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Some recent advances in the understanding of $\mathcal{N} = 4$ supersymmetric Yang-Mills thermodynamics

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The interest in the thermodynamics of supersymmetric Yang-Mills started after Maldacena proposed the duality between string theory on AdS backgrounds and the large-N limit of SYM theories. One of the motivations to study the thermal properties of $\mathcal{N} = 4$ supersymmetric Yang-Mills in four dimensions($\mathrm{SYM}_{4,4}$) is that at high temperatures, the weak-coupling limit of this theory has many similarities with high temperature quantum chromodynamics (QCD). In this proceedings contribution, I review recent work with my collaborators where we calculate the resummed perturbative free energy of $\mathcal{N} = 4$ supersymmetric Yang-Mills in four spacetime dimensions through second order in the 't Hooft coupling λ at finite temperature and zero chemical potential. We compare our final result with prior results obtained in the weak and strong-coupling limits and construct a generalized Padé approximant that interpolates between the weak-coupling result and the large-N_c strong-coupling result.

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1. Introduction

The perturbative expansion of the free energy of hot non-Abelian gauge theory and in our case $\mathcal{N}=4$ supersymmetric Yang-Mills in four dimensions with N_c colors and gauge coupling g can be written in the form

$$\lim_{\lambda\to 0}\mathcal{F}\sim T^4\left[a_0+a_2\lambda+a_3\lambda^{3/2}+\left(a_4+a_4'\log\lambda\right)\lambda^2+O(\lambda^{5/2})\right],\tag{1}$$

where $\lambda=g^2N_c$ is the 't Hooft coupling. The leading term in this expression is the free energy of an ideal plasma and the $O(\lambda)$ correction can be obtained by computing two-loop Feynman diagrams. The next contribution is $O(\lambda^2)$ and comes from three-loop contributions. However, a problem emerges because one finds uncanceled infrared divergences at the three-loop level if one uses bare propagators. This happens in QCD as well and there the infrared divergences can be eliminated by summing over the so-called ring diagrams [1]. The solution is same in $\mathrm{SYM}_{4,4}$ and the only difference with QCD is the number and types of degrees of freedom. In the weak-coupling limit, the free energy of $\mathrm{SYM}_{4,4}$ has been calculated through order $\lambda^{3/2}$ in [2] and in the opposite limit of strong coupling, the behavior of the $\mathrm{SYM}_{4,4}$ free energy was studied using the AdS/CFT correspondence in [3].

2. $\mathcal{N}=4$ supersymmetric Yang-Mills theory in 4-dimensions ($\mathrm{SYM}_{4,4}$)

The $\mathrm{SYM}_{4,4}$ theory can be obtained by dimensional reduction of $\mathrm{SYM}_{1,\mathcal{D}}$ in $\mathcal{D}=\mathcal{D}_{\max}=10$ with all fields being in the adjoint representation of $SU(N_c)$. The Lagrangian that generates the perturbative expansion for $\mathrm{SYM}_{4,4}$ in Minkowski-space can be expressed as

$$\begin{aligned}\mathcal{L}_{\mathrm{SYM}_{4,4}}=\mathrm{Tr}\Bigg[&-\frac{1}{2}G_{\mu\nu}^2+(D_\mu\Phi_A)^2+i\bar{\psi}_i D\psi_i-\frac{1}{2}g^2(i[\Phi_A,\Phi_B])^2\\&-ig\bar{\psi}_i\left[\alpha_{ij}^{\mathrm{p}}X_{\mathrm{p}}+i\beta_{ij}^{\mathrm{q}}\gamma_5 Y_{\mathrm{q}},\psi_j\right]\Bigg]+\mathcal{L}_{\mathrm{gf}}+\mathcal{L}_{\mathrm{gh}}+\Delta\mathcal{L}_{\mathrm{SYM}},\end{aligned}\tag{2}$$

with $\Phi_A\in(X_1,Y_1,X_2,Y_2,X_3,Y_3)$ and X_{p} and Y_{q} denote scalars and pseudoscalar fields, respectively.

