-p)N_I(p)$, and $\{A\}_k$ ($k=1$, 2, $\dots$) is a set of solutions of eq. (2.3). The weight $\exp(-S_B)$ is a Gaussian function of the variables $\{N\}$. To obtain configurations of $\{N\}$ and $\{A\}$, we generate complex random numbers $\{N(p)\}$ for each p_μ from the Gaussian distribution, and then, solve numerically the algebraic equation (2.3) with respect to $\{A\}$.

The momentum-cutoff regularization, however, breaks the locality of the theory.[1] In the 2D *massive* WZ model, one can argue the restoration of the locality in the continuum limit within perturbation theory [10]. For the *massless* case, it is not clear whether the locality is automatically restored so far. We believe that our numerical results support the validity of the present formulation.

3. Numerical measurement of the central charge

In a 2D SCFT, the central charge c appears in the two-point function of the EMT

$$\langle T(p)T(-p)\rangle=L_0L_1\frac{\pi c}{12}\frac{p_z^3}{p_{\bar{z}}},\tag{3.1}$$

where the EMT, $T(p)=T_{zz}(p)$, is given in the momentum space by [11]

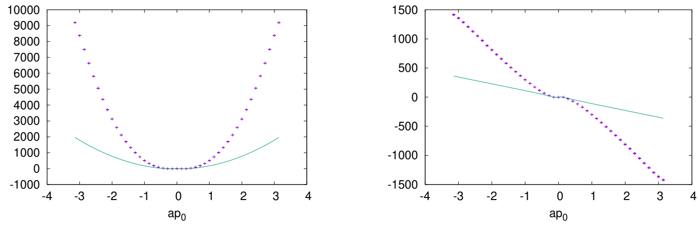
$$\begin{aligned}T(p)=\frac{\pi}{L_0L_1}\sum_q\sum_I\Big[&4(p-q)_zq_zA_I^*(p-q)A_I(q)\\&-iq_z\psi_{2I}(p-q)\bar{\psi}_{\dot{2}I}(q)+i(p-q)_z\psi_{2I}(p-q)\bar{\psi}_{\dot{2}I}(q)\Big].\end{aligned}\tag{3.2}$$

The IR behavior of the WZ model would be governed by relations as eq. (3.1) in SCFT. The central charge can be computed from the fit function (3.1) in the IR region.

Let us show the first main result of this paper, the numerical determination of the central charge in the A_2, A_3, D_3, D_4, and E_7 models, whose superpotentials are shown in table 1; for details of the computation, see [11, 12]. For the D_3-type theory with $L/a=44$, $a\lambda=0.3$ and $ap_1=\pi/22$, for example, we plot the correlation function $\langle T(p)T(-p)\rangle$ in figure 1 with the fitting curve (3.1); the central charge c is obtained from the fit in the IR region $2\pi/L\leq|p|<4\pi/L$. As is mentioned in [9, 11, 12], it is interesting to plot the "effective central charge," which changes as the function of $|p|=2\pi n/L$ with fitted momentum regions, $2\pi n/L\leq|p|<2\pi(n+1)/L$, for $n\in\mathbb{Z}_+$; it is analogous to the Zamolodchikov's c-function. Then figure 2 shows that the "effective central charge" connects the IR central charge to the UV one $c=3N_\Phi$ in the expected free $\mathcal{N}=2$ SCFT.

We tabulate the numerical results of the central charge for the maximal box size for each setup in table 2. These results are consistent with the expected values of the corresponding minimal models within the numerical errors. We have the numerical evidences of the following typical minimal models: the A_2, A_3, D_3, D_4, E_6 ($\cong A_2\otimes A_3$), and E_7-type theories.

[1] The present formulation is closely related to the 4D lattice formulation [14] based on the SLAC derivative [15, 16].



(a) Real part (b) Imaginary part

Figure 1: $\langle T(p)T(-p)\rangle$ for D_3, $L/a=44$, and $ap_1=\pi/22$. The fitting curve (3.1) is depicted at once.

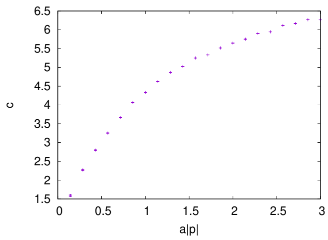


Figure 2: "Effective central charge" for D_3 and $L/a=44$.

Algebra	L/a	χ^2/d.o.f.	c	Expected value
A_2	36	1.017	1.061(36)(34)	1
A_3	30	0.916	1.415(36)(36)	1.5
D_3	44	3.598	1.595(31)(41)	1.5
D_4	42	1.177	2.172(48)(39)	2
E_7	24	1.364	2.638(47)(59)	2.666...

Table 2: The central charge obtained from the fit of the EMT correlator with the maximal box size for each setup. The fitted momentum range is $2\pi/L \leq |p| < 4\pi/L$. Numbers in the second parentheses indicate the systematic error associated with the finite-volume effect given in [11, 12].

