

L_∞ -algebras and membrane sigma models

Clay J. Grewcoe, Larisa Jonke*

Division of Theoretical Physics, Rudjer Bošković Institute,

Bijenička cesta 54, Zagreb, Croatia

E-mail: cgrewcoe@irb.hr, larisa@irb.hr

Membrane sigma-models have been used for the systematic description of closed strings in non-geometric flux backgrounds. In particular, the conditions for gauge invariance of the corresponding action functionals were related to the Bianchi identities for the fluxes. In this contribution we demonstrate how to express these Bianchi identities in terms of homotopy relations of the underlying L_∞ -algebra for the case of the Courant sigma-model. We argue that this result can be utilized in understanding the constraint structure of Double Field Theory and the corresponding membrane sigma-model.

*Corfu Summer Institute 2019 "School and Workshops on Elementary Particle Physics and Gravity"
(CORFU2019)*
31 August - 25 September 2019
Corfu, Greece

*Speaker.

1. Motivation and introduction

Our understanding of physics in quantum gravity regime is far from complete, but there are certain insights providing basis for further advances. In particular, to resolve the geometry of space-time in that regime one should use extended probes instead of point particles. This also implies that the concept of symmetries should be generalized, and more often than not, that the field content needs to be extended. In various concrete settings these observations were made precise which lead to many interesting developments in e.g. string theory, matrix/tensor models, and higher-spin (gravity) theories. In this contribution we would like to review part of the recent progress in the understanding of generalized symmetries relevant in Double Field Theory (DFT).

DFT is a proposal to incorporate T-duality symmetry of string theory as a symmetry of a field theory defined on the double configuration space [1, 2, 3]. Strings being extended objects can, unlike point particles, wind around non-contractible cycles. When compactified on toroidal backgrounds, string theory enjoys T-duality symmetry under the interchange of momentum and winding modes. String backgrounds that are T-dual to each other may correspond to target spaces with different geometry and topology. Moreover, a duality transformation can result in unconventional closed string geometries where string duality transformations are required as transition functions, and lead to non-geometric flux backgrounds, see Ref.[4] for a review and references therein. In order to account for this duality symmetry DFT is formulated on a configuration space extended by a set of dual coordinates conjugate to winding modes of a string, and it provided a natural setting for the description of non-geometric string backgrounds. However, the theory is constrained in the sense that, although all coordinates are doubled, the physical fields and parameters depend only on half of them. The strong constraint¹ makes DFT a well-defined theory, but limits its physical applications as we shall argue latter on.

The symmetries of DFT correspond to generalized diffeomorphisms, which combine standard diffeomorphisms with gauge transformations of a 2-form Kalb-Ramond field. These can naturally be described in the framework of generalized geometry defined on a doubled bundle [5]. Taking locally $TM \oplus T^*M$ as a doubled bundle over a target space manifold M one can define a skew-symmetric bracket on its sections known as Courant bracket, symmetric pairing that corresponds to the $O(d,d)$ metric and a map from the bundle to TM . These data, accompanied by a number of compatibility conditions, define a Courant algebroid (CA) [6, 7, 8]. On the solution of the strong constraint the relevant bracket of the symmetry algebra of DFT reduces to the Courant bracket. Furthermore, one can twist the Courant bracket by a generalized 3-form which can be related to (non-geometric) fluxes in DFT [9]. A particularly nice way to describe these fluxes in a systematic manner is by using the membrane sigma model [10, 11, 12, 13]. The starting point is a three dimensional topological model known as the Courant sigma model, which can be uniquely defined using CA data. One can show that the conditions for gauge invariance of this sigma model produce relations for background fluxes and their Bianchi identities, again on a given solution of the strong constraint. A membrane sigma model for DFT was proposed in Ref.[14], where it was demonstrated how the standard T-duality orbit of geometric and non-geometric flux backgrounds is

¹The weak constrained of DFT corresponds to level-matching condition in string theory and presents no additional constraint of the theory.

captured by its action functional in a unified way. However, this membrane sigma model is gauge invariant only after imposing the strong constraint.

We propose that in order to obtain a better insight into the interplay between the gauge symmetry of sigma models, T-duality symmetry and the strong constraint we analyse the aforementioned sigma models in L_∞ -algebra framework. L_∞ -algebras are generalizations of Lie algebras with infinitely-many higher brackets, related to each other by higher homotopy versions of the Jacobi identity [15, 16]. There are two aspects of an L_∞ -algebra structure that are relevant here. First, one can think of an L_∞ -algebra as a geometric structure corresponding to the BV-BRST complex. In that respect it is not surprising that one can use this framework to construct consistent deformations of a given (gauge-invariant, perturbative) theory, see e.g.[17]. Moreover, as one can incorporate both symmetries and dynamics in an L_∞ structure, even for non-Lagrangian theories, one hopes to clearly identify the role of constraints.

The first step toward this goal is to recast the Courant sigma model in terms of an L_∞ -algebra. This was done in Ref.[18] and here we shall review parts of this construction. In the next section we shall remind the reader of the rich gauge structure of the Courant sigma model. After a short introduction to L_∞ -algebras for classical field theories we shall formulate the model in terms of L_∞ -algebra(s). Construction of an improved DFT membrane sigma model is still work in progress, so we shall close this contribution with some insight obtained so far [19].

2. Courant sigma model

The Courant sigma model was constructed by examining consistent BV deformations of the abelian Chern-Simons gauge theory coupled with a BF theory in Ref. [20]. Related work in the context of open topological membranes can be found in Refs.[21, 22]. The model was also obtained in the framework of AKSZ construction of topological models satisfying the classical master equation [23]. This result is based on a theorem stating that there is a one-to-one correspondence between Courant algebroids and QP2-manifolds [24]. Specializing the general AKSZ construction to three worldvolume dimensions, this essentially means that given the data of a Courant algebroid one can construct a unique three dimensional sigma model, known as Courant sigma model [25].

