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Higgs-boson, tau-lepton, and Z-boson decay rates in fourth order and the five-loop β function of QCD

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The Higgs-boson decay rates into $b\bar{b}$ and into gg have been evaluated in N^{4}LO, corresponding to order α_s^4 for $b\bar{b}$ and order α_s^6 for gg final states. After inclusion of the four-loop term, nice stabilization of the series is observed. In a similar context the predictions for the $\tau-$ and the Z-decay rate, as well as the R-ratio measured in electron-positron annihilation are presented in order α_s^4. Similar methods are employed for the evaluation of the beta function which governs the running of the quark-gluon coupling in quantum chromodynamics. The five-loop term of this fundamental quantity has been evaluated and the result has quickly been confirmed and even extended to a general gauge group. This five-loop term leads to a further reduction of the theory uncertainty in α_s, evaluated at the Z-boson or Higgs-boson scale, if originally extracted from τ-lepton decays and subsequently evolved to m_Z or m_Z.

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Higgs-boson decays

The two dominant decay modes of the Higgs boson are the decay into two gluons and the decay into $b\bar{b}$. With branching ratios of approximately 8% and 65% respectively these are the two most important channels. The decay rate into two gluons is given by [1]

$$\Gamma(H \to gg) = K\,\frac{G_F m_Z^3}{36\pi\sqrt{2}}\left(\frac{\alpha_s^{(n_l)}(m_Z)}{\pi}\right)^2, \tag{1}$$

$$\begin{aligned} K \;&=\; 1 + 17.9167\,a_s' + (156.81 - 5.7083\ln\frac{m_t^2}{m_Z^2})(a_s')^2 \\ &+\; (467.68 - 122.44\ln\frac{m_t^2}{m_Z^2} + 10.94\ln^2\frac{m_t^2}{m_Z^2})\,(a_s')^3 \\ &=\; 1 + 0.65038 + 0.20095 + 0.01825, \end{aligned} \tag{2}$$

where $m_t = 175$ GeV, $m_Z = 125$ GeV and $a_s' = \alpha_s^{(5)}(m_Z)/\pi = 0.0363$ has been adopted. The next term, proportional α_s^6 and corresponding to N^{4}LO can be found in [2].

The dominant decay channel of the Higgs boson is the one into bottom quarks with a rate given by

$$\Gamma(H \to b\bar{b}) = \frac{G_F m_Z}{4\sqrt{2}\pi} m_b^2 \tilde{R}(s = m_Z^2). \tag{3}$$

Here $\tilde{R}$ stands for the absorptive part of the scalar correlator [3]

$$\begin{aligned} \tilde{R} \;&=\; 1 + 5.6667 a_s + 29.147 a_s^2 + 41.758 a_s^3 - 825.7 a_s^4 \\ &=\; 1 + 0.2041 + 0.0379 + 0.0020 - 0.0014, \end{aligned} \tag{4}$$

where $a_s(m_Z) = \alpha_s(m_Z)/\pi = 0.0360$ and $m_Z = 125$ GeV has been adopted for the numerical evaluation. For the b quark mass we start from the input value

$$m_b(10\,\text{GeV}) = \left(3610 - \frac{\alpha_s - 0.1189}{0.02} 12 \pm 11\right)\,\text{MeV}, \tag{5}$$

and evolve to $m_Z = 125$ GeV, arriving [4] at a value

$$m_b(m_Z) = (2771 \pm 8|_{m_b} \pm 15_{\alpha_s})\,\text{MeV}.$$

Last not least there are four-loop corrections to the hadronic decay rate of the Higgs boson which are induced by effective couplings of the Higgs boson to bottom quarks and to gluons and which are mediated by the top quark. These terms have been evaluated to order α_s^4 in Ref. [5] and we refer to this paper for details.

Hadronic $Z-$ and $\tau-$decay rates and the R-ratio in order α_s^4

Similar methods have been employed for the evaluation of $\mathcal{O}(\alpha_s^4)$ corrections to the ratio $R = \sigma(e^+e^- \to hadrons)/\sigma(e^+e^- \to \mu^+\mu^-)$ at low energies, for the decay rate of the Z-boson and for the decay rate of the τ lepton into hadrons [6, 7]. These results have been recently confirmed by an independent calculation [2].

In total, one finds for the QCD corrected decay rate of the Z boson (neglecting for the moment mass suppressed terms of $\mathcal{O}(m_b^2/m_\mathrm{Z}^2)$ and electroweak corrections)

$$R^{\mathrm{nc}} = 3\Big[\sum_f v_f^2 r_{\mathrm{NS}}^V + \big(\sum_f v_f\big)^2 r_\mathrm{S}^V + \sum_f a_f^2 r_{\mathrm{NS}}^A + r_{\mathrm{S;t,b}}^A\Big]\,. \tag{6}$$

