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Connection formulas related with Appell's hypergeometric function F_1

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We derive the connection formulas for the fundamental sets of solutions of the system of differential equations E_1. Here E_1 is satisfied by Appell's hypergeometric function F_1.

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Introduction

As the generalizations of Gauss hypergeometric function, Appell introduced the four functions F_1, F_2, F_3 and F_4, which are nowadays called Appell's hypergeometric functions in two variables [Ap1][Ap2]. The purpose of the present paper is to derive the connection formulas related with F_1 by using the intersection theory for the twisted homology groups.

Appell's hypergeometric function F_1 is the analytic continuation of Appell's hypergeometric series

$$F_1(\alpha,\beta,\beta',\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!}x^m y^n,\quad |x|<1,\ |y|<1,$$

where $(a)_n=\Gamma(a+n)/\Gamma(a)$ and $(\alpha)_0=1$. It satisfies the system of differential equations E_1 of rank 3 (The third equation below is missing in [AKdF][Er]. See also [MS][MN]):

$$(\mathrm{E}_1)\qquad \begin{cases} [\,\theta_x(\theta_x+\theta_y+\gamma-1)-x(\theta_x+\theta_y+\alpha)(\theta_x+\beta)\,]F=0,\\ [\,\theta_y(\theta_x+\theta_y+\gamma-1)-y(\theta_x+\theta_y+\alpha)(\theta_y+\beta')\,]F=0,\\ [\,x(\theta_x+\beta)\theta_y-y(\theta_y+\beta')\theta_x\,]F=0,\end{cases}$$

which is defined on the space $(\mathbb{P}^1(\mathbb{C}))^2\backslash S$, where S is the union of the singular loci

$$S=\{x=0\}\cup\{x=1\}\cup\{x=\infty\}\cup\{y=0\}\cup\{y=1\}\cup\{y=\infty\}\cup\{x=y\}.$$

The 60 solutions of E_1 in terms of F_1 are listed in [AKdF], which were obtained by Appell and Vavasseur [Va] (The solutions $z_1,\ldots,z_{60}$ in [AKdF] are listed in Appendix). These solutions were thought of as the analogue of Kummer's 24 solutions of Gauss' hypergeometric equation. However, it is not the case. These 60 solutions could not constitute any fundamental set of solutions around a point, and give only two members amongst three members which constitue the fundamental set of solutions around each point. This gap was filled by the idea of Erdélyi, who found the way in [Er] to derive the hidden 60 solutions in terms of Horn's hypergeometric function G_2 (the function G_2 was not found when [AKdF] was published). These 120 solutions give a complete list of 15 fundamental sets of solutions around the intersection of the singular loci S of E_1:

$$(0,\,0),\ (1,\,1),\ (\infty,\,\infty),\ (0,\,1),\ (1,\,0),\ (0,\,\infty),\ (\infty,\,0),\ (1,\,\infty),\ (\infty,\,1).$$

Each set consists of two F_1 and one G_2. One set is attached to the point around the normal crrossing point, and three sets are attached to the point around the intersection of three singular loci. Each F_1 has 6 different expressions as a result of tranformation formulas. Each G_2 has 4 different expressions. (Other solutions in terms of other series, Appell's F_2 or F_3 and Horn's H_2, are reduced to these 120 solutions.) On the other hand, Olsson gave some connection formulas for these sets of solutions in [Ol], where the formulas are derived by the connection formulas for Gauss' hypergeometric functions.

The purpose of this paper is to enlarge the results by Olsson. The present way to achieve it is by using the intersection theory for the twisted homology group. We refer the reader to [Mim] for another application of the intersection theory to the connection problem and to [Tak1][Tak2] for another approach by using the Euler-Darboux equation to the connection problem for F_1.

In Section 1 we give the preliminaries for the present paper. In Section 2, we give a complete list of the fundamental sets of solutions around the 9 intersection of the singular loci. In Section 3, we give some of connection formulas related with three fundamental sets of solutions around the point (0, 0). In Section 4, we give the integral representations for the functions which constitute the fundamental sets of solutions in Section 2. In Section 5, we derive the connection formulas by using the intersection numbers for the loaded cycles. In Appendix, the transformation formulas for the functions constituting the fundamental sets of solutions in Section 2.

In this paper, the symbols

$$e(A) = \exp(\pi\sqrt{-1}A), \quad \langle A \rangle = A - A^{-1}, \quad e_{ij\ldots k} = e(\mu_{ij\ldots k}),$$

$$d_{ij\ldots k} = d_{\mu_{ij\ldots k}} = e^2_{ij\ldots k} - 1, \quad d^{(-)}_{ij\ldots k} = d^{(-)}_{\mu_{ij\ldots k}} = e^{-2}_{ij\ldots k} - 1$$

and

$$\mu_{ij\ldots k} = \mu_i + \mu_j + \ldots + \mu_k$$

are used frequently.

1. Preliminaries

1.1 Twisted homology groups

Let $u(t) = \prod_i f_i(t)^{\alpha_i}$ be a multivalued function on $T \subset \mathbb{C}^m$, where $\alpha_i \in \mathbb{C}$ and T is the complement of the singular locus $\cup_i\{f_i(t) = 0\}$ in $\mathbb{C}^m$. Let $\mathcal{L}$ be the locally constant sheaf (the local system) defined by u: the sheaf consisting of the local solutions of $dL = L\omega$ for $\omega = du(t)/u(t)$. Let $\mathcal{L}^\vee$ be the locally constant sheaf defined by u^{-1}: the sheaf consisting of the local solutions of $dL = -L\omega$. (The convention for the symbols $\mathcal{L}$ and $\mathcal{L}^\vee$ is different from that in [AoKi] [KY], where $\mathcal{L}$ denotes $\mathcal{L}^\vee$ and $\mathcal{L}^\vee$ denotes $\mathcal{L}$.)

Let $H_m(T, \mathcal{L})$ be the m-th homology group with coefficients in $\mathcal{L}$, $H^{\mathrm{lf}}_m(T, \mathcal{L})$ the m-th locally finite homology group with coefficients in $\mathcal{L}$. Elements of these twisted homology groups, called *twisted cycles* or *loaded cycles*, are represented by ∂-closed loaded (finite or locally finite) chains

$$C = \sum_\Delta a_\Delta \Delta \otimes v_\Delta, \quad (a_\Delta \in \mathbb{C}),$$

where each Δ is an m-simplex and v_Δ a section of $\mathcal{L}$ on Δ. The subset $\cup_{\Delta \text{ satisfying } a_\Delta \neq 0} \Delta$ is called the *support* of C. The boundary operator ∂ is defined to be a $\mathbb{C}$-linear mapping satisfying $\partial(\Delta \otimes v) = \sum_{i=0}^m (-1)^i \Delta^i \otimes v|_{\Delta^i}$, where Δ is an m-simplex, Δ^i denotes the i-th face of Δ, and $v|_{\Delta^i}$ is the restriction of v on Δ^i.

If each factor $f_i(t)$ of $u(t)$ is defined over $\mathbb{R}$, and D is a simply connected region of the real manifold $T_{\mathbb{R}} = \mathbb{R}^m \setminus \cup_i \{f_i(t) = 0\}$ (the real locus of T), then it is convenient to load D with a section

$$u_D(t) = \prod_i (\varepsilon_i f_i(t))^{\alpha_i}$$

of $\mathcal{L}$ on D, and to make a loaded chain $D \otimes u_D(t)$, where $\varepsilon_i = \pm$ is so determined that $\varepsilon_i f_i(t)$ is positive on D, and the argument of $\varepsilon_i f_i(t)$ is assigned to be zero. This choice of a section is said to be *standard loading over D*. Sometimes, we omit the assignment of loading and denote just the support of the cycle for brevity.

Similarly, the function $u_p(t) = \prod_i (\varepsilon_i f_i(t))^{\alpha_i}$ for $p \in T_{\mathbb{R}}$ is also defined: $\varepsilon_i f_i(t)$ is positive at p and the argument of each $\varepsilon_i f_i(t)$ is assigned to be zero at p. Loading a region around p with $u_p(t)$ is said to be *standard loading over the point p*.

We also consider the standard loading with $u_D^{-1}(t)$ or $u_p^{-1}(t)$ to study $H_m^{\mathrm{lf}}(T, \mathcal{L}^\vee)$.

Example 1. In case $T = \mathbb{C}\setminus\{p, q\}$ with $p < q$ and $u(t) = (t-p)^{\lambda_p}(t-q)^{\lambda_q}$, we have $u_{(p,q)}(t) = (t-p)^{\lambda_p}(q-t)^{\lambda_q}$, $u_{(q,\infty)}(t) = (t-p)^{\lambda_p}(t-q)^{\lambda_q}$ and $u_{(\infty,p)}(t) = (p-t)^{\lambda_p}(q-t)^{\lambda_q}$. They are used like as $(p, q) \otimes u_{(p,q)} \in H_1^{\mathrm{lf}}(T, \mathcal{L})$ and $(p, q) \otimes u_{(p,q)}^{-1} \in H_1^{\mathrm{lf}}(T, \mathcal{L}^\vee)$.

Under some genericity condition on the exponents α_i, we have the isomorphism, called the *regularization*,

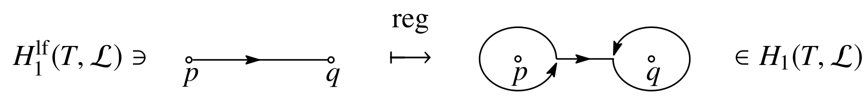
$$\mathrm{reg} \ : \ H_m^{\mathrm{lf}}(T, \mathcal{L}) \ \longrightarrow \ H_m(T, \mathcal{L}),$$

which is the inverse of the natural map $\iota \ : \ H_m(T, \mathcal{L}) \to H_m^{\mathrm{lf}}(T, \mathcal{L})$.

Example 2. In case $T = \mathbb{C}\setminus\{p, q\}$ with $p < q$ and $u(t) = (t-p)^{\lambda_p}(t-q)^{\lambda_q}$, where $\lambda_p, \lambda_q \in \mathbb{C}\setminus\mathbb{Z}$, the regularization $\mathrm{reg}\big((p, q) \otimes u_{(p,q)}(t)\big) \in H_1(T, \mathcal{L})$ of $(p, q) \otimes u_{(p,q)}(t) \in H_1^{\mathrm{lf}}(T, \mathcal{L})$ is given by

$$\mathrm{reg}\,(p, q) \otimes u_{(p,q)}(t) = \left\{ \frac{1}{d_p} S(p\,;\, p+\varepsilon) + [p+\varepsilon, q-\varepsilon] - \frac{1}{d_q} S(q\,;\, q-\varepsilon) \right\} \otimes u_{(p,q)}(t).$$

Here and in what follows ε means a sufficiently small positive number, $d_r = e(2\lambda_r) - 1$, $e(A) = \exp(\pi\sqrt{-1}A)$, and the symbol $S(r\,;\,a)$ denotes a positively oriented closed curve starting and ending at the point a, encircling the singular point r and lying outside the other singularities. Here the arguments of $t-p$ and $q-t$ on $S(p\,;\,p+\varepsilon)$ or $S(q\,;\,q-\varepsilon)$ are assigned naturally by analytic continuation, i.e. $\arg(t-p)$ takes values from 0 to 2π on $S(p\,;\,p+\varepsilon)$ and $\arg(q-t)$ takes values from 0 to 2π on $S(q\,;\,q-\varepsilon)$.