The classical action functional for this three-dimensional topological field theory is given as:

$$S[X, A, F] = \int_{\Sigma_3} F_i \wedge dX^i + \frac{1}{2} \eta_{IJ} A^I \wedge dA^J - \rho^i{}_I(X) A^I \wedge F_i + \frac{1}{6} T_{IJK}(X) A^I \wedge A^J \wedge A^K. \quad (2.1)$$

where $i = 1, \dots, d$ is a target space index, $I = 1, \dots, 2d$ is the pull-back bundle index. The field content is made up of: scalar fields considered as components of maps $X = (X^i) : \Sigma_3 \rightarrow M$, 1-forms $A \in \Omega^1(\Sigma_3, X^* E)$, and an auxiliary 2-form $F \in \Omega^2(\Sigma_3, X^* T^* M)$. Fields X^i are identified with the pull-backs of the coordinate functions, $X^i = X^*(x^i)$ on the target manifold M . Here, η is the $O(d, d)$ -invariant constant metric:

$$\eta = (\eta_{IJ}) = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}, \quad (2.2)$$

and $\rho^i{}_J(X)$ and $T_{IJK}(X)$ are functions corresponding to the anchor map and the twist of the Courant algebroid. The last term in the action (2.1) can be seen to generate a generalized Wess-Zumino term

(WZ). For example, choosing $\rho^i{}_j = \delta^i{}_j$, $\rho^{ij} = 0$ and $T_{IJK} = H_{ijk}$, $A^I = (q^i, p_j)$ produces:

$$S[X, A, F] = \int_{\Sigma_3} F_i \wedge (\mathrm{d}X^i - q^i) + q^i \wedge \mathrm{d}p_i + \frac{1}{6} H_{ijk} q^i \wedge q^j \wedge q^k.$$

On-shell, it is the topological sector of the string with a WZ-term.

Note that for a manifold with boundaries one can add both topological and non-topological terms [26, 21]:

$$S_b[X, A] = \int_{\partial\Sigma_3} \frac{1}{2} g_{IJ} A^I \wedge *A^J + \frac{1}{2} \mathcal{B}_{IJ} A^I \wedge A^J.$$

These boundary terms were used in the description of T-dual backgrounds² in Ref.[14], but in this contribution we shall assume a vanishing boundary.

The action (2.1) is invariant under the following infinitesimal gauge transformations [20] :

$$\begin{aligned} \delta_\varepsilon X^i &= \rho^i_J \varepsilon^J, \\ \delta_\varepsilon A^I &= \mathrm{d}\varepsilon^I + \eta^{IN} T_{NJK} A^J \varepsilon^K - \eta^{IJ} \rho^i{}_J t_i, \\ \delta_\varepsilon F_m &= -\mathrm{d}t_m - \partial_m \rho^j{}_J A^J \wedge t_j - \varepsilon^J \partial_m \rho^i{}_J F_i + \frac{1}{2} \varepsilon^J \partial_m T_{ILJ} A^I \wedge A^L, \end{aligned} \quad (2.3)$$

where ε^I and t_i are scalar and 1-form gauge parameters, respectively, provided:

$$\begin{aligned} \eta^{JK} \rho^i{}_J(X) \rho^j{}_K(X) &= 0, \\ 2\rho^m{}_{[K}(X) \partial_{m]} \rho^i{}_{J]}(X) - \rho^i{}_N \eta^{NM} T_{MKJ}(X) &= 0, \\ 3\eta^{MN} T_{M[JK}(X) T_{I]LN}(X) + 3\rho^m{}_{[I}(X) \partial_m T_{KJ]L}(X) + \rho^m{}_L \partial_m T_{IJK}(X) &= 0. \end{aligned} \quad (2.4)$$

Furthermore, the closure of the algebra of gauge transformations gives:

$$\begin{aligned} [\delta_1, \delta_2] X^i &= \rho^i{}_J \varepsilon_{12}^J, \quad \varepsilon_{12}^I = \eta^{IJ} T_{JKL} \varepsilon_1^K \varepsilon_2^L, \\ [\delta_1, \delta_2] A^I &= \delta_{12} A^I - \eta^{IJ} \partial_m T_{JKL} \varepsilon_1^K \varepsilon_2^L \mathcal{D}X^m, \quad t_{12i} = \partial_i T_{KLJ} \varepsilon_1^K \varepsilon_2^L A^J + 2\partial_i \rho^j{}_K \varepsilon_{[1}^K t_{2]j}, \end{aligned} \quad (2.5)$$

where $\mathcal{D}X^m := \mathrm{d}X^i - \rho^i{}_J A^J = 0$ on F equation of motion. We see that the gauge transformations (2.3) define a first-stage reducible gauge symmetry, typical for gauge theories that include differential forms with degree larger than one [28, 29]. Moreover, the algebra of gauge transformations closes only on-shell. In the following we shall reformulate CSM with its rich gauge structure as an L_∞ -algebra.

3. Courant sigma model as L_∞ -algebra

The gauge symmetry of the Courant sigma model is based on an open algebra, i.e., on an algebra that closes only on-shell. In order to properly define a CSM action functional, one should extend the classical action by additional fields (ghost etc.) into the BV-BRST framework [20, 25]. The natural algebraic structure underlying the BV-BRST construction is in fact an L_∞ -algebra, see [15, 16, 30]. In Ref.[18] we have constructed explicitly the L_∞ -algebra for BV CSM, but in this short note we restrict ourselves to classical theory. We shall start this section with a short overview of L_∞ -algebras following Refs.[31, 30] and then exemplify the general construction on the Courant sigma model following [18].

²Similarly, one can use appropriate boundary conditions for on-shell fields [27].