We are in general interested in supersymmetric field theories with supercharges in dimensions $D\le\mathcal{D}_{\max}$, with D being an integer. The evaluation of Feynman diagrams for theories that are obtained by dimensional reduction of $\mathrm{SYM}_{1,\mathcal{D}}$ can be carried out in a simple way that preserves the supersymmetry by taking all fields to be $\mathcal{D}$-dimensional tensors or spinors and all momentum to be $d=D-2\epsilon$ vectors. This scheme was introduced by W. Siegel and is called regularization by dimension reduction (RDR) [4] .

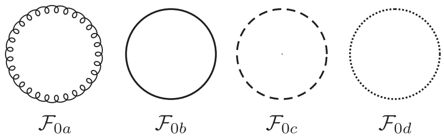
3. Resummation in $\mathrm{SYM}_{4,4}$

Since we want to obtain the thermodynamic functions up to $O(\lambda^2)$, we need to calculate Feynman diagrams through three loop order. However, at three loop level in QCD [1], infrared divergences appear that need to be canceled by summing over the ring diagrams appearing in the thermal mass counterterm. As detailed in ref. [1], in order to systematically resum the necessary diagrams, we need to modify the static bosonic propagators by incorporating gluon and scalar thermal masses, m_D and M, respectively.

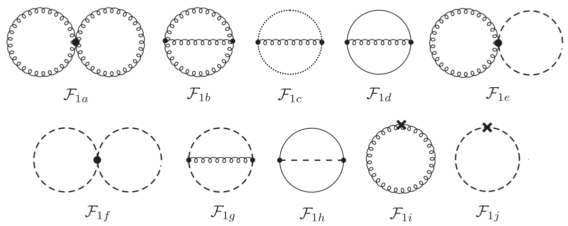
Following Arnold and Zhai, we introduce thermal masses, m_D and M, only for the zero Matsubara modes of the gluon and scalar fields. The resulting reorganized Lagrangian density in frequency space can be rewritten as

$$\mathcal{L}^{\text{resum}}_{\text{SYM}_{4,4}} = \{\mathcal{L}_{\text{SYM}_{4,4}} + \text{Tr}[m_D^2 A_0^2 \delta_{p_0} - M^2 \Phi_A^2 \delta_{p_0}]\} - \text{Tr}[m_D^2 A_0^2 \delta_{p_0} - M^2 \Phi_A^2 \delta_{p_0}], \tag{3}$$

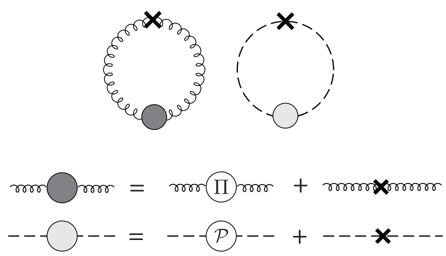
Then we absorb the two A_0^2 and Φ^2 terms in the curly brackets into our unperturbed Lagrangian $\mathcal{L}_0$, and treat the two terms outside the curly brackets as a perturbation.



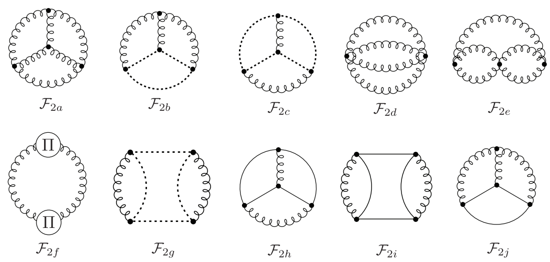
(a) One-loop diagrams contributing to the $\text{SYM}_{4,4}$ free energy



(b) Two-loop contributions to the $\text{SYM}_{4,4}$ free energy. The crosses are the thermal counterterms produced by the last two terms of (3).