Example 3. In case $T = \mathbb{C}\setminus\{p, q\}$ with $p < q$ and $u(t) = (t-p)^{\lambda_p}(t-q)^{\lambda_q}$, where $\lambda_q, \lambda_\infty \in \mathbb{C}\setminus\mathbb{Z}$ and $\lambda_\infty = 2 - \lambda_p - \lambda_q$, the regularization $\mathrm{reg}\big((q, \infty) \otimes u_{(q,\infty)}(t)\big) \in H_1(T, \mathcal{L})$ of $(q, \infty) \otimes u_{(q,\infty)}(t) \in$

$H_1^{\mathrm{lf}}(T, \mathcal{L})$ is given by

$$\mathrm{reg}\,(q,\ \infty) \otimes u_{(q,\,\infty)}(t) = \left\{\frac{1}{d_q} S(q\,;\, q+\varepsilon) + [q+\varepsilon,\ R] - \frac{1}{d_\infty} S(\infty\,;\, R)\right\} \otimes u_{(q,\,\infty)}(t),$$

where R is a sufficiently large number. Here the curve $S(\infty\,;\, R)$ starting and ending at the point $t = R$ around the point at infinity ∞ and lying outside the other singularities is regarded as the negatively oriented curve $S^{-1}(\{p,\ q\}, R)$ encircling the singular points p and q. The arguments of $t-p$ and $t-q$ on $S^{-1}(\{p,\ q\}, R)$ are assigned naturally by analytic continuation, i.e. both $\arg(t-p)$ and $\arg(t-q)$ take values from 0 to -2π on $S^{-1}(\{p,\ q\}, R)$.

Example 4. In case $T = \mathbb{C}\backslash\{p,\ q,\ r\}$ with $r \in \mathbb{R},\ \mathrm{Re}\,(p) < r,\ \mathrm{Re}\,(q) < r$ and $u(t) = (t-p)^{\lambda_p}(t-q)^{\lambda_q}(t-r)^{\lambda_r}$, where $\lambda_{pq},\ \lambda_r \in \mathbb{C}\backslash\mathbb{Z}$ and $\lambda_{pq} = \lambda_p + \lambda_q$, we define the symbol $(\{p,\ q\},\ r) \otimes u_{(r-\varepsilon)}(t)$ to denote the loaded cycle

$$(\{p,\ q\},\ r) \otimes u_{r-\varepsilon}(t) = \left\{\frac{1}{d_{pq}}\, S(\{p,\ q\}\,;\, r-\varepsilon) + [r-\varepsilon,\ r)\right\} \otimes u_{r-\varepsilon}(t) \in H_1^{\mathrm{lf}}(T, \mathcal{L}),$$

where $d_{pq} = e(2\lambda_{pq}) - 1$ and $S(\{p,\ q\}\,;\, r-\varepsilon)$ denotes the positively oriented closed curve starting and ending at the point $r-\varepsilon$, encircling the singular points p and q, and lying outside the other singularities. The arguments of $t-p$ and $t-q$ on $S(\{p,\ q\}\,;\, r-\varepsilon)$ take values from 0 to 2π.

The regularization $\mathrm{reg}\big((\{p,\ q\},\ r) \otimes u_{r-\varepsilon}(t)\big) \in H_1(T, \mathcal{L})$ of $(\{p,\ q\},\ r) \otimes u_{r-\varepsilon}(t) \in H_1^{\mathrm{lf}}(T, \mathcal{L})$ is given by

$$\begin{aligned}&\mathrm{reg}\big((\{p,\ q\},\ r) \otimes u_{r-\varepsilon}(t)\big)\\ &= \left\{\frac{1}{d_{pq}}\, S(\{p,\ q\}\,;\, r-\varepsilon) + [r-\varepsilon,\ r-\frac{\varepsilon}{2}] - \frac{1}{d_r}\, S(r\,;\, r-\frac{\varepsilon}{2})\right\} \otimes u_{r-\varepsilon}(t),\end{aligned}$$

where the argument of $r-t$ on $S(r\,;\, r-\frac{\varepsilon}{2})$ takes values from 0 to 2π.

Example 5. In case $T = \mathbb{C}\backslash\{p,\ q,\ r\}$ with $p \in \mathbb{R},\ p < \mathrm{Re}\,(q),\ p < \mathrm{Re}\,(r)$ and $u(t) = (t-p)^{\lambda_p}(t-q)^{\lambda_q}(t-r)^{\lambda_r}$, where $\lambda_p,\ \lambda_{qr} \in \mathbb{C}\backslash\mathbb{Z}$ and $\lambda_{qr} = \lambda_q + \lambda_r$, we define the symbol $(p,\ \{q,\ r\}) \otimes u_{(p,\,q)}(t)$ to denote the loaded cycle

$$(p,\ \{q,\ r\}) \otimes u_{p+\varepsilon}(t) = \left\{(p,\ p+\varepsilon] - \frac{1}{d_{qr}}\, S(\{q,\ r\}\,;\, p+\varepsilon)\right\} \otimes u_{p+\varepsilon}(t) \in H_1^{\mathrm{lf}}(T, \mathcal{L}),$$

where $d_{qr} = e(2\lambda_{qr}) - 1$ and $S(\{q,\ r\}\,;\, p+\varepsilon)$ denotes the positively oriented closed curve starting and ending at the point $p+\varepsilon$, encircling the points q and r, and lying outside the other singularities. The arguments of $q-t$ and $r-t$ on $S(\{q,\ r\}\,;\, p+\varepsilon)$ take values from 0 to 2π.

The regularization $\mathrm{reg}\big((p,\ \{q,\ r\}) \otimes u_{p+\varepsilon}(t)\big) \in H_1(T, \mathcal{L})$ of $(p,\ \{q,\ r\}) \otimes u_{p+\varepsilon}(t) \in H_1^{\mathrm{lf}}(T, \mathcal{L})$ is given by

$$\begin{aligned}&\mathrm{reg}\big((p,\ \{q,\ r\}) \otimes u_{p+\varepsilon}(t)\big)\\ &= \left\{\frac{1}{d_p}\, S(p\,;\, p+\frac{\varepsilon}{2}) + [p+\frac{\varepsilon}{2},\ p+\varepsilon] - \frac{1}{d_{qr}}\, S(\{q,\ r\}\,;\, p+\varepsilon)\right\} \otimes u_{p+\varepsilon}(t),\end{aligned}$$

where the argument of $t - p$ on $S(p\,;\, p + \frac{\varepsilon}{2})$ takes values from 0 to 2π.

Example 6. In case $T = \mathbb{C}\backslash\{p,\, q,\, r\}$ and $u(t) = (t-p)^{\lambda_p}(t-q)^{\lambda_q}(t-r)^{\lambda_r}$; when $p \in \mathbb{R},\ p < \mathrm{Re}\,(q),\ p < \mathrm{Re}\,(r)$ and $\lambda_{qr} \in \mathbb{C}\backslash\mathbb{Z}$, we define the symbol $(p,\, \{q,\, r\}) \otimes u^{-1}_{p+\varepsilon}(t)$ to denote the loaded cycle

$$(p,\, \{q,\, r\}) \otimes u^{-1}_{p+\varepsilon}(t) = \left\{(p,\, p+\varepsilon] - \frac{1}{d^{(-)}_{qr}}\, S(\{q,\, r\}\,;\, p+\varepsilon)\right\} \otimes u^{-1}_{p+\varepsilon}(t) \in H^{\mathrm{lf}}_1(T, \mathcal{L}^\vee),$$

and when $r \in \mathbb{R},\ \mathrm{Re}\,(p) < r,\ \mathrm{Re}\,(q) < r$ and $\lambda_{pq} \in \mathbb{C}\backslash\mathbb{Z}$, we define the symbol $(\{p,\, q\},\, r) \otimes u^{-1}_{r-\varepsilon}(t)$ to denote the loaded cycle

$$(\{p,\, q\},\, r) \otimes u^{-1}_{r-\varepsilon}(t) = \left\{\frac{1}{d^{(-)}_{pq}}\, S(\{p,\, q\}\,;\, r-\varepsilon) + [r-\varepsilon,\, r)\right\} \otimes u^{-1}_{r-\varepsilon}(t) \in H^{\mathrm{lf}}_1(T, \mathcal{L}^\vee),$$

where $d^{(-)}_{pq} = e(-2\lambda_{pq}) - 1$.

1.2 Intersection numbers

The Poincaré duality gives the non-degenerate pairing, called the *intersection form* of loaded (twisted) cycles,

$$I\ :\ H_m(T, \mathcal{L})\ \times\ H^{\mathrm{lf}}_m(T, \mathcal{L}^\vee) \longrightarrow \mathbb{C}$$

defined by

$$I\left(\sum_\rho a_\rho\, \rho \otimes v_\rho, \sum_\sigma b_\sigma\, \sigma \otimes w_\sigma\right) = \sum_{\rho,\,\sigma} a_\rho\, b_\sigma \sum_{x \in \rho\cap\sigma} I_x(\rho, \sigma) v_\rho(x) w_\sigma(x),$$

where $a_\rho, b_\sigma \in \mathbb{C}$, each ρ or σ is an m-simplex, v_ρ a section of $\mathcal{L}$ on ρ, w_σ a section of $\mathcal{L}^\vee$ on σ, and $I_x(\rho, \sigma)$ the topological intersection number at x.

Combining this form with the regularization mapping yields the *intersection number* :

$$\cdot\ :\ H^{\mathrm{lf}}_m(T, \mathcal{L}) \times H^{\mathrm{lf}}_m(T, \mathcal{L}^\vee) \longrightarrow \mathbb{C},$$

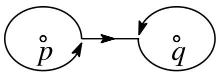
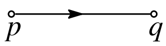
which is defined by

$$C \cdot C' = I(\mathrm{reg}\, C,\ C')$$

for $C \in H^{\mathrm{lf}}_m(T, \mathcal{L})$ and $C' \in H^{\mathrm{lf}}_m(T, \mathcal{L}^\vee)$.

Examples 7. In case $T = \mathbb{C}\backslash\{p,\, q\}$ with $p < q$ and $u(t) = (t-p)^{\lambda_p}(t-q)^{\lambda_q}$, where $\lambda_p,\ \lambda_q \in \mathbb{C}\backslash\mathbb{Z}$, we have

$$(p,\, q)\cdot(p,\, q) = I\Big(\mathrm{reg}\,\big((p,\, q)\otimes u_{(p,\, q)}(t)\big),\, (p,\, q)\otimes u^{-1}_{(p,\, q)}(t)\Big)$$

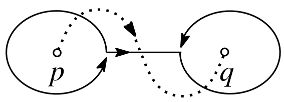
$= I\Big(\Big\{$  $\Big\}\otimes u_{(p,q)}(t),\ \Big\{$  $\Big\}\otimes u^{-1}_{(p,q)}(t)\Big)$

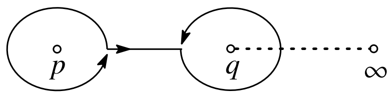
$$= -\frac{1}{d_p} - 1 + \frac{-1}{d_q} = -\frac{d_{pq}}{d_p d_q} = -\frac{\langle e(\lambda_{pq})\rangle}{\langle e(\lambda_p)\rangle\langle e(\lambda_q)\rangle}$$

and

$$(p,\, q)\cdot(q, \infty) = I\Big(\mathrm{reg}\,\big((p,\, q)\otimes u_{(p,q)}(t)\big),\, (q,\, \infty)\otimes u^{-1}_{(q,\, \infty)}(t)\Big)$$
$$= \frac{e_q}{d_q} = \frac{1}{\langle e_q\rangle},$$

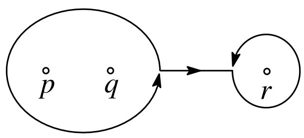
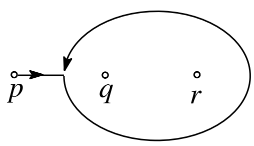
where $\langle\, A\,\rangle = A - A^{-1}$.