3.1 L_∞-algebra for classical field theories

A L_∞-algebra or strong homotopy Lie algebra (L, μ_i) is a graded vector space L with a collection of graded totally antisymmetric multilinear maps:

$$\mu_i : \underbrace{L \times \cdots \times L}_{i\text{-times}} \rightarrow L .$$

of degree 2 – i which satisfy the homotopy Jacobi identities:

$$\sum_{j+k=i} \sum_{\sigma} \chi(\sigma; l_1, \dots, l_i) (-1)^k \mu_{k+1}(\mu_j(l_{\sigma(1)}, \dots, l_{\sigma(j)}), l_{\sigma(j+1)}, \dots, l_{\sigma(i)}) = 0 ; \quad (3.1)$$

$$\forall i \in \mathbb{N} \quad \forall l_i \in L .$$

where χ(σ; l₁, …, l_i) is the graded Koszul sign that includes the sign from the parity of the permutation σ ordered as: σ(1) < … < σ(j) and σ(j+1) < … < σ(i). Let us first examine a few relations resulting from the general expression (3.1). Taking the free summation index i = 1 we get μ₁²(l) = 0, i.e., the map μ₁ is differential on L. For i = 2 we have:

$$\mu_1(\mu_2(l_1, l_2)) = \mu_2(\mu_1(l_1), l_2) + (-1)^{1+|l_1||l_2|} \mu_2(\mu_1(l_2), l_1) , \quad (3.2)$$

where |l_i| is the L_∞-degree of l_i, specifying that μ₁ is a derivation of (graded) bracket μ₂. Finally, for i = 3 we get the generalized Jacobi identity for bracket μ₂ controlled by μ₃:

$$\begin{aligned} \mu_1(\mu_3(l_1, l_2, l_3)) &= \mu_2(\mu_2(l_1, l_2), l_3) - (-1)^{|l_2||l_3|} \mu_2(\mu_2(l_1, l_3), l_2) + \\ &\quad + (-1)^{|l_1|(|l_2|+|l_3|)} \mu_2(\mu_2(l_2, l_3), l_1) - \mu_3(\mu_1(l_1), l_2, l_3) + \\ &\quad + (-1)^{|l_1||l_2|} \mu_3(\mu_1(l_2), l_1, l_3) - (-1)^{|l_3|(|l_1|+|l_2|)} \mu_3(\mu_1(l_3), l_1, l_2) , \end{aligned} \quad (3.3)$$

Standard examples include a Lie algebra with μ₂ being the Lie bracket and all other maps vanishing, and a differential graded Lie algebra obtained for μ_i = 0 for i ≥ 3.

Elements of graded vector space L = ⊕_i L_i form a cochain complex as μ₁ is a differential:

$$\dots \xrightarrow{\mu_1} L_i \xrightarrow{\mu_1} L_{i+1} \xrightarrow{\mu_1} \dots$$

In order to describe classical field theory we associate with each subspace a particular field content, including fields, gauge parameters, equations of motion and Noether identities.

$$\dots \xrightarrow{\mu_1} L_{-1} \xrightarrow{\mu_1} L_0 \xrightarrow{\mu_1} L_1 \xrightarrow{\mu_1} L_2 \xrightarrow{\mu_1} L_3 \xrightarrow{\mu_1} \dots$$

h. gauge gauge physical eoms Noether
parameters parameters fields identities

Take a ∈ L₁ to be the gauge potential. The corresponding curvature is:

$$f \equiv \mu_1(a) + \frac{1}{2} \mu_2(a, a) + \dots = \sum_{i \geq 1} \frac{1}{i!} \mu_i(a, \dots, a) .$$

If f = 0 we obtain the generalized Maurer-Cartan equation. Next, for c₀ ∈ L₀ being the level zero gauge parameter, we can write gauge transformations as

$$\begin{aligned} \delta_{c_0} a &= \sum_{i \geq 0} \frac{1}{i!} \mu_{i+1}(a, \dots, a, c_0) , \\ \delta_{c_0} f &= \sum_{i \geq 0} \frac{1}{i!} \mu_{i+2}(a, \dots, a, f, c_0) . \end{aligned} \quad (3.4)$$

If a theory contains higher gauge symmetries we will have higher (level k) gauge parameters $c_{-k} \in \mathsf{L}_{-k}$, $k > 0$, with infinitesimal gauge transformations given by:

$$\delta_{c_{-k-1}} c_{-k} = \sum_{i \geq 0} \frac{1}{i!} \mu_{i+1}(a, \dots, a, c_{-k-1}) .$$

The algebra of gauge transformations:

$$[\delta_{c_0}, \delta_{c'_0}]a = \delta_{c''_0}a + \sum_{i \geq 0} \frac{1}{i!} (-1)^i \mu_{i+3}(f, a, \dots, a, c_0, c'_0) ,$$

with:

$$c''_0 = \sum_{i \geq 0} \frac{1}{i!} \mu_{i+2}(a, \dots, a, c_0, c'_0) .$$

closes up to $f = 0$, which can be realized either as a constraint or as an equation of motion, the latter resulting in an open algebra. However, to implement $f = 0$ as a variational equation of motion resulting from an action functional, we need additional input – a cyclic pairing satisfying a certain compatibility condition. In that case we define a cyclic L_∞ -algebra as an L_∞ -algebra endowed with a graded symmetric non-degenerate bilinear pairing $\langle \cdot, \cdot \rangle_{\mathsf{L}} : \mathsf{L} \times \mathsf{L} \rightarrow \mathbb{R}$ that satisfies the cyclicity condition:

$$\langle l_1, \mu_i(l_2, \dots, l_{i+1}) \rangle_{\mathsf{L}} = (-1)^{i+i(|l_1|+|l_{i+1}|)|l_{i+1}|} \sum_{j=1}^i |l_j| \langle l_{i+1}, \mu_i(l_1, \dots, l_i) \rangle_{\mathsf{L}} .$$

Then one can define an action whose stationary point is the Maurer-Cartan equation:

$$S_{\text{MC}}[a] \equiv \sum_{i \geq 1} \frac{1}{(i+1)!} \langle a, \mu_i(a, \dots, a) \rangle_{\mathsf{L}} . \quad (3.5)$$