(c) Three-loop gluon and scalar counterterm diagrams in $\text{SYM}_{4,4}$



(d) Three-loop vacuum diagrams contributing to the $\text{SYM}_{1,10}$ free energy

Figure 1: Feynman diagrams up to 3-loop order. The dashed lines indicate a scalar field and dotted lines indicate a ghost field. The crosses are the thermal counter-terms.

3.1 The resummed one-loop free energy

The resummed one loop self energy can be written as

$$F^{\text{resum}}_{\text{1-loop}} = d_A \mathcal{F}_{0a} + d_F \mathcal{F}_{0b} + d_S \mathcal{F}_{0c} + d_A \mathcal{F}_{0d}\,, \tag{4}$$

with $d_F = 4d_A$ and $d_S = 6d_A$. By using resummed gluonic and scalar propagators, imposing $D = 4$, $m_D^2 = 2\lambda T^2$, $M^2 = \lambda T^2$, and truncating at $\mathcal{O}(\epsilon^0)$ one obtains

$$F^{\text{resum}}_{\text{1-loop}} = -d_A \left(\frac{\pi^2 T^4}{6}\right)\left[1 + \frac{3 + \sqrt{2}}{\pi^3}\lambda^{3/2}\right]. \tag{5}$$

3.2 The resummed two-loop free energy

The $\mathrm{SYM}_{4,4}$ two-loop free energy can be written as

$$F^{\text{resum}}_{\text{2-loop}} = d_A\Big\{\lambda[\mathcal{F}_{1a}+\mathcal{F}_{1b}+\mathcal{F}_{1c}+\mathcal{F}_{1d}+\mathcal{F}_{1e}+\mathcal{F}_{1f}+\mathcal{F}_{1g}+\mathcal{F}_{1h}]+\mathcal{F}_{1i}+\mathcal{F}_{1j}\Big\}. \tag{6}$$

By using resummed gluonic and scalar propagators one obtains

$$F^{\text{resum}}_{\text{2-loop}} = -d_A\left(\frac{\pi^2T^4}{6}\right)\left[-\frac{3}{2\pi^2}\lambda-\frac{3}{2\pi^4}\left(\frac{23}{8}+\frac{3\sqrt{2}}{4}+\frac{15\log 2}{4}-\log\lambda\right)\lambda^2\right]. \tag{7}$$

3.3 The resummed three-loop free energy

The calculation of the massless three-loop vacuum Feynman diagrams in $\mathrm{SYM}_{4,4}$ can be accomplished more simply in the corresponding $\mathrm{SYM}_{1,10}$ theory. As a result of this equivalence, one can consider the much smaller set of $\mathrm{SYM}_{1,10}$ graphs presented in fig.1(d), which are topologically equivalent to three-loop QCD vacuum graphs. The three-loop results in $\mathrm{SYM}_{4,4}$ can be obtained by imposing $\mathcal{D}=\mathcal{D}_{\max}=10$, $d=4-2\epsilon$ in the $\mathrm{SYM}_{1,\mathcal{D}}$ theory.

$$F^{\text{vacuum}}_{\text{3-loop}} = d_A\lambda^2[\mathcal{F}_{2a}+\mathcal{F}_{2b}+\mathcal{F}_{2c}+\mathcal{F}_{2d}+\mathcal{F}_{2e}+\mathcal{F}_{2f}+\mathcal{F}_{2g}+\mathcal{F}_{2h}+\mathcal{F}_{2i}+\mathcal{F}_{2j}]\big|^{\mathcal{D}=10}_{d=4-2\epsilon}. \tag{8}$$

Infrared divergences are generated in eq.8 due to 3-momentum integrations. These divergences are canceled by thermal mass counterterm diagrams in fig.1(c).