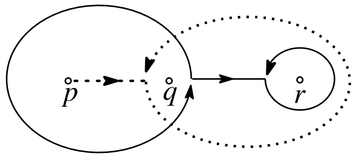
Examples 8. In case $T = \mathbb{C}\backslash\{p,\, q,\, r\}$ with $p < q < r$ and $u(t) = (t-p)^{\lambda_p}(t-q)^{\lambda_q}(t-r)^{\lambda_r}$, where $\mu_{pq},\ \mu_{qr},\ \mu_r \in \mathbb{C}\backslash\mathbb{Z}$, we have

$$(\{p,\, q\},\, r)\cdot(p,\, \{q,\, r\})$$
$$= I\Big(\mathrm{reg}\,\big((\{p,\, q\},\, r)\otimes u_{(q,\, r)}(t)\big),\, (p,\, \{q,\, r\})\otimes u^{-1}_{(p,\, q)}(t)\Big)$$

$= I\Bigg(\Bigg\{$  $\Bigg\}\otimes u_{(q,\, r)}(t),\ \Bigg\{$  $\Bigg\}\otimes u^{-1}_{(p,q)}(t)\Bigg)$

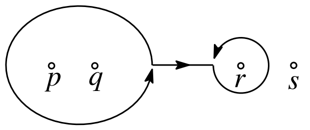
$$= \frac{1}{d_{pq}}\frac{-1}{d^{(-)}_{qr}}(e_q^{-1}e_r^{-2} - e^2_{pq}e_q^{-1}) = \frac{-\langle e_{pqr}\rangle}{\langle e_{pq}\rangle\langle e_{qr}\rangle},$$

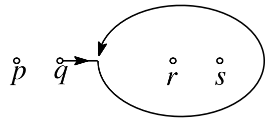
where $d^{(-)}_{qr} = e(-2\lambda_{qr}) - 1,\ e_{ij\cdots k} = e(\lambda_{ij\cdots k})$ and $\lambda_{ij\cdots k} = \lambda_i + \lambda_j + \cdots + \lambda_k$.



Examples 9. In case $T = \mathbb{C}\backslash\{p, q, r, s\}$ with $p < q < r < s$ and $u(t) = (t-p)^{\lambda_p}(t-q)^{\lambda_q}(t-r)^{\lambda_r}(t-s)^{\lambda_s}$, where and $\lambda_{pq}, \lambda_{rs}, \lambda_r \in \mathbb{C}\backslash\mathbb{Z}$, we have

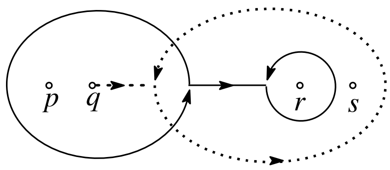
$$(\{p, q\}, r) \cdot (q, \{r, s\})$$
$$= I\Big(\mathrm{reg}\left((\{p, q\}, r) \otimes u_{(q,r)}(t)\right), (q, \{r, s\}) \otimes u^{-1}_{(q,r)}(t)\Big)$$

$= I\Bigg(\Bigg\{$ 

$\Bigg\} \otimes u_{(q,r)}, \Bigg\{$ 

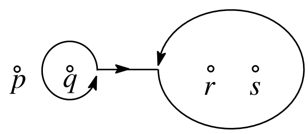
$\Bigg\} \otimes u^{-1}_{(q,r)}\Bigg)$

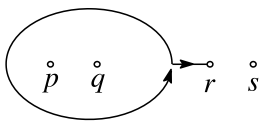
$$= \frac{1}{d_{pq}} \frac{-1}{d^{(-)}_{rs}}(e^{-2}_{rs} - e^2_{pq}) = \frac{-\langle e_{pqrs}\rangle}{\langle e_{pq}\rangle\langle e_{rs}\rangle}.$$



Examples 10. In case $T = \mathbb{C}\backslash\{p, q, r, s\}$ with $p < q < r < s$ and $u(t) = (t-p)^{\lambda_p}(t-q)^{\lambda_q}(t-r)^{\lambda_r}(t-s)^{\lambda_s}$, where $\lambda_{pq}, \lambda_{rs}, \lambda_q \in \mathbb{C}\backslash\mathbb{Z}$, we have

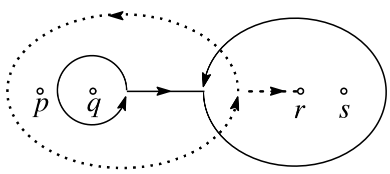
$$(q, \{r, s\}) \cdot (\{p, q\}, r)$$
$$= I\Big(\mathrm{reg}\left((q, \{r, s\}) \otimes u_{(q,r)}(t)\right), (\{p, q\}, r) \otimes u^{-1}_{(q,r)}(t)\Big)$$

$= I\Bigg(\Bigg\{$ 

$\Bigg\} \otimes u_{(q,r)}, \Bigg\{$ 

$\Bigg\} \otimes u^{-1}_{(q,r)}\Bigg)$

$$= \frac{1}{d^{(-)}_{pq}} \frac{-1}{d_{rs}}(e^{-2}_{pq} - e^2_{rs}) = \frac{-\langle e_{pqrs}\rangle}{\langle e_{pq}\rangle\langle e_{rs}\rangle}.$$



We refer the reader to [KY][MY1][MY2] for related works on the intersection numbers for loaded cycles.

1.3 Several hypergeometric functions in two variables and their transformation formulas

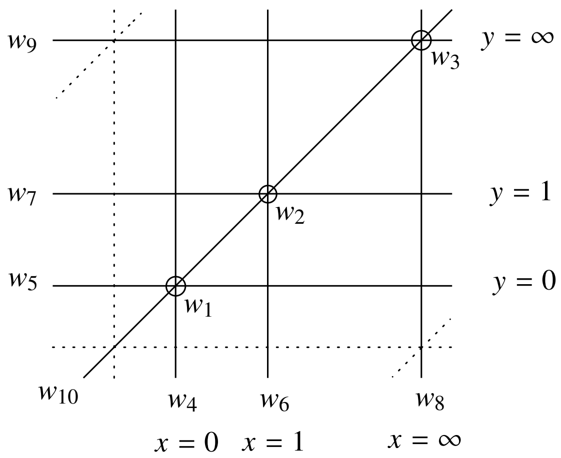
Appell's hypergeometric function F_1 stisfies

$$F_1(\alpha, \beta, \beta', \gamma; x, y) = F_1(\alpha, \beta', \beta, \gamma; y, x)$$

and the following transformation formulas.

$$\begin{aligned}
F_1(\alpha, \beta, \beta', \gamma; x, y) &= (1-x)^{-\beta}(1-y)^{-\beta'} F_1\left(\gamma-\alpha, \beta, \beta', \gamma; \frac{x}{x-1}, \frac{y}{y-1}\right) \\
&= (1-x)^{-\alpha} F_1\left(\alpha, \gamma-\beta-\beta', \beta', \gamma; \frac{x}{x-1}, \frac{x-y}{x-1}\right) \\
&= (1-y)^{-\alpha} F_1\left(\alpha, \beta, \gamma-\beta-\beta', \gamma; \frac{y-x}{y-1}, \frac{y}{y-1}\right) \\
&= (1-x)^{\gamma-\alpha-\beta}(1-y)^{-\beta'} F_1\left(\gamma-\alpha, \gamma-\beta-\beta', \beta', \gamma; x, \frac{y-x}{y-1}\right) \\
&= (1-x)^{-\beta}(1-y)^{\gamma-\alpha-\beta'} F_1\left(\gamma-\alpha, \beta, \gamma-\beta-\beta', \gamma; \frac{x-y}{x-1}, y\right).
\end{aligned}$$

The 60 solutions of E_1 in terms of F_1 are reduced to 10 solutions by these transformation formulas. Each fundamental set of solutions listed in the next section contains two of the 10 solutions in terms of F_1. The 60 solutions in terms of F_1 are listed in Appendix, where w_1 gives the solution around the point $(0,0)$, w_2 that around the point $(1,1)$, w_3 that around the point (∞,∞), w_4 that around the line $x = 0$, w_5 that around the line $y = 0$, w_6 that around the line $x = 1$, w_7 that around the line $y = 1$, w_8 that around the line $x = \infty$, w_9 that around the line $y = \infty$, and w_{10} that around the line $x = y$. For each $1 \le j \le 10$, the function w_j corresponds to $z_j = z_{10+j} = z_{20+j} = \cdots = z_{50+j}$ of pages 62-64 in [AKdF].



Horn's hypergeometric function G_2 is the analytic continuation of the series ([Ho])

$$G_2(\,\alpha,\,\beta,\,\gamma,\,\delta\,;\,x,\,y\,) = \sum_{m,n\geq 0} (\alpha)_m(\beta)_n(\gamma)_{n-m}(\delta)_{m-n}\frac{x^m}{m!}\frac{y^n}{n!}, \quad |x|<1,\ |y|<1,$$

where $(a)_{m-n} = \Gamma(a+m-n)/\Gamma(a)$. It stisfies

$$G_2(\,\alpha,\,\beta,\,\gamma,\,\delta\,;\,x,\,y\,) = G_2(\,\beta,\,\alpha,\,\delta,\,\gamma\,;\,y,\,x\,)$$

and the following transformation formulas.

$$\begin{aligned}
&G_2(\,\alpha,\,\alpha',\,\beta,\,\beta'\,;\,x,\,y\,)\\
&= (1+x)^{-\beta'}(1-xy)^{-\alpha'}G_2\left(1-\alpha-\beta,\,\alpha',\,\beta,\,\beta'\,;\,\frac{-x}{x+1},\,y\frac{1+x}{1-xy}\right)\\
&= (1+y)^{-\beta}(1-xy)^{-\alpha}G_2\left(\alpha,\,1-\alpha'-\beta',\,\beta,\,\beta'\,;\,x\frac{1+y}{1-xy},\,\frac{-y}{y+1}\right)\\
&= (1+x)^{-\beta'}(1+y)^{-\beta}(1-xy)^{1-\alpha-\alpha'}\\
&\quad\times G_2\left(1-\alpha-\beta,\,1-\alpha'-\beta',\,\beta,\,\beta'\,;\,-x\frac{1+y}{1+x},\,-y\frac{1+x}{1+y}\right).
\end{aligned}$$

The 60 solutions of E_1 in terms of G_2 are reduced to 15 solutions by these transfrmation formulas. Each fundamental set of solutions listed in the next section contains one of the 15 solutions in terms of G_2, and, converesely, each one of the 15 solutions in terms of G_2 is included into such a fundamental set of solutions. In Appendix, the 60 solutions in terms of G_2 are listed.

On the other hand, Appell's hypergeometric function F_2, F_3 defined by

$$F_2(\,\alpha,\ \beta,\ \beta',\ \gamma,\ \gamma'\ ;\ x,\ y\,) = \sum_{m,n\geq 0} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{m!n!(\gamma)_m(\gamma')_n} x^m y^n, \quad |x|+|y|<1,$$

$$F_3(\,\alpha,\ \alpha',\ \beta,\ \beta',\ \gamma\ ;\ x,\ y\,) = \sum_{m,n\geq 0} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{m!n!(\gamma)_{m+n}} x^m y^n, \quad |x|<1,\ |y|<1$$

and Horn's hypergeometric function H_2 dfined by

$$H_2(\,\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon\ ;\ x,\ y\,) = \sum_{m,n\geq 0} \frac{(\alpha)_{m-n}(\beta)_m(\gamma)_n(\delta)_n}{m!\,n!\,(\varepsilon)_m} x^m y^n, \quad |x|<1,\ |y|<(|x|+1)^{-1}$$

may appear as solutions of E_1. Nevertheless, the following equalities reduce them into the exppressions in terms of F_1 or G_2.

$$\begin{aligned}
&F_1(\,\alpha,\ \beta,\ \beta',\ \gamma\ ;\ x,\ y\,)\\
&= \left(\frac{x}{y}\right)^{\beta'} F_2\left(\beta+\beta',\ \alpha,\ \beta',\ \gamma,\ \beta+\beta'\ ;\ x,\ 1-\frac{x}{y}\right)\\
&= \left(\frac{y}{x}\right)^{\beta} F_2\left(\beta+\beta',\ \alpha,\ \beta,\ \gamma,\ \beta+\beta'\ ;\ y,\ 1-\frac{y}{x}\right)\\
&= (1-y)^{-\beta'} F_3\left(\alpha,\ \gamma-\alpha,\ \beta,\ \beta',\ \gamma\ ;\ x,\ \frac{y}{y-1}\right)\\
&= (1-y)^{-\beta} F_3\left(\gamma-\alpha,\ \alpha,\ \beta,\ \beta',\ \gamma\ ;\ \frac{x}{x-1},\ y\right)\\
&= \left(1-\frac{y}{x}\right)^{-\beta'} H_2\left(\beta,\ \alpha,\ \beta',\ 1-\beta,\ \gamma\ ;\ x,\ \frac{y}{x-y}\right)\\
&= \left(1-\frac{x}{y}\right)^{-\beta} H_2\left(\beta',\ \alpha,\ \beta,\ 1-\beta',\ \gamma\ ;\ y,\ \frac{x}{y-x}\right),\\
&G_2(\,\alpha,\ \alpha',\ \beta,\ \beta'\ ;\ x,\ y\,)\\
&= (1+y)^{-\alpha'} H_2\left(\beta',\ \alpha,\ \alpha',\ 1-\beta-\beta',\ 1-\beta\ ;\ -x,\ \frac{-y}{y+1}\right)\\
&= (1+x)^{-\alpha}(1+y)^{-\alpha'} F_2\left(1-\beta-\beta',\ \alpha,\ \alpha',\ 1-\beta,\ 1-\beta'\ ;\ \frac{x}{x+1},\ \frac{y}{y+1}\right).
\end{aligned}$$

Each fundamental set of solutions in the next section consists of two F_1's and one G_2.