As a conclusion of this short summary of L_∞ -structures we introduce an important class of L_∞ -algebras induced by the tensor product of an L_∞ -algebra with a differential graded commutative algebra. In the following sections we shall use an L_∞ -algebra obtained as a tensor product of the de Rham complex on a manifold M , $(\Omega^\bullet(M), d)$ with an L_∞ -algebra:

$$\mathsf{L}' \equiv \Omega^\bullet(M, \mathsf{L}) \equiv \bigoplus_{k \in \mathbb{Z}} \Omega_k^\bullet(M, \mathsf{L}), \quad \Omega_k^\bullet(M, \mathsf{L}) \equiv \bigoplus_{i+j=k} \Omega^i(M) \otimes \mathsf{L}_j ,$$

where the higher products are defined as:

$$\begin{aligned} \mu'_1(\alpha_1 \otimes l_1) &= d\alpha_1 \otimes l_1 + (-1)^{|\alpha_1|} \alpha_1 \otimes \mu_1(l_1) , \\ \mu'_i(\alpha_1 \otimes l_1, \dots, \alpha_i \otimes l_i) &= (-1)^{i \sum_{j=1}^i |\alpha_j| + \sum_{j=0}^{i-2} |\alpha_{i-j}| \sum_{k=1}^{i-j-1} |l_k|} (\alpha_1 \wedge \dots \wedge \alpha_i) \otimes \mu_i(l_1, \dots, l_i) , \\ &\forall i \geq 2, \quad \alpha_1, \dots, \alpha_i \in \Omega^\bullet(M), \quad l_1, \dots, l_i \in \mathsf{L} . \end{aligned} \quad (3.6)$$

This tensored algebra (L', μ') is cyclic provided (L, μ) is cyclic and M is an oriented, compact cycle.

3.2 L_∞ -algebra for Courant sigma model

In the following we shall apply the above formalism to cast the Courant sigma model into an L_∞ structure. In action (2.1) we first identify (X, A, F) as physical fields, then perturbatively expand structure functions $\rho(X)$ and $T(X)$ around $X = 0$, resulting in an action with an infinite number of interaction terms:³

$$\begin{aligned} S[X, A, F] = & \int_{\Sigma_3} F_i \wedge dX^i + \frac{1}{2} \eta_{IJ} A^I \wedge dA^J - \rho^i{}_I A^I \wedge F_i + \frac{1}{6} T_{IJK} A^I \wedge A^J \wedge A^K - \\ & - X^{i_1} \partial_{i_1} \rho^i{}_I A^I \wedge F_i - \dots - \frac{1}{n!} X^{i_1} \dots X^{i_n} \partial_{i_1} \dots \partial_{i_n} \rho^i{}_I A^I \wedge F_i - \dots + \\ & + \frac{1}{12} X^{i_1} X^{i_2} \partial_{i_1} \partial_{i_2} T_{IJK} A^I \wedge A^J \wedge A^K + \dots + \\ & + \frac{1}{6 \cdot n!} X^{i_1} \dots X^{i_n} \partial_{i_1} \dots \partial_{i_n} T_{IJK} A^I \wedge A^J \wedge A^K + \dots . \end{aligned} \quad (3.7)$$

In Sect. 2 we have seen that the gauge symmetry is mediated by two gauge parameters, one scalar ε and one 1-form t . Although not exemplified at the classical level, we also have a higher gauge parameter v of level 1, which mediates the transformation of level zero parameters. Therefore, the complete (classical) field content is:

$$\begin{aligned} a &= X + A + F \in \Omega^0(M, L_1) \oplus \Omega^1(M, L_0) \oplus \Omega^2(M, L_{-1}) , \\ c_0 &= \varepsilon + t \in \Omega^0(M, L_0) \oplus \Omega^1(M, L_{-1}) , \\ c_{-1} &= v \in \Omega^0(M, L_{-1}) , \end{aligned}$$

Introducing notation $\mathcal{D}E$ for the equation of motion of physical field $E = (X, A, F)$ we can represent the content of graded vector space L in the following table.

L'_{-1}	$\xrightarrow{\mu'_1}$	L'_0	$\xrightarrow{\mu'_1}$	L'_1	$\xrightarrow{\mu'_1}$	L'_2	$\xrightarrow{\mu'_1}$	L'_3
h. gauge parameters		gauge parameters		physical fields		equations of motion		Noether identities
L_{-1}	v_i	t_i		F_i		$\mathcal{D}F_i$		
$\mu_1 \downarrow$								
L_0		ε^I		A^I		$\mathcal{D}A^I$		
$\mu_1 \downarrow$				X^i		$\mathcal{D}X^i$		
L_1								

Once we have placed the fields in their appropriate homogeneous subspaces, we must define all the

³We introduce the shorthand $f(0) \equiv f$ and $\partial f|_0 \equiv \partial f$ for any function f of X evaluated at 0, and when the full function is meant the argument will be explicitly written.