The relative importance of the different terms is best seen from the results of the various r-ratios introduced above. In numerical form [7]

$$\begin{aligned} r_{\mathrm{NS}} &= 1 + a_s + 1.4092\,a_s^2 - 12.7671\,a_s^3 - 79.9806\,a_s^4\,, \\ r_\mathrm{S}^V &= -0.4132\,a_s^3 - 4.9841\,a_s^4\,, \\ r_{\mathrm{S:t,b}}^A &= (-3.0833 + l_t)\,a_s^2 + (-15.9877 + 3.7222\,l_t + 1.9167\,l_t^2)\,a_s^3 \\ &\quad + (49.0309 - 17.6637\,l_t + 14.6597\,l_t^2 + 3.6736\,l_t^3)\,a_s^4\,, \end{aligned} \tag{7}$$

with $a_s = \alpha_\mathrm{s}(m_\mathrm{Z})/\pi$ and $l_t = \ln(m_\mathrm{Z}^2/m_\mathrm{t}^2)$. Using for the pole mass m_t the value 172 GeV, the axial singlet contribution in numerical form is given by

$$r_{\mathrm{S;t,b}}^A = -4.3524\,a_s^2 - 17.6245\,a_s^3 + 87.5520\,a_s^4\,. \tag{8}$$

Let us recall the basic aspects of these results:

- The non-singlet term dominates all different channels. It starts in Born approximation and is identical for τ decay, for $\sigma(e^+e^- \to \mathit{hadrons})$ through the vector current (virtual photon) and for $\Gamma(Z \to \mathit{hadrons})$ through vector and axial current.
- The singlet axial term starts in order α_s^2, is present in $Z \to \mathit{hadrons}$ and depends on $\ln(m_\mathrm{Z}^2/m_\mathrm{t}^2)$. Its origin is the strong imbalance between the masses of top and bottom quarks [8].
- The singlet vector term is present both in $\gamma^* \to \mathit{hadrons}$ and $Z \to \mathit{hadrons}$ and starts in $\mathcal{O}(\alpha_\mathrm{s}^3)$.
- All three terms are known up to order α_s^4 and the total rate is remarkably stable under scale variations.

The perturbative corrections to the τ decay rate can be obtained either from fixed order perturbation theory or with "Contour Improvement" [9, 10]. Within the two schemes one finds for the perturbative corrections [6]

$$\delta_0^{FO} = a_s + 5.202\,a_s^2 + 26.366\,a_s^3 + 127.079\,a_s^4, \tag{9}$$

$$\delta_0^{CI} = 1.364\,a_s + 2.54\,a_s^2 + 9.71\,a_s^3 + 64.29\,a_s^4. \tag{10}$$

Using the input discussed in [6], one obtains

$$\alpha_\mathrm{s}(m_\tau) = 0.332 \pm 0.005|_{exp} \pm 0.015|_{th}\,. \tag{11}$$

Applying four-loop running and matching this corresponds to

$$\alpha_\mathrm{s}(m_\mathrm{Z}) = 0.1202 \pm 0.0019 \tag{12}$$

nicely consistent with other determinations.

Five-Loop Running of the QCD Coupling Constant

Asymptotic freedom, manifested by a decreasing coupling with increasing energy, can be considered as the basic prediction of nonabelian gauge theories [11, 12]. The dominant, leading order prediction was quickly followed by the corresponding two-loop [13, 14] and three-loop [15, 16] results. The next, four-loop calculation was performed almost twenty years later [17] and confirmed in [18]. These results have moved the theory from qualitative agreement with experiment, as observed on the basis of the early results, to precise quantitative predictions, valid over a wide kinematic range, from τ-lepton decays up to LHC results.

There are, of course, a number of phenomenological applications of the five-loop result. On the one hand there is the relation between Z-boson and τ-lepton decay rates into hadrons, which involves the strong coupling at two vastly different scales. On the other hand there is the Higgs boson decay rate into bottom quarks and into gluons, which are sensitive to the five-loop running of the QCD coupling.

Let us start with the definition of the beta function

$$\beta(a_s) = \mu^2 \frac{d}{d\mu^2} a_s(\mu) = -\sum_{i\geq 0} \beta_i a_s^{i+2} \tag{13}$$

which describes the running of the quark-gluon coupling $a_s \equiv \alpha_s/\pi$ as a function of the normalization scale μ.

The QCD β-function in five-loop order reads [19, 20, 21]

$$\begin{aligned}
\beta_0 &= \frac{1}{4}\left\{11 - \frac{2}{3}n_f,\right\}, \quad \beta_1 = \frac{1}{4^2}\left\{102 - \frac{38}{3}n_f\right\}, \quad \beta_2 = \frac{1}{4^3}\left\{\frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2\right\}, \\
\beta_3 &= \frac{1}{4^4}\left\{\frac{149753}{6} + 3564\zeta_3 - \left[\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right] n_f + \left[\frac{50065}{162} + \frac{6472}{81}\zeta_3\right] n_f^2 + \frac{1093}{729}n_f^3\right\},
\end{aligned}$$