2. Fundamental sets of solutions

We give a list of fundamental sets of solutions around each intersection of the singular loci:

$$(0,\ 0),\ (1,\ 1),\ (\infty,\ \infty),\ (0,\ 1),\ (1,\ 0),\ (0,\ \infty),\ (\infty,\ 0),\ (1,\ \infty),\ (\infty,\ 1).$$

In what follows, for $a \neq b$, the symbol $f_j^{(a,\,b)}$ denotes a function valid on a domain around the point $(a,\ b)$, where the two singular loci intersect, and the symbol $f_j^{(a,\,a)\,x=a}$ a function valid on the hypercone whose vertex is at $(a,\ a)$, and whose axis is $x = a$. A domain around the point $(a,\ a)$, where the three singular loci intersect, is covered by the three hypercones (sectors) such as (1) $|x-a| < |y-a|$ i.e. "near" $x = a$; (2) $|y-a| < |x-a|$ i.e. "near" $y = a$ (3) $|x-y| < |x-a|$ and $|x-y| < |y-a|$ i.e. "near" $x = y$. Therefore we give 15 fundamental sets of solutions.

- Around the point $(0, 0)$ i.e. $|x| < 1$ and $|y| < 1$;

(i) in case $|x| < |y|$; if $\gamma,\ \beta' - \gamma \notin \mathbb{Z}$,

$$f_1^{(0,\,0)\,x=0} = x^{1+\beta'-\gamma} y^{-\beta'} F_1\left(1+\beta+\beta'-\gamma,\ 1+\alpha-\gamma,\ \beta',\ 2+\beta'-\gamma\ ;\ x,\ \frac{x}{y} \right),$$

$$f_2^{(0,\,0)\,x=0} = y^{1-\gamma}\, G_2\left(\beta,\ 1+\alpha-\gamma,\ 1+\beta'-\gamma,\ \gamma-1\ ;\ -\frac{x}{y},\ -y \right),$$

$$f_3^{(0,\,0)\,x=0} = F_1(\ \alpha,\ \beta,\ \beta',\ \gamma\ ;\ x,\ y\ )$$

give a fundamental set of solutions,

(ii) in case $|x-y| < |y|,\ |x-y| < |x|$; if $\gamma,\ \beta+\beta' \notin \mathbb{Z}$,

$$\begin{aligned} f_1^{(0,\,0)\,x=y} &= y^{\beta+\beta'-\gamma}(1-y)^{\gamma-\alpha-1}(y-x)^{1-\beta-\beta'} \\ &\quad \times F_1\left(1-\beta',\ \gamma-\beta-\beta',\ 1+\alpha-\gamma,\ 2-\beta-\beta'\ ;\ \frac{y-x}{y},\ \frac{y-x}{y-1} \right), \\ f_2^{(0,\,0)\,x=y} &= x^{1-\gamma}(1-x)^{\gamma-\alpha-1}\, G_2\left(1+\alpha-\gamma,\ \beta',\ \gamma-1,\ 1-\beta-\beta'\ ;\ \frac{x}{1-x},\ \frac{y-x}{x} \right), \\ f_3^{(0,\,0)\,x=y} &= F_1(\ \alpha,\ \beta,\ \beta',\ \gamma\ ;\ x,\ y\ ) \end{aligned}$$

give a fundamental set of solutions, and

(iii) in case $|y| < |x|$; if $\gamma,\ \beta-\gamma \notin \mathbb{Z}$,

$$f_1^{(0,\,0)\,y=0} = x^{-\beta} y^{1+\beta-\gamma}\, F_1\left(1+\beta+\beta'-\gamma,\ \beta,\ 1+\alpha-\gamma,\ 2+\beta-\gamma\ ;\ \frac{y}{x},\ y \right),$$

$$f_2^{(0,\,0)\,y=0} = x^{1-\gamma}\, G_2\left(1+\alpha-\gamma,\ \beta',\ \gamma-1,\ 1+\beta-\gamma\ ;\ -x,\ -\frac{y}{x} \right),$$

$$f_3^{(0,\,0)\,y=0} = F_1(\ \alpha,\ \beta,\ \beta',\ \gamma\ ;\ x,\ y\ )$$

give a fundamental set of solutions.

- Around the point $(1, 1)$ i.e. $|x - 1| < 1$ and $|y - 1| < 1$;

(i) in case $|x - 1| < |y - 1|$; if $\alpha + \beta + \beta' - \gamma,\ \gamma - \alpha - \beta \notin \mathbb{Z}$,

$$f_1^{(1,\,1)\,x=1} =(1-x)^{\gamma-\alpha-\beta}(1-y)^{-\beta'} F_1\left(\gamma-\alpha,\ \gamma-\beta-\beta',\ \beta',\ 1+\gamma-\alpha-\beta\,;\ 1-x,\ \frac{1-x}{1-y}\right),$$

$$f_2^{(1,\,1)\,x=1} =(1-y)^{\gamma-\alpha-\beta-\beta'} G_2\left(\beta,\ \gamma-\beta-\beta',\ \gamma-\alpha-\beta,\ \alpha+\beta+\beta'-\gamma\,;\ \frac{x-1}{1-y},\ y-1\right),$$

$$f_3^{(1,\,1)\,x=1} =F_1(\,\alpha,\ \beta,\ \beta',\ 1+\alpha+\beta+\beta'-\gamma\,;\ 1-x,\ 1-y\,)$$

give a fundamental set of solutions,

(ii) in case $|x - y| < |y - 1|,\ |x - y| < |x - 1|$; if $\alpha + \beta + \beta' - \gamma,\ \beta + \beta' \notin \mathbb{Z}$,

$$f_1^{(1,\,1)\,x=y} =y^{\beta+\beta'-\gamma}(1-y)^{\gamma-\alpha-1}(y-x)^{1-\beta-\beta'} \times F_1\left(1-\beta',\ \gamma-\beta-\beta',\ 1+\alpha-\gamma,\ 2-\beta-\beta'\,;\ \frac{y-x}{y},\ \frac{y-x}{y-1}\right),$$

$$f_2^{(1,\,1)\,x=y} =y^{\beta+\beta'-\gamma}(1-y)^{\gamma-\alpha-\beta-\beta'} G_2\left(\beta,\ \gamma-\beta-\beta',\ 1-\beta-\beta',\ \alpha+\beta+\beta'-\gamma\,;\ \frac{y-x}{1-y},\ \frac{1-y}{y}\right),$$

$$f_3^{(1,\,1)\,x=y} =F_1(\,\alpha,\ \beta,\ \beta',\ 1+\alpha+\beta+\beta'-\gamma\,;\ 1-x,\ 1-y\,)$$

gives a fundamental set of solutions, and

(iii) in case $|y - 1| < |x - 1|$; if $\alpha + \beta + \beta' - \gamma,\ \gamma - \alpha - \beta' \notin \mathbb{Z}$,

$$f_1^{(1,\,1)\,y=1} =(1-x)^{-\beta}(1-y)^{\gamma-\alpha-\beta'} F_1\left(\gamma-\alpha,\ \beta,\ \gamma-\beta-\beta',\ 1+\gamma-\alpha-\beta'\,;\ \frac{1-y}{1-x},\ 1-y\right),$$

$$f_2^{(1,\,1)\,y=1} =(1-x)^{\gamma-\alpha-\beta-\beta'} G_2\left(\gamma-\beta-\beta',\ \beta',\ \alpha+\beta+\beta'-\gamma,\ \gamma-\alpha-\beta'\,;\ x-1,\ \frac{y-1}{1-x}\right),$$

$$f_3^{(1,\,1)\,y=1} =F_1(\,\alpha,\ \beta,\ \beta',\ 1+\alpha+\beta+\beta'-\gamma\,;\ 1-x,\ 1-y\,)$$

give a fundamental set of solutions.

- Aroud the point (∞, ∞) i.e. $|x| > 2$ and $|y| > 2$;;

(i) in case $|y| < |x|$; if $\beta + \beta' - \alpha,\ \alpha - \beta \notin \mathbb{Z}$,

$$f_1^{(\infty,\,\infty)\,x=\infty} = x^{-\alpha} F_1\left(\alpha,\, 1+\alpha-\gamma,\, \beta',\, 1+\alpha-\beta\,;\, \frac{1}{x},\, \frac{y}{x}\right),$$

$$f_2^{(\infty,\,\infty)\,x=\infty} = x^{-\beta} y^{\beta-\alpha} G_2\left(\beta,\, 1+\alpha-\gamma,\, \alpha-\beta,\, \beta+\beta'-\alpha\,;\, -\frac{y}{x},\, -\frac{1}{y}\right),$$

$$f_3^{(\infty,\,\infty)\,x=\infty} = x^{-\beta} y^{-\beta'} F_1\left(1+\beta+\beta'-\gamma,\, \beta,\, \beta',\, 1+\beta+\beta'-\alpha\,;\, \frac{1}{x},\, \frac{1}{y}\right)$$

give a fundamental set of solutions,

(ii) in case $|x-y| < |y|,\ |x-y| < |x|\,;\ \beta+\beta'-\alpha,\ \beta+\beta' \notin \mathbb{Z}$,

$$\begin{aligned} f_1^{(\infty,\,\infty)\,x=y} &= y^{-\alpha}\left(1-\frac{1}{y}\right)^{\gamma-\alpha-1}\left(1-\frac{x}{y}\right)^{1-\beta-\beta'} \\ &\quad \times F_1\left(1-\beta',\, \gamma-\beta-\beta',\, 1+\alpha-\gamma,\, 2-\beta-\beta'\,;\, \frac{y-x}{y},\, \frac{y-x}{y-1}\right), \\ f_2^{(\infty,\,\infty)\,x=y} &= x^{\beta'-\alpha}\left(1-\frac{1}{x}\right)^{\gamma-\alpha-\beta} y^{-\beta'}\left(1-\frac{1}{y}\right)^{-\beta'} \\ &\quad \times G_2\left(\gamma-\beta-\beta',\, \beta',\, \beta+\beta'-\alpha,\, 1-\beta-\beta'\,;\, -\frac{1}{x},\, \frac{y-x}{1-y}\right), \\ f_3^{(\infty,\,\infty)\,x=y} &= x^{-\beta} y^{-\beta'} F_1\left(1+\beta+\beta'-\gamma,\, \beta,\, \beta',\, 1+\beta+\beta'-\alpha\,;\, \frac{1}{x},\, \frac{1}{y}\right) \end{aligned}$$

give a fundamental set of solutions, and

(iii) in case $|x| < |y|\,;\ \beta+\beta'-\alpha,\ \alpha-\beta' \notin \mathbb{Z}$,

$$f_1^{(\infty,\,\infty)\,y=\infty} = y^{-\alpha} F_1\left(\alpha,\, \beta,\, 1+\alpha-\gamma,\, 1+\alpha-\beta'\,;\, \frac{x}{y},\, \frac{1}{y}\right),$$

$$f_2^{(\infty,\,\infty)\,y=\infty} = x^{\beta'-\alpha} y^{-\beta'} G_2\left(1+\alpha-\gamma,\, \beta',\, \beta+\beta'-\alpha,\, \alpha-\beta'\,;\, -\frac{1}{x},\, -\frac{x}{y}\right),$$

$$f_3^{(\infty,\,\infty)\,y=\infty} = x^{-\beta} y^{-\beta'} F_1\left(1+\beta+\beta'-\gamma,\, \beta,\, \beta',\, 1+\beta+\beta'-\alpha\,;\, \frac{1}{x},\, \frac{1}{y}\right)$$

give a fundamental set of solutions.