products. Our selection for the nonvanishing higher products of L is:

$$\begin{aligned} \mathsf{L}_1 \ni & \quad \mu_n(l_{(1)1}, \dots, l_{(1)n-1}, l_{(0)}) = l_{(1)1}^{i_1} \cdots l_{(1)n-1}^{i_{n-1}} \partial_{i_1} \cdots \partial_{i_{n-1}} \rho^i l_{(0)}^I, \\ \mathsf{L}_0 \ni & \quad \mu_n(l_{(1)1}, \dots, l_{(1)n-1}, l_{(-1)}) = -l_{(1)1}^{i_1} \cdots l_{(1)n-1}^{i_{n-1}} \partial_{i_1} \cdots \partial_{i_{n-1}} \rho^i J l_{(-1)i} \eta^{IJ}, \\ \mathsf{L}_{-1} \ni & \quad \mu_m(l_{(1)1}, \dots, l_{(1)m-2}, l_{(-1)}, l_{(0)}) = -l_{(1)1}^{i_1} \cdots l_{(1)m-2}^{i_{m-2}} \partial_{i_1} \cdots \partial_{i_{m-2}} \partial_i \rho^j J l_{(-1)j} l_{(0)}^I, \\ \mathsf{L}_0 \ni & \quad \mu_m(l_{(1)1}, \dots, l_{(1)m-2}, l_{(0)1}, l_{(0)2}) = l_{(1)1}^{i_1} \cdots l_{(1)m-2}^{i_{m-2}} \partial_{i_1} \cdots \partial_{i_{m-2}} T_{JKL} l_{(0)1}^K l_{(0)2}^L \eta^{IJ}, \\ \mathsf{L}_{-1} \ni & \quad \mu_r(l_{(1)1}, \dots, l_{(1)r-3}, l_{(0)1}, l_{(0)2}, l_{(0)3}) = l_{(1)1}^{i_1} \cdots l_{(1)r-3}^{i_{r-3}} \partial_{i_1} \cdots \partial_{i_{r-3}} \partial_i T_{IJK} l_{(0)1}^I l_{(0)2}^J l_{(0)3}^K, \end{aligned} \quad (3.8)$$

where $n \geq 1$, $m \geq 2$ and $r \geq 3$, and $l_{(i)} \in \mathsf{L}_i$. We defined the products by comparing the general expression for the MC action (3.5) with (3.7), but they can also be obtained from the (expanded) homological vector Q [23, 32] defined for a Courant algebroid described as a QP2-manifold in [25]. The products for tensored algebra (L', μ'_i) can be easily deduced from the general expression (3.6) and the above expressions for (L, μ_i) . Using expressions (3.8) for products one can recast the gauge transformation (2.3) of the Courant sigma model in general form (3.4).

Next, we define a consistent inner product:

$$\langle l_{(0)1}, l_{(0)2} \rangle \equiv \eta_{IJ} l_{(0)1}^I l_{(0)2}^J, \quad \langle l_{(1)}, l_{(-1)} \rangle \equiv l_{(1)}^i l_{(-1)i}, \quad \langle l_{(-1)}, l_{(1)} \rangle \equiv -l_{(-1)}^i l_{(1)i}, \quad (3.9)$$

resulting in the MC action:

$$\begin{aligned} S_{\text{MC}}[X, A, F] = & \langle dX, F \rangle + \frac{1}{2} \langle A, dA \rangle + \sum_{n \geq 0} \frac{1}{n!} \langle A, \mu_{n+1}(X, \dots, X, F) \rangle + \\ & + \frac{1}{6} \sum_{n \geq 0} \frac{1}{n!} \langle A, \mu_{n+2}(X, \dots, X, A, A) \rangle. \end{aligned} \quad (3.10)$$

Note that the physical field A is a section of the pull-back bundle X^*E where there is no natural structure defined; in particular the bracket of sections is not the Courant bracket. Thus one can think of L_∞ -products as defining relations for the relevant structures on sections of the pull-back bundle. Likewise, the inner product (3.9) defines a pairing on the pull-back bundle X^*E .

The equations of motions are obtained directly from the action (3.10):

$$\begin{aligned} f_1 &= dX - \sum_{n \geq 0} \frac{1}{n!} \mu_{n+1}(X, \dots, X, A), \\ f_0 &= dA + \sum_{n \geq 0} \frac{1}{n!} \mu_{n+1}(X, \dots, X, F) + \frac{1}{2} \sum_{n \geq 0} \frac{1}{n!} \mu_{n+2}(X, \dots, X, A, A), \\ f_{-1} &= dF - \sum_{n \geq 0} \frac{1}{n!} \mu_{n+2}(X, \dots, X, F, A) - \frac{1}{3!} \sum_{n \geq 0} \frac{1}{n!} \mu_{n+3}(X, \dots, X, A, A, A). \end{aligned}$$

With the aid of (3.8) it becomes obvious these correspond to the equations of motion of action (2.1) or (3.7):

$$\begin{aligned} \mathcal{D}X^i &= dX^i - \rho^i{}_J(X)A^J, \\ \mathcal{D}A^I &= dA^I - \eta^{IJ} \rho^i{}_J(X)F_i + \frac{1}{2} \eta^{IJ} T_{JKL}(X)A^K \wedge A^L, \\ \mathcal{D}F_i &= dF_i + \partial_i \rho^j{}_J(X)A^J \wedge F_j - \frac{1}{3!} \partial_i T_{IJK}(X)A^I \wedge A^J \wedge A^K. \end{aligned} \quad (3.11)$$

Finally, we need to check homotopy Jacobi identities (3.1) of products (3.8). As explicitly shown in [18], there exist 7 non-trivial identities of which 3 are independent. These three sets of homotopy conditions for the higher products are actually all terms in the Taylor expansions of classical gauge invariance conditions (2.4) and they belong, as expected, to the space L'_3 of Noether identities.

4. On the relation between CA and DFT

As already mentioned in the introduction, the DFT symmetry algebra reduces to the one of a CA on the solutions of strong constraint. However, to understand the geometric origin of DFT data and strong constraint itself, one has to go a step further. Starting with the observation that in DFT one doubles the target space, while a CA is defined on a doubled bundle, in Ref.[14] it was proposed that one should construct a ‘large’ CA defined over $2d$ dimensional manifold and then recover DFT data by suitable projection. One starts from a CA structure defined on a bundle $\mathbb{E} = (T \oplus T^*)\mathcal{M}$ over the doubled configuration space \mathcal{M} with sections $\mathbb{A} \in \mathbb{E}$

$$\mathbb{A} := \mathbb{A}_V + \mathbb{A}_F = \mathbb{A}^I \partial_I + \tilde{\mathbb{A}}_I d\mathbb{X}^I, I = 1, \dots, 2d.$$