$$\begin{aligned}
\beta_4 = \frac{1}{4^5}\Bigg\{ & \frac{8157455}{16} + \frac{621885}{2}\zeta_3 - \frac{88209}{2}\zeta_4 - 288090\zeta_5 \\
+\ & n_f \left[-\frac{336460813}{1944} - \frac{4811164}{81}\zeta_3 + \frac{33935}{6}\zeta_4 + \frac{1358995}{27}\zeta_5\right] \\
+\ & n_f^2 \left[\frac{25960913}{1944} + \frac{698531}{81}\zeta_3 - \frac{10526}{9}\zeta_4 - \frac{381760}{81}\zeta_5\right] \\
+\ & n_f^3 \left[-\frac{630559}{5832} - \frac{48722}{243}\zeta_3 + \frac{1618}{27}\zeta_4 + \frac{460}{9}\zeta_5\right] \\
+\ & n_f^4 \left[\frac{1205}{2916} - \frac{152}{81}\zeta_3\right]\Bigg\},
\end{aligned}$$

where n_f denotes the number of active quark flavors. As expected from the three and four-loop results, the higher transcendentalities ζ_6 and ζ_7 that could be present at five-loop order are actually absent.

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The coefficients are surprisingly small. For example, for the particular cases of $n_f = 3,4,5,$ and 6 we get:

$$\begin{aligned}
\overline{\beta}(n_f = 3) &= 1 + 1.78\,a_s + 4.47\,a_s^2 + 20.99\,a_s^3 + 56.59\,a_s^4, \\
\overline{\beta}(n_f = 4) &= 1 + 1.54\,a_s + 3.05\,a_s^2 + 15.07\,a_s^3 + 27.33\,a_s^4, \\
\overline{\beta}(n_f = 5) &= 1 + 1.26\,a_s + 1.47\,a_s^2 + 9.83\,a_s^3 + 7.88\,a_s^4, \\
\overline{\beta}(n_f = 6) &= 1 + 0.93\,a_s - 0.29\,a_s^2 + 5.52\,a_s^3 + 0.15\,a_s^4,
\end{aligned}$$

where $\overline{\beta} \equiv \frac{\beta(a_s)}{-\beta_0 a_s^2} = 1 + \sum_{i\geq 1} \bar{\beta}_i a_s^i$.

At this point it may be useful to present the impact of the five-loop term on the running of the strong coupling from low energies, say $\mu = m_\tau$, up to the high energy region $\mu = m_Z$, by comparing the predictions based on three and four versus five-loop results [1]. We start from the scale of m_τ with $\alpha_s^{(3)}(m_\tau) = 0.33$ (as given in [23]) and evolve the coupling up to 3 GeV. At this point the four-loop matching from 3 to 4 flavours is performed. The strong coupling now runs up to $\mu = 10$ GeV and, at this point, the number of active quark flavours is switched from the 4 to 5. Subsequently, the strong coupling runs again up to m_Z and, finally, up to the Higgs mass $m_Z = 125$ GeV. The relevant values of α_s are listed in Table 1. The combined uncertainty in $\alpha_s^{(5)}(m_Z)$ induced by running and matching can be conservatively estimated by the shift in $\alpha_s^{(5)}(m_Z)$ produced by the use of five-loop running (and, consequently) four-loop matching instead of four-loop running (and three-loop matching). It amounts to a minute $6 \cdot 10^{-5}$ which is by a factor of three less than the similar shift made by the use of four-loop running instead of the three-loop one (see Table 1). Note that the final value of $\alpha_s^{(5)}(m_Z)$ which follows from $\alpha_s^{(3)}(m_\tau)$ is in remarkably good agreement with the fit to electroweak precision data (collected in Z boson decays), namely [24]:

$$\alpha_s^{(5)}(m_Z) = 0.1196 \pm 0.0030. \tag{14}$$

Table 1: Running of α_s from $\mu = m_\tau$ to $\mu = m_Z$. For the threshold values of c and b quarks we have chosen [25, 26] $m_c(3\ \text{GeV}) = 0.986$ GeV and $m_b(10\ \text{GeV}) = 3.160$ GeV respectively.

# of loops	$\alpha_s^{(3)}(m_\tau)$	$\alpha_s^{(5)}(m_Z)$	$\alpha_s^{(5)}(m_Z)$
3	0.33 ± 0.014	0.1200 ± 0.0016	0.1145 ± 0.0014
4	0.33 ± 0.014	0.1199 ± 0.0016	0.1143 ± 0.0014
5	0.33 ± 0.014	0.1198 ± 0.0016	0.1143 ± 0.0014

Thus, exact result for the five-loop term of the QCD β-function allows to relate the strong coupling constant α_s, as determined with N^{3}LO accuracy at low energies, say m_τ with the strong coupling as evaluated at high scales, say m_Z or m_H. Including the exact five-loop term has little influence on the central value of the prediction, a consequence of partial cancellations between various contributions from matching and running. However, the five-loop result leads to a considerable further reduction of the theory uncertainty and allows to combine values from low and high energies of appropriate order.

[1] For all practical examples in this paper we have used an extended version of the package RunDec [22].

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