- Around the point $(0, 1)\,$; if $\gamma-\alpha-\beta',\ \beta'-\gamma \notin \mathbb{Z}$,

$$f_1^{(0,\,1)} = x^{1+\beta'-\gamma} y^{-\beta'} F_1\left(1+\beta+\beta'-\gamma,\, 1+\alpha-\gamma,\, \beta',\, 2+\beta'-\gamma\,;\, x,\, \frac{x}{y}\right),$$

$$f_2^{(0,\,1)} = y^{-\beta'}(1-x)^{-\beta} G_2\left(\beta,\, \beta',\, 1+\beta'-\gamma,\, \gamma-\alpha-\beta'\,;\, \frac{x}{1-x},\, \frac{1-y}{y}\right),$$

$$f_3^{(0,\,1)} = (1-x)^{-\beta}(1-y)^{\gamma-\alpha-\beta'} F_1\left(\gamma-\alpha,\, \beta,\, \gamma-\beta-\beta',\, 1+\gamma-\alpha-\beta'\,;\, \frac{1-y}{1-x},\, 1-y\right)$$

give a fundamental set of solutions.

• Around the point $(1, 0)$; if $\gamma - \alpha - \beta,\ \beta - \gamma \notin \mathbb{Z}$,

$$
\begin{aligned}
f_1^{(1,0)} =& x^{-\beta} y^{1+\beta-\gamma} F_1\left(1+\beta+\beta'-\gamma,\ \beta,\ 1+\alpha-\gamma,\ 2+\beta-\gamma\ ;\ \frac{y}{x},\ y\right),\\
f_2^{(1,0)} =& x^{-\beta}(1-y)^{-\beta'} G_2\left(\beta,\ \beta',\ \gamma-\alpha-\beta,\ 1+\beta-\gamma\ ;\ \frac{1-x}{x},\ \frac{y}{1-y}\right),\\
f_3^{(1,0)} =& (1-x)^{\gamma-\alpha-\beta}(1-y)^{-\beta'} F_1\left(\gamma-\alpha,\ \gamma-\beta-\beta',\ \beta',\ 1+\gamma-\alpha-\beta\ ;\ 1-x,\ \frac{1-x}{1-y}\right)
\end{aligned}
$$

give a fundamental set of solutions.

• Around the point $(0, \infty)$; if $\alpha - \beta',\ \beta' - \gamma \notin \mathbb{Z}$,

$$
\begin{aligned}
f_1^{(0,\infty)} =& y^{-\alpha} F_1\left(\alpha,\ \beta,\ 1+\alpha-\gamma,\ 1+\alpha-\beta'\ ;\ \frac{x}{y},\ \frac{1}{y}\right),\\
f_2^{(0,\infty)} =& y^{-\beta'} G_2\left(\beta,\ \beta',\ 1+\beta'-\gamma,\ \alpha-\beta'\ ;\ -x,\ -\frac{1}{y}\right),\\
f_3^{(0,\infty)} =& x^{1+\beta'-\gamma} y^{-\beta'} F_1\left(1+\beta+\beta'-\gamma,\ 1+\alpha-\gamma,\ \beta',\ 2+\beta'-\gamma\ ;\ x,\ \frac{x}{y}\right)
\end{aligned}
$$

give a fundamental set of solutions.

• Around the point $(\infty, 0)$; if $\alpha - \beta,\ \beta - \gamma \notin \mathbb{Z}$,

$$
\begin{aligned}
f_1^{(\infty,0)} =& x^{-\alpha} F_1\left(\alpha,\ 1+\alpha-\gamma,\ \beta',\ 1+\alpha-\beta\ ;\ \frac{1}{x},\ \frac{y}{x}\right),\\
f_2^{(\infty,0)} =& x^{-\beta} G_2\left(\beta,\ \beta',\ \alpha-\beta,\ 1+\beta-\gamma\ ;\ -\frac{1}{x},\ -y\right),\\
f_3^{(\infty,0)} =& x^{-\beta} y^{1+\beta-\gamma} F_1\left(1+\beta+\beta'-\gamma,\ \beta,\ 1+\alpha-\gamma,\ 2+\beta-\gamma\ ;\ \frac{y}{x},\ y\right)
\end{aligned}
$$

gives a fundamental set of solutions.

• Around the point $(1, \infty)$; if $\gamma - \alpha - \beta,\ \alpha - \beta' \notin \mathbb{Z}$,

$$
\begin{aligned}
f_1^{(1,\infty)} =& (1-x)^{\gamma-\alpha-\beta} y^{-\beta'} \left(1-\frac{1}{y}\right)^{-\beta'} F_1\left(\gamma-\alpha,\ \gamma-\beta-\beta',\ \beta',\ 1+\gamma-\alpha-\beta\ ;\ 1-x,\ \frac{1-x}{1-y}\right),\\
f_2^{(1,\infty)} =& y^{-\beta'}\left(1-\frac{1}{y}\right)^{-\beta'} G_2\left(\beta,\ \beta',\ \gamma-\alpha-\beta,\ \alpha-\beta'\ ;\ x-1,\ \frac{1}{y-1}\right),\\
f_3^{(1,\infty)} =& y^{-\alpha} F_1\left(\alpha,\ \beta,\ 1+\alpha-\gamma,\ 1+\alpha-\beta'\ ;\ \frac{x}{y},\ \frac{1}{y}\right)
\end{aligned}
$$

give a fundamental set of solutions.

- Around the point $(\infty, 1)$; if $\gamma-\alpha-\beta',\ \alpha-\beta\notin\mathbb{Z}$,

$$
\begin{aligned}
f_1^{(\infty,1)} =& x^{-\beta}\left(1-\frac{1}{x}\right)^{-\beta}(1-y)^{\gamma-\alpha-\beta'}F_1\left(\gamma-\alpha,\,\beta,\,\gamma-\beta-\beta',\,1+\gamma-\alpha-\beta'\,;\,\frac{1-y}{1-x},\,1-y\right),\\
f_2^{(\infty,1)} =& x^{-\beta}\left(1-\frac{1}{x}\right)^{-\beta}G_2\left(\beta,\,\beta',\,\alpha-\beta,\,\gamma-\alpha-\beta'\,;\,\frac{1}{x-1},\,y-1\right),\\
f_3^{(\infty,1)} =& x^{-\alpha}F_1\left(\alpha,\,1+\alpha-\gamma,\,\beta',\,1+\alpha-\beta\,;\,\frac{1}{x},\,\frac{y}{x}\right)
\end{aligned}
$$

give a fundamental set of solutions.

3. Connection formulas

We give connection formulas, each of which expresses a fundamental set of solutions around (0, 0) in terms of that around another point located at the boundary, which contains also (0, 0), of a simply connected real region of $(\mathbb{P}^1(\mathbb{C}))^2\backslash S$. Each formula gives the analytic continuation of the functions around $(0,0)$ along a path on the simply connected real region.

- The functions around (0, 0) in case $|x|<|y|$ in terms of those around (1, 1) in case $|y-1|<|x-1|$:

$$
\begin{aligned}
f_1^{(0,0)\,x=0}(x,y) =& \frac{\Gamma(2+\beta'-\gamma,\,\alpha+\beta+\beta'-\gamma)}{\Gamma(1+\beta+\beta'-\gamma,\,1+\alpha+\beta'-\gamma)}\,f_2^{(1,1)\,y=1}(x,y)\\
&+\frac{\Gamma(2+\beta'-\gamma,\,\gamma-\alpha-\beta-\beta')}{\Gamma(1-\alpha,\,1-\beta)}\,f_3^{(1,1)\,y=1}(x,y),
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
f_2^{(0,0)\,x=0}(x,y) =& \frac{\Gamma(2-\gamma,\,\alpha+\beta'-\gamma)}{\Gamma(1+\alpha-\gamma,\,1+\beta'-\gamma)}\,f_1^{(1,1)\,y=1}(x,y)\\
&+\frac{\Gamma(2-\gamma,\,\alpha+\beta+\beta'-\gamma,\,\gamma-\alpha-\beta',\,\gamma-\beta')}{\Gamma(1-\beta',\,\beta,\,1-\alpha,\,\alpha)}\,f_2^{(1,1)\,y=1}(x,y)\\
&+\frac{\Gamma(2-\gamma,\,\gamma-\alpha-\beta-\beta',\,\gamma-\beta')}{\Gamma(1-\beta',\,1-\alpha,\,\gamma-\beta-\beta')}\,f_3^{(1,1)\,y=1}(x,y),
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
f_3^{(0,0)\,x=0}(x,y) =& \frac{\Gamma(\gamma,\,\alpha+\beta'-\gamma)}{\Gamma(\alpha,\,\beta')}\,f_1^{(1,1)\,y=1}(x,y)\\
&+\frac{\Gamma(\gamma,\,\alpha+\beta+\beta'-\gamma,\,\gamma-\alpha-\beta')}{\Gamma(\gamma-\alpha,\,\alpha,\,\beta)}\,f_2^{(1,1)\,y=1}(x,y)\\
&+\frac{\Gamma(\gamma,\,\gamma-\alpha-\beta-\beta')}{\Gamma(\gamma-\alpha,\,\gamma-\beta-\beta')}\,f_3^{(1,1)\,y=1}(x,y),
\end{aligned}
\tag{3.3}
$$

where

$$f_1^{(0,0)\,x=0}(x,y) = x^{1+\beta'-\gamma}y^{-\beta'}F_1\left(1+\beta+\beta'-\gamma,\ 1+\alpha-\gamma,\ \beta',\ 2+\beta'-\gamma\,;\ x,\ \frac{x}{y}\right),$$

$$f_2^{(0,0)\,x=0}(x,y) = y^{1-\gamma}G_2\left(1+\alpha-\gamma,\ \beta,\ \gamma-1,\ 1+\beta'-\gamma\,;\ -y,\ \frac{-x}{y}\right),$$

$$f_3^{(0,0)\,x=0}(x,y) = F_1(\,\alpha,\ \beta,\ \beta',\ \gamma\,;\ x,\ y\,)$$

and

$$\begin{aligned} f_1^{(1,1)\,y=1}(x,y) &= (1-x)^{-\beta}(1-y)^{\gamma-\alpha-\beta'} \\ &\qquad \times F_1\left(\gamma-\alpha,\ \beta,\ \gamma-\beta-\beta',\ 1+\gamma-\alpha-\beta'\,;\ \frac{1-y}{1-x},\ 1-y\right), \\ f_2^{(1,1)\,y=1}(x,y) &= (1-x)^{\gamma-\alpha-\beta-\beta'} \\ &\qquad \times G_2\left(\gamma-\beta-\beta',\ \beta',\ \alpha+\beta+\beta'-\gamma,\ \gamma-\alpha-\beta'\,;\ x-1,\ \frac{1-y}{x-1}\right), \\ f_3^{(1,1)\,y=1}(x,y) &= F_1(\,\alpha,\ \beta,\ \beta',\ 1+\alpha+\beta+\beta'-\gamma\,;\ 1-x,\ 1-y\,). \end{aligned}$$

Here the arguments of $x,\ y,\ 1-x$ and $1-y$ of the factors $x^*,\ y^*,\ (1-x)^*$ and $(1-y)^*$ are assigned to be zero on the real region $0<x<y<1$. Note that (3.3) is the first equality of (19) in [Ol].