Introducing an invariant $O(d, d)$ metric⁴ one can define the following splitting: $\mathbb{E} = (T \oplus T^*)\mathcal{M} = L_+ \oplus L_-$;

$$\mathbb{A} = \mathbb{A}_+^I e_I^+ + \mathbb{A}_-^I e_I^-, \quad \text{where} \quad e_I^\pm = \partial_I \pm \eta_{IJ} d\mathbb{X}^J.$$

Using the projection:

$$p : \mathbb{E} \rightarrow L_+ ; \quad (\mathbb{A}_V, \mathbb{A}_F) \mapsto \mathbb{A}_+ := A,$$

one can systematically project the bracket, bilinear and structure functions (anchor map and twist) of the large CA to obtain DFT vectors, the C-bracket and generalized Lie derivative with properties defining a DFT algebroid. Using the data of this DFT algebroid a membrane sigma model was proposed:

$$S[\mathbb{X}, A, F] = \int_{\Sigma_3} \left(F_I \wedge d\mathbb{X}^I + \eta_{IJ} A^I \wedge dA^J - \hat{\rho}^I{}_J(\mathbb{X}) A^J \wedge F_I + \frac{1}{3} \hat{T}_{IJK}(\mathbb{X}) A^I \wedge A^J \wedge A^K \right),$$

where one introduced maps $\mathbb{X} = (\mathbb{X}^I) : \Sigma_3 \rightarrow \mathcal{M}$, 1-form $A \in \Omega^1(\Sigma_3, \mathbb{X}^* L_+)$, and an auxiliary 2-form $F \in \Omega^2(\Sigma_3, \mathbb{X}^* T^*\mathcal{M})$. The structure functions $\hat{\rho}$ and \hat{T} obtained through the projection were then related to DFT fluxes and their Bianchi identities using the flux formulation of DFT [33]. This was realized by parametrizing the projected anchor map ρ_+ in terms of the generalized bein components of DFT. The identification imposes:⁵

$$\eta^{IJ} \hat{\rho}^K{}_I \hat{\rho}^L{}_J = \eta^{KL}, \tag{4.1}$$

and reveals the source of the strong constraint in DFT as coming from constraining the generalized bein of DFT to be an element of $O(d, d)$. Further consistency conditions defining the DFT

⁴This metric should not be confused with $O(2d, 2d)$ invariant metric on the large Courant algebroid.

⁵Additionally, it implies the helpful identity $\eta_{KL} \hat{\rho}^K{}_I \hat{\rho}^L{}_J = \eta_{IJ}$, heavily used in calculations.

algebroid:

$$\begin{aligned} 2\widehat{\rho}^L_{[I}\partial_L\widehat{\rho}^K_{J]} - \widehat{\rho}^K_M\eta^{MN}\widehat{T}_{NIJ} &= \eta_{LL'}\widehat{\rho}^L_{[I}\partial^K\widehat{\rho}^{L'}_{J]} , \\ 3\eta^{MN}\widehat{T}_{M[IJ}\widehat{T}_{KL]N} - 4\widehat{\rho}^M_{[I}\partial_M\widehat{T}_{JKL]} &= 3\eta_{MM'}\eta_{NN'}\eta^{PP'}\widehat{\rho}^M_{[I}\partial_P\widehat{\rho}^{M'}_{J}\widehat{\rho}^N_{K}\partial_{P'}\widehat{\rho}^{N'}_{L]} , \end{aligned} \quad (4.2)$$

were used to demonstrate that one should use the strong constraint to guarantee the gauge invariance of the DFT sigma model and (on-shell) closure of the gauge algebra.

The constructed membrane sigma model allows for the description of a standard T-duality orbit of geometric and non-geometric flux backgrounds in a unified way [14]. However, the necessity to enforce strong constraint limits its physical applications. As mentioned in the introduction, solving the strong constraint effectively reduces physical field dependence to only half of the coordinates of doubled space. For example, when the compactification scale is much bigger than the string size, the winding modes are heavy and at low energies this corresponds to the usual situation where there is no dependence on dual coordinates. Oppositely, in the T-dual description, if the compactification scale is small, then the momentum modes are heavy, and DFT only depends on dual coordinates. However, in order to describe the process of decompactification one should have a consistent description that includes both momentum and winding modes/coordinates. And precisely this process of decompactification is at heart of the string-gas cosmology proposal [34, 35]. Furthermore, recent developments in string theory have suggested that the low-energy effective dynamics of closed strings in non-geometric flux compactifications may be governed by noncommutative and even nonassociative deformations of gravity [36, 37, 11, 38, 39]. However these backgrounds are, in general, not solutions of the strong constraint [38, 14]. Finally, in the approach of the generalized Scherk-Schwarz compactification in DFT it was shown that one could consistently relax this constraint [40, 33].

Here we want to argue that recasting the problem into an L_∞ -framework could indicate a way of relaxing this strong constraint. The first observation we make is that the relevant structure for the case of a DFT algebroid is that of curved L_∞ -algebras. These are L_∞ -algebras that include a μ_0 map that corresponds to a constant element⁶ of the homogeneous subspace of degree 2 in our conventions. The homotopy relations are again written in similar form as (3.1), however, now every expression of a fixed order in i has an additional term coming from μ_0 . In particular:

$$\begin{aligned} i = 1 : \mu_1^2(l) &= \mu_2(\mu_0, l) , \\ i = 2 : \mu_1(\mu_2(l_1, l_2) - \mu_2(\mu_1(l_1), l_2) - (-1)^{1+|l_1||l_2|}\mu_2(\mu_1(l_2), l_1)) &= -\mu_3(\mu_0, l_1, l_2) , \end{aligned}$$

Importantly, map μ_1 is no longer a differential and there is no (co)chain complex underlying a curved L_∞ -algebra.