$$\begin{aligned} F^{\text{resum}}_{\text{3-loop}} &= \mathcal{F}^{\text{vacuum}}_{\text{3-loop}}+\mathcal{F}^{\text{sct}}_{\text{3-loop}}+\mathcal{F}^{\text{bct}}_{\text{3-loop}} \\ &= -d_A\left(\frac{\pi^2T^4}{6}\right)\frac{\lambda^2}{2\pi^4}\left[\frac{27}{8}+3\gamma+3\frac{\zeta'(-1)}{\zeta(-1)}+5\log 2-6\log\pi\right]. \end{aligned} \tag{9}$$

4. $\mathrm{SYM}_{4,4}$ thermodynamic functions to $\mathcal{O}(\lambda^2)$

Combining eqs. (5), (7), and (9), we obtain our final result for the resummed free energy in the RDR scheme through $\mathcal{O}(\lambda^2)$ [1].

$$\begin{aligned} \mathcal{F} = -d_A\left(\frac{\pi^2T^4}{6}\right)\Bigg\{1-\frac{3}{2}\frac{\lambda}{\pi^2}+\left(3+\sqrt{2}\right)\left(\frac{\lambda}{\pi^2}\right)^{3/2} \\ +\left[-\frac{42}{16}-\frac{9\sqrt{2}}{8}+\frac{3}{2}\gamma_E+\frac{3}{2}\frac{\zeta'(-1)}{\zeta(-1)}-\frac{25}{8}\log 2+\frac{3}{2}\log\frac{\lambda}{\pi^2}\right]\left(\frac{\lambda}{\pi^2}\right)^2\Bigg\}. \end{aligned} \tag{10}$$

5. Conclusions and Outlook

In this work, I reviewed the computation of the thermodynamic function of $\mathrm{SYM}_{4,4}$ to $\mathcal{O}(\lambda^2)$. The final result, presented in eq. (10), extends our knowledge of weak-coupling $\mathrm{SYM}_{4,4}$ thermodynamics to include terms at $\mathcal{O}(\lambda^2)$ and $\mathcal{O}(\lambda^2\log\lambda)$. With the $\mathcal{O}(\lambda^2)$ and $\mathcal{O}(\lambda^2\log\lambda)$ coefficients in the $\mathrm{SYM}_{4,4}$ free energy, we then constructed a large-N_c Padé approximant that interpolates between the weak- and strong-coupling limits. Fig. 2 summarizes our findings.

[1] We have noticed a small error in (10) recently. The finite part should be $-\frac{42}{16}$ instead of $-\frac{45}{16}$ as quoted in our work [5].

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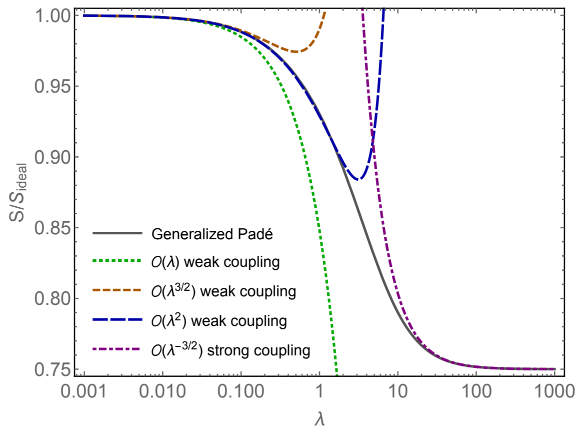

Figure 2: The entropy density $\mathcal{S}$ normalized by the $\mathcal{S}_{\rm ideal}$ in SYM$_{4,4}$ as a function of the t 'Hooft coupling λ.

We have recently rederived the final result (10) using effective field theory techniques [6]. We are also working on computing the coefficient of $\lambda^{5/2}$ in the SYM$_{4,4}$ free energy using effective field theory methods. Finally, we also plan to pursue a three-loop HTLpt calculation of SYM$_{4,4}$ thermodynamics, extending our prior two-loop HTLpt result [7].

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