• The functions around $(0,0)$ in case $|y|<|x|$ in terms of those around $(1,1)$ in case $|x-1|<|y-1|$:

$$\begin{aligned} f_1^{(0,0)\,y=0}(x,y) &= \frac{\Gamma(2+\beta-\gamma,\ \alpha+\beta+\beta'-\gamma)}{\Gamma(1+\beta+\beta'-\gamma,\ 1+\alpha+\beta-\gamma)}\, f_2^{(1,1)\,x=1}(x,y) \\ &\quad + \frac{\Gamma(2+\beta-\gamma,\ \gamma-\alpha-\beta-\beta')}{\Gamma(1-\alpha,\ 1-\beta')}\, f_3^{(1,1)\,x=1}(x,y), \end{aligned} \tag{3.4}$$

$$\begin{aligned} f_2^{(0,0)\,y=0}(x,y) &= \frac{\Gamma(2-\gamma,\ \alpha+\beta-\gamma)}{\Gamma(1+\alpha-\gamma,\ 1+\beta-\gamma)}\, f_1^{(1,1)\,x=1}(x,y) \\ &\quad + \frac{\Gamma(2-\gamma,\ \alpha+\beta+\beta'-\gamma,\ \gamma-\alpha-\beta,\ \gamma-\beta)}{\Gamma(1-\beta,\ \beta',\ 1-\alpha,\ \alpha)}\, f_2^{(1,1)\,x=1}(x,y) \\ &\quad + \frac{\Gamma(2-\gamma,\ \gamma-\alpha-\beta-\beta',\ \gamma-\beta)}{\Gamma(1-\beta,\ 1-\alpha,\ \gamma-\beta-\beta')}\, f_3^{(1,1)\,x=1}(x,y), \end{aligned} \tag{3.5}$$

$$f_3^{(0,0)\,y=0}(x,y) = \frac{\Gamma(\gamma,\ \alpha+\beta-\gamma)}{\Gamma(\alpha,\ \beta)}\, f_1^{(1,1)\,x=1}(x,y)$$

$$+\frac{\Gamma(\gamma,\ \alpha+\beta+\beta'-\gamma,\ \gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha,\ \alpha,\ \beta')}\, f_2^{(1,1)\,x=1}(x,y)$$

$$+\frac{\Gamma(\gamma,\ \gamma-\alpha-\beta-\beta')}{\Gamma(\gamma-\alpha,\ \gamma-\beta-\beta')}\, f_3^{(1,1)\,x=1}(x,y), \tag{3.6}$$

where

$$f_1^{(0,0)\,y=0}(x,y) = y^{1+\beta-\gamma}x^{-\beta}F_1\left(1+\beta+\beta'-\gamma,\ 1+\alpha-\gamma,\ \beta,\ 2+\beta-\gamma\,;\ y,\ \frac{y}{x}\right),$$

$$f_2^{(0,0)\,y=0}(x,y) = x^{1-\gamma}G_2\left(1+\alpha-\gamma,\ \beta',\ \gamma-1,\ 1+\beta-\gamma\,;\ -x,\ \frac{-y}{x}\right),$$

$$f_3^{(0,0)\,y=0}(x,y) = F_1(\,\alpha,\ \beta',\ \beta,\ \gamma\,;\ y,\ x\,)$$

and

$$f_1^{(1,1)\,x=1}(x,y) = (1-y)^{-\beta'}(1-x)^{\gamma-\alpha-\beta}$$
$$\times F_1\left(\gamma-\alpha',\ \beta',\ \gamma-\beta-\beta',\ 1+\gamma-\alpha-\beta\,;\ \frac{1-x}{1-y},\ 1-x\right),$$

$$f_2^{(1,1)\,x=1}(x,y) = (1-y)^{\gamma-\alpha-\beta-\beta'}$$
$$\times G_2\left(\gamma-\beta-\beta',\ \beta,\ \alpha+\beta+\beta'-\gamma,\ \gamma-\alpha-\beta'\,;\ y-1,\ \frac{1-x}{y-1}\right),$$

$$f_3^{(1,1)\,x=1}(x,y) = F_1(\,\alpha,\ \beta',\ \beta,\ 1+\alpha+\beta+\beta'-\gamma\,;\ 1-y,\ 1-x\,).$$

Here the arguments of $x,\ y,\ 1-x$ and $1-y$ of the factors $x^*,\ y^*,\ (1-x)^*$ and $(1-y)^*$ are assigned to be zero on the real region $0<y<x<1$. Note that (3.6) is the second equality of (19) in [Ol].

• The functions around $(0,\ 0)$ in case $|x-y|<|x|,\ |x-y|<|y|$ in terms of those around $(1,\ 1)$ in case $|x-y|<|x-1|,\ |x-y|<|y-1|$:

$$f_1^{(0,0)\,x=y}(x,y) = f_1^{(1,1)\,x=y}(x,y), \tag{3.7}$$

$$f_2^{(0,0)\,x=y}(x,y) = \frac{\Gamma(\alpha+\beta+\beta'-\gamma,\ 2-\gamma)}{\Gamma(1+\alpha-\gamma,\ 1+\beta+\beta'-\gamma)}\, f_2^{(1,1)\,x=y}(x,y)$$
$$+\frac{\Gamma(\gamma-\alpha-\beta-\beta',\ 2-\gamma)}{\Gamma(1-\alpha,\ 1-\beta-\beta')}\, f_3^{(1,1)\,x=y}(x,y), \tag{3.8}$$

$$f_3^{(0,0)\,x=y}(x,y) = \frac{\Gamma(\alpha+\beta+\beta'-\gamma,\ \gamma)}{\Gamma(\alpha,\ \beta+\beta')}\, f_2^{(1,1)\,x=y}(x,y)$$
$$+\frac{\Gamma(\gamma-\alpha-\beta-\beta',\ \gamma)}{\Gamma(\gamma-\beta-\beta',\ \gamma-\alpha)}\, f_3^{(1,1)}(x,y), \tag{3.9}$$

where

$$f_1^{(0,0)\,x=y}(x,y) = y^{\beta+\beta'-\gamma}(1-y)^{\gamma-\alpha-1}(y-x)^{1-\beta-\beta'}$$
$$\times F_1\left(1-\beta',\,\gamma-\beta-\beta',\,1+\alpha-\gamma,\,2-\beta-\beta'\,;\,\frac{y-x}{y},\,\frac{y-x}{y-1}\right),$$
$$f_2^{(0,0)\,x=y}(x,y) = x^{1-\gamma}(1-x)^{\gamma-\alpha-1}$$
$$\times G_2\left(1+\alpha-\gamma,\,\beta',\,\gamma-1,\,1-\beta-\beta'\,;\,\frac{x}{1-x},\,\frac{y-x}{x}\right),$$
$$f_3^{(0,0)\,x=y}(x,y) = F_1(\,\alpha,\,\beta,\,\beta',\,\gamma\,;\,x,\,y\,)$$

and

$$f_1^{(1,1)\,x=y}(x,y) = y^{\beta+\beta'-\gamma}(1-y)^{\gamma-\alpha-1}(y-x)^{1-\beta-\beta'}$$
$$\times F_1\left(1-\beta',\,\gamma-\beta-\beta',\,1+\alpha-\gamma,\,2-\beta-\beta'\,;\,\frac{y-x}{y},\,\frac{y-x}{y-1}\right),$$
$$f_2^{(1,1)\,x=y}(x,y) = y^{\beta+\beta'-\gamma}(1-y)^{\gamma-\alpha-\beta-\beta'}$$
$$\times G_2\left(\beta,\,\gamma-\beta-\beta',\,1-\beta-\beta',\,\alpha+\beta+\beta'-\gamma\,;\,\frac{y-x}{1-y},\,\frac{1-y}{y}\right),$$
$$f_3^{(1,1)}(x,y) = F_1(\alpha,\,\beta,\,\beta',\,1+\alpha+\beta+\beta'-\gamma\,;\,1-x,\,1-y\,).$$

Here the arguments of $x,\ y,\ 1-x,\ 1-y$ and $y-x$ of the factors $x^*,\ y^*,\ (1-x)^*,\ (1-y)^*$ and $(y-x)^*$ are assigned to be zero on the real region $0<x<y<1$. Note that (3.9) is the first equality of (23) in [Ol].

- The functions around $(0,\,0)$ in case $|x|<|y|$ in terms of those around $(0,\,1)$:

$$f_1^{(0,0)\,x=0}(x,y) = f_1^{(0,1)}(x,y), \tag{3.10}$$

$$f_2^{(0,0)\,x=0}(x,y) = \frac{\Gamma(\gamma-\alpha-\beta',\,2-\gamma)}{\Gamma(1-\alpha,\,1-\beta')}\,f_2^{(0,1)}(x,y) + \frac{\Gamma(\alpha+\beta'-\gamma,\,2-\gamma)}{\Gamma(1+\alpha-\gamma,\,1+\beta'-\gamma)}\,f_3^{(0,1)}(x,y), \tag{3.11}$$

$$f_3^{(0,0)\,x=0}(x,y) = \frac{\Gamma(\gamma-\alpha-\beta',\,\gamma)}{\Gamma(\gamma-\beta',\,\gamma-\alpha)}\,f_2^{(0,1)}(x,y) + \frac{\Gamma(\alpha+\beta'-\gamma,\,\gamma)}{\Gamma(\alpha,\,\beta')}\,f_3^{(0,1)}(x,y), \tag{3.12}$$

where

$$f_1^{(0,0)\,x=0}(x,y) = x^{1+\beta'-\gamma}y^{-\beta'}F_1\left(1+\beta+\beta'-\gamma,\ 1+\alpha-\gamma,\ \beta',\ 2+\beta'-\gamma\,;\ x,\ \frac{x}{y}\right),$$

$$f_2^{(0,0)\,x=0}(x,y) = y^{1-\gamma}G_2\left(1+\alpha-\gamma,\ \beta,\ \gamma-1,\ 1+\beta'-\gamma\,;\ -y,\ \frac{-x}{y}\right),$$

$$f_3^{(0,0)\,x=0}(x,y) = F_1(\,\alpha,\ \beta,\ \beta',\ \gamma\,;\ x,\ y\,)$$

and

$$f_1^{(0,1)}(x,y) = x^{1+\beta'-\gamma}y^{-\beta'}F_1\left(1+\beta+\beta'-\gamma,\ 1+\alpha-\gamma,\ \beta',\ 2+\beta'-\gamma\,;\ x,\ \frac{x}{y}\right),$$

$$f_2^{(0,1)}(x,y) = (1-x)^{-\beta}y^{-\beta'}\,G_2\left(\beta,\ \beta',\ 1+\beta'-\gamma,\ \gamma-\alpha-\beta'\,;\ \frac{x}{1-x},\ \frac{1-y}{y}\right),$$

$$f_3^{(0,1)}(x,y) = (1-x)^{-\beta}(1-y)^{\gamma-\alpha-\beta'} \times F_1\left(\gamma-\alpha,\ \beta,\ \gamma-\beta-\beta',\ 1+\gamma-\alpha-\beta'\,;\ \frac{1-y}{1-x},\ 1-y\right).$$

Here the arguments of $x,\ y,\ 1-x$ and $1-y$ of the factors $x^*,\ y^*,\ (1-x)^*$ and $(1-y)^*$ are assigned to be zero on the real region $0<x<y<1$. Note that (3.12) is the first equality of (15) in [Ol].