In DFT framework the strong condition is expressed as a differential equation:

$$\eta^{IJ}\partial_I\partial_J(\dots) = 0 ,$$

where derivatives act on products of fields. This lead us to propose a curved L_∞ - algebra of DFT defined on the following graded space.

$$\begin{array}{ccc} L_{-1} & \oplus & L_0 & \oplus & L_2 \\ f \in C^\infty(\mathcal{M}) & & e \in \Gamma(L_+) & & \mu_0 \end{array}$$

⁶From the point of view of QP manifolds this constant comes from the expansion of homological vector Q outside of the fixed point.

Following the construction of the L_∞ - algebra for a CA [41] one can define a curved L_∞ - algebra for the DFT algebroid [19] where:

$$\mu_0 = \eta^{IJ} \partial_I \otimes \partial_J ,$$

is a constant symmetric 2-vector, and homotopy relations reproduce the defining relations of DFT algebroid (4.2). The physical relevance of constant map μ_0 in the context of DFT is still under investigation. Note, however, that in Ref. [15], Zwiebach identified L_∞ -algebras as the structure underlying closed string field theory and related the μ_0 map to the backgrounds which are not conformal. In DFT context, this nicely fits with observation that on one hand the non-associative backgrounds violate the strong constraint of DFT, and that on the other hand they are incompatible with on-shell conformal symmetry of string theory, see discussion in Ref.[42]. To further the understanding of the constraint structure of DFT we propose to use the homotopy relations of L_∞ -algebra to bootstrap the gauge invariant membrane sigma model of relevance to the unconstrained theory. We plan to report on progress of this direction in forthcoming work [19].

Acknowledgments

We thank the organizers of Humboldt Kolleg Frontiers in Physics: From the Electroweak to the Planck Scales and the Alexander von Humboldt Foundation. The work was supported by the Croatian Science Foundation under the Project IP-2019-04-4168 and by the European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (KK.01.1.1.06).

References

- [1] W. Siegel, “Two vierbein formalism for string inspired axionic gravity,” Phys. Rev. D **47** (1993) 5453 [[arXiv:hep-th/9302036](#)].
- [2] W. Siegel, “Superspace duality in low-energy superstrings,” Phys. Rev. D **48** (1993) 2826 [[arXiv:hep-th/9305073](#)].
- [3] C. Hull and B. Zwiebach, “Double Field Theory,” JHEP **09** (2009) 099 [[arXiv:0904.4664 \[hep-th\]](#)].
- [4] E. Plauschinn, “Non-geometric backgrounds in string theory,” Phys. Rept. **798** (2019), 1-122 [[arXiv:1811.11203 \[hep-th\]](#)].
- [5] C. Hull and B. Zwiebach, “The gauge algebra of double field theory and Courant brackets,” JHEP **09** (2009) 090 [[arXiv:0908.1792 \[hep-th\]](#)].
- [6] T. J. Courant, “Dirac manifolds,” Trans. Am. Math. Soc. **319** (1990) 631.
- [7] Z.-J. Liu, A. Weinstein and P. Xu, “Manin Triples for Lie Bialgebroids,” J. Diff. Geom. **45** (1997) 547 [[arXiv:dg-ga/9508013](#)].
- [8] P. Ševera, “Letters to Alan Weinstein about Courant algebroids,” [arXiv:1707.00265 \[math.DG\]](#).
- [9] R. Blumenhagen, A. Deser, E. Plauschinn and F. Rennecke, “Bianchi Identities for Non-Geometric Fluxes - From Quasi-Poisson Structures to Courant Algebroids,” Fortsch. Phys. **60** (2012) 1217 [[arXiv:1205.1522 \[hep-th\]](#)].