4. Continuum-limit analysis of the scaling dimension

In the same way, we can also compute the scaling dimension $h+\bar{h}$ from the scalar correlator,

$$\langle A(x)A^*(0)\rangle = \frac{1}{z^{2h}\bar{z}^{2\bar{h}}}, \tag{4.1}$$

for large $|x|=\sqrt{x^2}$, where $z=x_0+ix_1$, $\bar{z}=x_0-ix_1$ and the conformal weights $(h,\bar{h})$ are supposed to meet the spinless condition $h=\bar{h}$; see table 3. It was found [11] that, although the measured scaling dimension tends to approach an expected value as the grid size L/a increases, the approach to the $L/a\to\infty$ limit appears not quit smooth. To obtain a result in the continuum and the infinite volume, we develop a systematic method of the continuum and thermodynamic limit. To do this, let us consider a numerical determination of the scaling dimension, which is the finite-size scaling analysis given in [8]. In this analysis, we observe the susceptibility of the scalar field A, defined by

$$\chi(L_\mu) = \frac{1}{a^2}\int_{L_0L_1} d^2x\,\langle A(x)A^*(0)\rangle = \frac{1}{a^2L_0L_1}\left\langle |A(p=0)|^2\right\rangle. \tag{4.2}$$

From the long-distance behavior (4.1), we have the finite-volume scaling of the scalar susceptibility for large L_μ, as $\chi\propto(L_0L_1)^{1-h-\bar{h}}$. Numerically simulating the scalar correlator for some different volumes, one can read the exponent, $1-h-\bar{h}$, from the slope of $\ln\chi(L_\mu)$ as a linear function of $\ln(L_0L_1)$. In what follows, for simplicity, we set the physical box size $L=L_0=L_1$,.

We develop this finite-volume scaling into an analysis method with the continuum limit [13]. In what follows, for simplicity, we consider the A_n-type LG model with the superpotential, $W(\Phi)=$

Algebra	L	χ^2/d.o.f.	$1-h-\bar{h}$	Expected value
A_2	36	0.506	0.682(10)(7)	0.666...
A_3	30	0.358	0.747(11)(12)	0.75

Table 3: Scaling dimension $1-h-\bar{h}$ obtained from the fit of the scalar correlator [11].

$\lambda\Phi^{n+1}/(n+1)$. Our strategy of the continuum limit is as follows: We regard $\ln\chi(L)$ as the same kind of the running coupling $\bar{g}^2(L)$ defined on a lattice [17]. The lattice parameter $a\lambda$ is tuned so that $\ln\chi(L)$ is kept fixed; we set $\ln\chi(L)=u$. Then, computing $\ln\chi(2L)$ for $2L/a$ and $a\lambda$, we observe the a-dependence of $\ln\chi(2L)|_a$; we denote $\Sigma(u,a/L)=\ln\chi(sL)|_a$, where the statistical error of Σ is defined by a square root of the sum of the squared errors of $\ln\chi(L)$ and $\ln\chi(2L)$. With a to-be-determined fit function, the scaling dimension is given by

$$1-h-\bar{h}=\frac{1}{\ln s^2}\left[\lim_{a/L\to 0}\Sigma(u,a/L)-u\right]. \tag{4.3}$$

To study the conformal behavior, note that the unique mass scale λ in the A-type theory should be sufficiently larger than $1/L$ [8], hence $\lambda L\to\infty$ as the continuum limit $a/L\to 0$. This implies that the above extrapolation method carries out the thermodynamic limit. One can apply our continuum-extrapolation method to other lattice formulations, e.g., that in [8].

Let us show the result of the precision measurement of the scaling dimension for the A_2-type theory with the cubic superpotential Φ^3; for details of the computation, see [13]. From table 4 in [13], we simply applies a linear function of a/L to eq. (4.3), then we have

$$1-h-\bar{h}=0.6699(77)(87), \tag{4.4}$$

with $\chi^2/\mathrm{d.o.f.}=1.417$. This is the second main result in this paper. Here, a number in the second parentheses indicates the systematic error defined by the deviation between this central value and a result with a slightly different fitted region; see [13]. This result is rather consistent with the expected exact value $1-h-\bar{h}=2/3=0.6666\ldots$ within the statistical error.

5. Conclusion

In this paper, we numerically studied the IR behavior of the 2D $\mathcal{N}=2$ WZ model corresponding to the ADE minimal models, by using the supersymmetry-preserving formulation with the momentum cutoff [10]. First, we numerically measured the central charge of various typical minimal models: A_2, A_3, D_3, D_4, E_6 ($\cong A_2\otimes A_3$), and E_7-type theories [11, 12]. Second, we gave the continuum-extrapolation method through the finite-size scaling to determine the scaling dimension; then, we performed the precision measurement of the scaling dimension [13]. Although the theoretical background of the formulation [10] is not clear so far, our results are consistent with the conjectured correspondence between the WZ model and the minimal series of SCFT, and thus, support the validity of the approach.

For a possible application of the present numerical approach to the Calabi–Yau compactification, the simulation of the LG theory which corresponds to the A_4 minimal model or a simpler non-minimal SCFT will be an important starting point.

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