- The functions around $(0,\ 0)$ in case $|x|<|y|$ in terms of those around $(\infty,\ 1)$:

$$f_1^{(0,0)\,x=0}(x,y) = \frac{\Gamma(\alpha-\beta,\ 1+\beta'-\gamma)}{\Gamma(1-\beta,\ 1+\alpha+\beta'-\gamma)}\,f_2^{(\infty,\,1)}(x,y) + \frac{\Gamma(\beta-\alpha,\ 1+\beta'-\gamma)}{\Gamma(1+\beta+\beta'-\gamma,\ 1-\alpha)}\,f_3^{(\infty,\,1)}(x,y), \tag{3.13}$$

$$f_2^{(0,0)\,x=0}(x,y) = \frac{\Gamma(\alpha+\beta'-\gamma,\ 2-\gamma)}{\Gamma(1+\alpha-\gamma,\ 1+\beta'-\gamma)}\,f_1^{(\infty,\,1)}(x,y) + \frac{\Gamma(\gamma-\beta',\ \alpha-\beta,\ \alpha+\beta'-\gamma,\ 2-\gamma)}{\Gamma(1-\beta',\ 1-\alpha,\ \alpha,\ \gamma-\beta-\beta')}\,f_2^{(\infty,\,1)}(x,y) + \frac{\Gamma(\gamma-\beta',\ \beta-\alpha,\ 2-\gamma)}{\Gamma(1-\alpha,\ \beta,\ 1-\beta')}\,f_3^{(\infty,\,1)}(x,y), \tag{3.14}$$

$$f_3^{(0,0)}(x,y) = \frac{\Gamma(\alpha+\beta'-\gamma,\ \gamma)}{\Gamma(\alpha,\ \beta')}\,f_1^{(\infty,\,1)}(x,y) + \frac{\Gamma(\alpha-\beta,\ \gamma-\alpha-\beta',\ \gamma)}{\Gamma(\gamma-\alpha,\ \alpha,\ \gamma-\beta-\beta')}\,f_2^{(\infty,\,1)}(x,y) + \frac{\Gamma(\beta-\alpha,\ \gamma)}{\Gamma(\beta,\ \gamma-\alpha)}\,f_3^{(\infty,\,1)}(x,y), \tag{3.15}$$

where

$$f_1^{(0,0)\,x=0}(x,y) = (-x)^{1+\beta'-\gamma} y^{-\beta'} F_1\left(1+\beta+\beta'-\gamma,\, 1+\alpha-\gamma,\, \beta',\, 2+\beta'-\gamma\,;\, x,\, \frac{x}{y}\right),$$

$$f_2^{(0,0)\,x=0}(x,y) = y^{1-\gamma}\, G_2\left(\beta,\, 1+\alpha-\gamma,\, 1+\beta'-\gamma,\, \gamma-1\,;\, -\frac{x}{y},\, -y\right),$$

$$f_3^{(0,0)\,x=0}(x,y) = F_1(\,\alpha,\, \beta,\, \beta',\, \gamma\,;\, x,\, y\,),$$

and

$$f_1^{(\infty,\,1)}(x,y) = (-x)^{-\beta}\left(1-\frac{1}{x}\right)^{-\beta}(1-y)^{\gamma-\alpha-\beta'}$$
$$\times F_1\left(\gamma-\alpha,\, \beta,\, \gamma-\beta-\beta',\, 1+\gamma-\alpha-\beta'\,;\, \frac{1-y}{1-x},\, 1-y\right),$$

$$f_2^{(\infty,\,1)}(x,y) = (-x)^{-\beta}\left(1-\frac{1}{x}\right)^{-\beta} G_2\left(\beta,\, \beta',\, \alpha-\beta,\, \gamma-\alpha-\beta'\,;\, \frac{1}{x-1},\, y-1\right),$$

$$f_3^{(\infty,\,1)}(x,y) = (-x)^{-\alpha} F_1\left(\alpha,\, 1+\alpha-\gamma,\, \beta',\, 1+\alpha-\beta\,;\, \frac{1}{x},\, \frac{y}{x}\right).$$

Here the arguments of $-x$, y, $1-y$ and $1-\dfrac{1}{x}$ of the factors $(-x)^*$, y^*, $(1-y)^*$ and $\left(1-\dfrac{1}{x}\right)^*$ are assigned to be zero on the real region $\infty < x < 0 < y < 1$. Note that (3.15) is the second equality of (21) in [Ol].

- The functions around $(0,0)$ in case $|y| < |x|$ in terms of those around $(\infty, 0)$:

$$f_1^{(0,0)\,y=0}(x,y) = f_3^{(\infty,\,0)}(x,y), \tag{3.16}$$

$$f_2^{(0,0)\,y=0}(x,y) = \frac{\Gamma(\beta-\alpha,\, 2-\gamma)}{\Gamma(1+\beta-\gamma,\, 1-\alpha)}\, f_1^{(\infty,\,0)}(x,y) + \frac{\Gamma(\alpha-\beta,\, 2-\gamma)}{\Gamma(1+\alpha-\gamma,\, 1-\beta)}\, f_2^{(\infty,\,0)}(x,y), \tag{3.17}$$

$$f_3^{(0,0)\,y=0}(x,y) = \frac{\Gamma(\beta-\alpha,\, \gamma)}{\Gamma(\beta,\, \gamma-\alpha)}\, f_1^{(\infty,\,0)}(x,y) + \frac{\Gamma(\alpha-\beta,\, \gamma)}{\Gamma(\alpha,\, \gamma-\beta)}\, f_2^{(\infty,\,0)}(x,y), \tag{3.18}$$

where

$$f_1^{(0,0)\,y=0}(x,y) = (-y)^{1+\beta-\gamma}(-x)^{-\beta} F_1\left(1+\beta+\beta'-\gamma,\ 1+\alpha-\gamma,\ \beta,\ 2+\beta-\gamma\,;\ y,\ \frac{y}{x}\right),$$

$$f_2^{(0,0)\,y=0}(x,y) = (-x)^{1-\gamma} G_2\left(1+\alpha-\gamma,\ \beta',\ \gamma-1,\ 1+\beta-\gamma\,;\ -x,\ \frac{-y}{x}\right),$$

$$f_3^{(0,0)\,y=0}(x,y) = F_1(\,\alpha,\ \beta',\ \beta,\ \gamma\,;\ y,\ x\,)$$

and

$$f_1^{(\infty,\,0)}(x,y) = (-x)^{-\alpha}\, F_1\left(\alpha,\ 1+\alpha-\gamma,\ \beta',\ 1+\alpha-\beta\,;\ \frac{1}{x},\ \frac{y}{x}\right),$$

$$f_2^{(\infty,\,0)}(x,y) = (-x)^{-\beta}\, G_2\left(\beta,\ \beta',\ \alpha-\beta,\ 1+\beta-\gamma\,;\ -\frac{1}{x},\ -y\right),$$

$$f_3^{(\infty,\,0)}(x,y) = (-x)^{-\beta}(-y)^{1+\beta-\gamma}\, F_1\left(1+\beta+\beta'-\gamma,\ \beta,\ 1+\alpha-\gamma,\ 2+\beta-\gamma\,;\ \frac{y}{x},\ y\right).$$

Here the arguments of $-x,\ -y,\ 1-x$ and $1-y$ of the factors $(-x)^*$, $(-y)^*$, $(1-x)^*$ and $(1-y)^*$ are assigned to be zero on the real region $\infty < x < y < 0$. Note that (3.18) is the second equality of (17) in [Ol].

- The functions around $(0,\ 0)$ in case $|y| < |x|$ in terms of those around $(\infty,\ \infty)$ in case $|y| < |x|$:

$$\begin{aligned} f_1^{(0,0)\,y=0}(x,y) &= \frac{\Gamma(\beta+\beta'-\alpha,\ 2+\beta-\gamma)}{\Gamma(1+\beta+\beta'-\gamma,\ 1+\beta-\alpha)}\, f_2^{(\infty,\,\infty)\,x=\infty}(x,y) \\ &+ \frac{\Gamma(\alpha-\beta-\beta',\ 2+\beta-\gamma)}{\Gamma(1-\beta',\ 1+\alpha-\gamma)}\, f_3^{(\infty,\,\infty)\,x=\infty}(x,y), \end{aligned} \tag{3.19}$$

$$\begin{aligned} f_2^{(0,0)\,y=0}(x,y) &= \frac{\Gamma(\beta-\alpha,\ 2-\gamma)}{\Gamma(1-\alpha,\ 1+\beta-\gamma)}\, f_1^{(\infty,\,\infty)\,x=\infty}(x,y) \\ &+ \frac{\Gamma(2-\gamma,\ \gamma-\beta,\ \beta+\beta'-\alpha,\ \alpha-\beta)}{\Gamma(1-\beta,\ \gamma-\alpha,\ 1+\alpha-\gamma,\ \beta')}\, f_2^{(\infty,\,\infty)\,x=\infty}(x,y) \\ &+ \frac{\Gamma(\alpha-\beta-\beta',\ \gamma-\beta,\ 2-\gamma)}{\Gamma(1+\alpha-\gamma,\ \gamma-\beta-\beta',\ 1-\beta)}\, f_3^{(\infty,\,\infty)\,x=\infty}(x,y), \end{aligned} \tag{3.20}$$

$$\begin{aligned} f_3^{(0,0)}(x,y) &= \frac{\Gamma(\beta-\alpha,\ \gamma)}{\Gamma(\beta,\ \gamma-\alpha)}\, f_1^{(\infty,\,\infty)\,x=\infty}(x,y) \\ &+ \frac{\Gamma(\gamma,\ \alpha-\beta,\ \beta+\beta'-\alpha)}{\Gamma(\gamma-\alpha,\ \alpha,\ \beta')}\, f_2^{(\infty,\,\infty)\,x=\infty}(x,y) \\ &+ \frac{\Gamma(\alpha-\beta-\beta',\ \gamma)}{\Gamma(\alpha,\ \gamma-\beta-\beta')}\, f_3^{(\infty,\,\infty)\,x=\infty}(x,y), \end{aligned} \tag{3.21}$$

where

$$f_1^{(0,0)\,y=0}(x,y) = (-y)^{1+\beta-\gamma}(-x)^{-\beta} F_1\left(1+\beta+\beta'-\gamma,\ 1+\alpha-\gamma,\ \beta,\ 2+\beta-\gamma\,;\ y,\ \frac{y}{x} \right),$$

$$f_2^{(0,0)\,y=0}(x,y) = (-x)^{1-\gamma} G_2\left(1+\alpha-\gamma,\ \beta',\ \gamma-1,\ 1+\beta-\gamma\,;\ -x,\ \frac{-y}{x} \right),$$

$$f_3^{(0,0)\,y=0}(x,y) = F_1(\,\alpha,\ \beta',\ \beta,\ \gamma\,;\ y,\ x\,)$$

and

$$f_1^{(\infty,\,\infty)\,x=\infty}(x,y) = (-x)^{-\alpha} F_1\left(\alpha,\ 1+\alpha-\gamma,\ \beta',\ 1+\alpha-\beta\,;\ \frac{1}{x},\ \frac{y}{x} \right),$$

$$f_2^{(\infty,\,\infty)\,x=\infty}(x,y) = (-x)^{-\beta}\,(-y)^{\beta-\alpha}\, G_2\left(\beta,\ 1+\alpha-\gamma,\ \alpha-\beta,\ \beta+\beta'-\gamma\,;\ -\frac{y}{x},\ -\frac{1}{x} \right),$$

$$f_3^{(\infty,\,\infty)\,x=\infty}(x,y) = (-x)^{-\beta}\,(-y)^{-\beta'}\, F_1(\,1+\beta+\beta'-\gamma,\ \beta,\ \beta',\ 1+\beta+\beta'-\alpha\,;\ x^{-1},\ y^{-1}\,).$$

Here the arguments of $-x$ and $-y$ of the factors $(-x)^*$ and $(-y)^*$ are assigned to be zero on the real region $\infty < x < y < 0$. Note that (3.21) is the second equality of (22) in [Ol].