- [10] N. Halmagyi, “Non-geometric backgrounds and the first order string sigma-model,” [arXiv:0906.2891 \[hep-th\]](https://arxiv.org/abs/0906.2891).
- [11] D. Mylonas, P. Schupp and R. J. Szabo, “Membrane Sigma-Models and Quantization of Non-Geometric Flux Backgrounds,” *JHEP* **1209** (2012) 012 [[arXiv:1207.0926 \[hep-th\]](https://arxiv.org/abs/1207.0926)].
- [12] A. Chatzistavrakidis, L. Jonke and O. Lechtenfeld, “Sigma models for genuinely non-geometric backgrounds,” *JHEP* **1511** (2015) 182 [[arXiv:1505.05457 \[hep-th\]](https://arxiv.org/abs/1505.05457)].
- [13] T. Bessho, M. A. Heller, N. Ikeda and S. Watamura, “Topological Membranes, Current Algebras and H-flux - R-flux Duality based on Courant Algebroids,” *JHEP* **1604** (2016) 170 [[arXiv:1511.03425 \[hep-th\]](https://arxiv.org/abs/1511.03425)].
- [14] A. Chatzistavrakidis, L. Jonke, F. S. Khoo and R. J. Szabo, “Double Field Theory and Membrane Sigma-Models,” *JHEP* **1807** (2018) 015 [[arXiv:1802.07003 \[hep-th\]](https://arxiv.org/abs/1802.07003)].
- [15] B. Zwiebach, “Closed string field theory: Quantum action and the B-V master equation,” *Nucl. Phys. B* **390** (1993) 33 [[arXiv:hep-th/9206084](https://arxiv.org/abs/hep-th/9206084)].
- [16] T. Lada and J. Stasheff, “Introduction to SH Lie algebras for physicists,” *Int. J. Theor. Phys.* **32** (1993) 1087 [[arXiv:hep-th/9209099](https://arxiv.org/abs/hep-th/9209099)].
- [17] R. Blumenhagen, I. Brunner, V. Kupriyanov and D. Lüst, “Bootstrapping non-commutative gauge theories from L_∞ algebras,” *JHEP* **05** (2018), 097 [[arXiv:1803.00732 \[hep-th\]](https://arxiv.org/abs/1803.00732)].
- [18] C. J. Grewcoe and L. Jonke, “Courant sigma model and L_∞ -algebras,” [arXiv:2001.11745 \[hep-th\]](https://arxiv.org/abs/2001.11745).
- [19] C. J. Grewcoe and L. Jonke, in progress.
- [20] N. Ikeda, “Chern-Simons gauge theory coupled with BF theory,” *Int. J. Mod. Phys. A* **18** (2003) 2689 [[arXiv:hep-th/0203043](https://arxiv.org/abs/hep-th/0203043)].
- [21] J.-S. Park, “Topological open p -branes,” in: *Symplectic Geometry and Mirror Symmetry*, eds. K. Fukaya, Y.-G. Oh, K. Ono and G. Tian (World Scientific, 2001), pp. 311–384 [[arXiv:hep-th/0012141](https://arxiv.org/abs/hep-th/0012141)].
- [22] C. Hofman and J.-S. Park, “BV quantization of topological open membranes,” *Commun. Math. Phys.* **249** (2004) 249–271 [[arXiv:hep-th/0209214](https://arxiv.org/abs/hep-th/0209214)].
- [23] M. Alexandrov, M. Kontsevich, A. Schwarz, and O. Zaboronsky “The Geometry of the master equation and topological quantum field theory,” *Int. J. Mod. Phys. A* **12** (1997) 1405 [[arXiv:hep-th/9502010](https://arxiv.org/abs/hep-th/9502010)].
- [24] D. Roytenberg, “On the structure of graded symplectic supermanifolds and Courant algebroids,” *Contemp. Math.* **315** (2002) 169–186 [[arXiv:math/0203110 \[math.SG\]](https://arxiv.org/abs/math/0203110)].
- [25] D. Roytenberg, “AKSZ-BV Formalism and Courant Algebroid-induced Topological Field Theories,” *Lett. Math. Phys.* **79** (2007) 143 [[arXiv:hep-th/0608150](https://arxiv.org/abs/hep-th/0608150)].
- [26] A. S. Cattaneo and G. Felder, “On the AKSZ formulation of the Poisson sigma model,” *Lett. Math. Phys.* **56** (2001), 163–179 [[arXiv:math/0102108 \[math\]](https://arxiv.org/abs/math/0102108)].
- [27] P. Ševera, “Poisson-Lie T-duality as a boundary phenomenon of Chern-Simons theory,” *JHEP* **05** (2016), 044 [[arXiv:1602.05126 \[hep-th\]](https://arxiv.org/abs/1602.05126)].
- [28] M. Henneaux, “Lectures on the Antifield-BRST Formalism for Gauge Theories,” *Nucl. Phys. Proc. Suppl.* **18A** (1990) 47.

- [29] J. Gomis, J. Paris and S. Samuel, “Antibracket, antifields and gauge theory quantization,” Phys. Rept. **259** (1995) 1 [[arXiv:hep-th/9412228](#)].
- [30] B. Jurčo, L. Raspollini, C. Sämann and M. Wolf, “ L_∞ -Algebras of Classical Field Theories and the Batalin-Vilkovisky Formalism,” Fortsch. Phys. **67** (2019) no.7, 1900025 [[arXiv:1809.09899 \[hep-th\]](#)].
- [31] O. Hohm and B. Zwiebach, “ L_∞ Algebras and Field Theory,” Fortsch. Phys. **65** (2017) no.3-4, 1700014 [[arXiv:1701.08824 \[hep-th\]](#)].
- [32] Th. Voronov, “Higher derived brackets and homotopy algebras,” J. Pure and Appl. Algebra. **202** (2005) 133 [[arXiv:math/0304038 \[math.QA\]](#)].
- [33] D. Geissbühler, D. Marqués, C. Núñez and V. Penas, “Exploring double field theory,” JHEP **1306** (2013) 101 [[arXiv:1304.1472 \[hep-th\]](#)].
- [34] R. H. Brandenberger and C. Vafa, “Superstrings in the Early Universe,” Nucl. Phys. B **316** (1989), 391-410
- [35] R. Brandenberger, R. Costa, G. Franzmann and A. Weltman, “Dual spacetime and nonsingular string cosmology,” Phys. Rev. D **98** (2018) no.6, 063521 [[arXiv:1805.06321 \[hep-th\]](#)].
- [36] R. Blumenhagen and E. Plauschinn, “Nonassociative Gravity in String Theory?,” J. Phys. A **44** (2011), 015401 [[arXiv:1010.1263 \[hep-th\]](#)].
- [37] R. Blumenhagen, A. Deser, D. Lüst, E. Plauschinn and F. Rennecke, “Non-geometric Fluxes, Asymmetric Strings and Nonassociative Geometry,” J. Phys. A **44** (2011), 385401 [[arXiv:1106.0316 \[hep-th\]](#)].
- [38] R. Blumenhagen, M. Fuchs, F. Hassler, D. Lüst and R. Sun, “Non-associative Deformations of Geometry in Double Field Theory,” JHEP **04** (2014), 141 [[arXiv:1312.0719 \[hep-th\]](#)].
- [39] D. Mylonas, P. Schupp and R. J. Szabo, “Non-Geometric Fluxes, Quasi-Hopf Twist Deformations and Nonassociative Quantum Mechanics,” J. Math. Phys. **55** (2014), 122301 [[arXiv:1312.1621 \[hep-th\]](#)].
- [40] M. Graña and D. Marques, “Gauged Double Field Theory,” JHEP **04** (2012), 020 [[arXiv:1201.2924 \[hep-th\]](#)].
- [41] D. Roytenberg and A. Weinstein, “Courant Algebroids and Strongly Homotopy Lie Algebras,” Lett. Math. Phys. **46** (1998), 81 [[arXiv:math/9802118 \[math.QA\]](#)].
- [42] R. Blumenhagen and M. Fuchs, “Towards a Theory of Nonassociative Gravity,” JHEP **07** (2016), 019 [[arXiv:1604.03253 \[hep-th\]](#)].