• The functions around $(0,\,0)$ in case $|x-y| < |x|$ and $|x-y| < |y|$ in terms of those around $(\infty,\,\infty)$ in case $|x-y| < |x|$ and $|x-y| < |y|$:

$$f_1^{(0,0)\,x=y}(x,y) = f_1^{(\infty,\,\infty)\,x=y}(x,y), \tag{3.22}$$

$$\begin{aligned} f_2^{(0,0)\,x=y}(x,y) &= \frac{\Gamma(\beta+\beta'-\alpha,\ 2-\gamma)}{\Gamma(1-\alpha,\ 1+\beta+\beta'-\gamma)}\, f_2^{(\infty,\,\infty)\,x=y}(x,y) \\ &\quad + \frac{\Gamma(\alpha-\beta-\beta',\ 2-\gamma)}{\Gamma(1+\alpha-\gamma,\ 1-\beta-\beta')}\, f_3^{(\infty,\,\infty)\,x=y}(x,y), \end{aligned} \tag{3.23}$$

$$\begin{aligned} f_3^{(0,0)\,x=y}(x,y) &= \frac{\Gamma(\beta+\beta'-\alpha,\ \gamma)}{\Gamma(\gamma-\alpha,\ \beta+\beta')}\, f_2^{(\infty,\,\infty)\,x=y}(x,y) \\ &\quad + \frac{\Gamma(\alpha-\beta-\beta',\ \gamma)}{\Gamma(\gamma-\beta-\beta',\ \alpha)}\, f_3^{(\infty,\,\infty)\,x=y}(x,y), \end{aligned} \tag{3.24}$$

where

$$f_1^{(0,0)\,x=y}(x, y) = (-y)^{\beta+\beta'-\gamma}(1-y)^{\gamma-\alpha-1}(y-x)^{1-\beta-\beta'}$$
$$\times F_1\left(1-\beta',\, \gamma-\beta-\beta',\, 1+\alpha-\gamma,\, 2-\beta-\beta'\,;\, \frac{y-x}{y},\, \frac{y-x}{y-1}\right),$$
$$f_2^{(0,0)\,x=y}(x, y) = (-x)^{1-\gamma}(1-x)^{\gamma-\alpha-1}$$
$$\times G_2\left(1+\alpha-\gamma,\, \beta',\, \gamma-1,\, 1-\beta-\beta'\,;\, \frac{x}{1-x},\, \frac{y-x}{x}\right),$$
$$f_3^{(0,0)\,x=y}(x, y) = F_1(\,\alpha,\, \beta,\, \beta',\, \gamma\,;\, x,\, y\,),$$

and

$$f_1^{(\infty,\,\infty)\,x=y} =(-y)^{-\alpha}\left(1-\frac{1}{y}\right)^{\gamma-\alpha-1}\left(1-\frac{x}{y}\right)^{1-\beta-\beta'}$$
$$\times\, F_1\left(1-\beta',\, \gamma-\beta-\beta',\, 1+\alpha-\gamma,\, 2-\beta-\beta'\,;\, \frac{y-x}{y},\, \frac{y-x}{y-1}\right),$$
$$f_2^{(\infty,\,\infty)\,x=y} =(-x)^{\beta'-\alpha}\left(1-\frac{1}{x}\right)^{\gamma-\alpha-\beta}(-y)^{-\beta'}\left(1-\frac{1}{y}\right)^{-\beta'}$$
$$\times\, G_2\left(\gamma-\beta-\beta',\, \beta',\, \beta+\beta'-\alpha,\, 1-\beta-\beta'\,;\, -\frac{1}{x},\, \frac{y-x}{1-y}\right),$$
$$f_3^{(\infty,\,\infty)\,x=y} =(-x)^{-\beta}(-y)^{-\beta'}F_1\left(1+\beta+\beta'-\gamma,\, \beta,\, \beta',\, 1+\beta+\beta'-\alpha\,;\, \frac{1}{x},\, \frac{1}{y}\right).$$

Here the arguments of $-x,\ -y,\ 1-x,\ 1-y,\ y-x,\ 1-\dfrac{1}{x},\ 1-\dfrac{1}{y}$ and $1-\dfrac{x}{y}$ of the factors $(-x)^*,\ (-y)^*,\ (1-x)^*,\ (1-y)^*,\ (y-x)^*,\ \left(1-\dfrac{1}{x}\right)^*,\ \left(1-\dfrac{1}{y}\right)^*$ and $\left(1-\dfrac{x}{y}\right)^*$ are assigned to be zero on the real region $\infty < x < y < 0$. Note that (3.24) is the second equality of (22) in [Ol].

• The functions around $(0,\ 0)$ in case $|x| < |y|$ in terms of those around $(\infty,\ \infty)$ in case $|x| < |y|$:

$$f_1^{(0,0)\,x=0}(x, y) = \frac{\Gamma(\beta+\beta'-\alpha,\, 2+\beta'-\gamma)}{\Gamma(1+\beta+\beta'-\gamma,\, 1+\beta'-\alpha)}\, f_2^{(\infty,\,\infty)\,y=\infty}(x, y)$$
$$+\, \frac{\Gamma(\alpha-\beta-\beta',\, 2+\beta'-\gamma)}{\Gamma(1-\beta,\, 1+\alpha-\gamma)}\, f_3^{(\infty,\,\infty)\,y=\infty}(x, y), \tag{3.25}$$
$$f_2^{(0,0)\,x=0}(x, y) = \frac{\Gamma(\beta'-\alpha,\, 2-\gamma)}{\Gamma(1-\alpha,\, 1+\beta'-\gamma)}\, f_1^{(\infty,\,\infty)\,y=\infty}(x, y)$$

$$+\frac{\Gamma(2-\gamma,\,\gamma-\beta',\,\beta+\beta'-\alpha,\,\alpha-\beta')}{\Gamma(1-\beta',\,\gamma-\alpha,\,1+\alpha-\gamma,\,\beta)}\,f_2^{(\infty,\,\infty)\,y=\infty}(x,y)$$

$$+\frac{\Gamma(\alpha-\beta-\beta',\,\gamma-\beta',\,2-\gamma)}{\Gamma(1+\alpha-\gamma,\,\gamma-\beta-\beta',\,1-\beta')}\,f_3^{(\infty,\,\infty)\,y=\infty}(x,y), \tag{3.26}$$

$$f_3^{(0,0)}(x,y)=\frac{\Gamma(\beta'-\alpha,\,\gamma)}{\Gamma(\beta',\,\gamma-\alpha)}\,f_1^{(\infty,\,\infty)\,y=\infty}(x,y)$$

$$+\frac{\Gamma(\gamma,\,\alpha-\beta',\,\beta+\beta'-\alpha)}{\Gamma(\gamma-\alpha,\,\alpha,\,\beta)}\,f_2^{(\infty,\,\infty)\,y=\infty}(x,y)$$

$$+\frac{\Gamma(\alpha-\beta-\beta',\,\gamma)}{\Gamma(\alpha,\,\gamma-\beta-\beta')}\,f_3^{(\infty,\,\infty)\,y=\infty}(x,y), \tag{3.27}$$

where

$$f_1^{(0,0)\,x=0}(x,y)=(-x)^{1+\beta'-\gamma}\,(-y)^{-\beta'}\,F_1\left(1+\beta+\beta'-\gamma,\,1+\alpha-\gamma,\,\beta',\,2+\beta'-\gamma\,;\,x,\,\frac{x}{y}\right),$$

$$f_2^{(0,0)\,x=0}(x,y)=(-y)^{1-\gamma}\,G_2\left(\beta,\,1+\alpha-\gamma,\,1+\beta'-\gamma,\,\gamma-1\,;\,\frac{-x}{y},\,-y\right),$$

$$f_3^{(0,0)\,x=0}(x,y)=F_1(\,\alpha,\,\beta,\,\beta',\,\gamma\,;\,x,\,y\,)$$

and

$$f_1^{(\infty,\,\infty)\,y=\infty}=(-y)^{-\alpha}F_1\left(\alpha,\,\beta,\,1+\alpha-\gamma,\,1+\alpha-\beta'\,;\,\frac{x}{y},\,\frac{1}{y}\right),$$

$$f_2^{(\infty,\,\infty)\,y=\infty}=(-x)^{\beta'-\alpha}\,(-y)^{\beta'}\,G_2\left(1+\alpha-\gamma,\,\beta',\,\beta+\beta'-\alpha,\,\alpha-\beta'\,;\,-\frac{1}{x},\,-\frac{x}{y}\right),$$

$$f_3^{(\infty,\,\infty)\,y=\infty}=(-x)^{-\beta}\,(-y)^{-\beta'}\,F_1\left(1+\beta+\beta'-\gamma,\,\beta,\,\beta',\,1+\beta+\beta'-\alpha\,;\,\frac{1}{x},\,\frac{1}{y}\right).$$

Here the arguments of $-x$ and $-y$ of the factors $(-x)^*$ and $(-y)^*$ are assigned to be zero on the real region $\infty<y<x<0$. Note that (3.27) is the first equality of (22) in [Ol].

- The functions around $(0,0)$ in case $|x|<|y|$ in terms of those around $(0,0)$ in case $|x-y|<|x|$, $|x-y|<|y|$:

$$f_1^{(0,0)\,x=0}(x,y) = \frac{\Gamma(2+\beta'-\gamma,\,\beta+\beta'-1)}{\Gamma(1+\beta+\beta'-\gamma,\,\beta')}\,f_1^{(0,0)\,x=y}(x,y) + \frac{\Gamma(2+\beta'-\gamma,\,1-\beta-\beta')}{\Gamma(1-\beta,\,2-\gamma)}\,f_2^{(0,0)\,x=y}(x,y), \tag{3.28}$$

$$f_2^{(0,0)\,x=0}(x,y) = \frac{\Gamma(\gamma-\beta',\,\beta+\beta'-1)}{\Gamma(\gamma-1,\,\beta)}\,f_1^{(0,0)\,x=y}(x,y) + \frac{\Gamma(\gamma-\beta',\,1-\beta-\beta')}{\Gamma(1-\beta',\,\gamma-\beta-\beta')}\,f_2^{(0,0)\,x=y}(x,y), \tag{3.29}$$

$$f_3^{(0,0)\,x=0}(x,y) = f_3^{(0,0)\,x=y}(x,y), \tag{3.30}$$

where

$$f_1^{(0,0)\,x=0}(x,y) = x^{1+\beta'-\gamma}y^{-\beta'}F_1\left(1+\beta+\beta'-\gamma,\,1+\alpha-\gamma,\,\beta',\,2+\beta'-\gamma\,;\,x,\,\frac{x}{y}\right),$$

$$f_2^{(0,0)\,x=0}(x,y) = y^{1-\gamma}G_2\left(\beta,\,1+\alpha-\gamma,\,1+\beta'-\gamma,\,\gamma-1\,;\,\frac{-x}{y},\,-y\right),$$

$$f_3^{(0,0)\,x=0}(x,y) = F_1(\,\alpha,\,\beta,\,\beta',\,\gamma\,;\,x,\,y\,)$$

and

$$f_1^{(0,0)\,x=y}(x,y) = y^{\beta+\beta'-\gamma}(1-y)^{\gamma-\alpha-1}(y-x)^{1-\beta-\beta'} \times F_1\left(1-\beta',\,\gamma-\beta-\beta',\,1+\alpha-\gamma,\,2-\beta-\beta'\,;\,\frac{y-x}{y},\,\frac{y-x}{y-1}\right),$$

$$f_2^{(0,0)\,x=y}(x,y) = x^{1-\gamma}(1-x)^{\gamma-\alpha-1}G_2\left(1+\alpha-\gamma,\,\beta',\,\gamma-1,\,1-\beta-\beta'\,;\,\frac{x}{1-x},\,\frac{y-x}{x}\right),$$

$$f_3^{(0,0)}(x,y) = F_1(\,\alpha,\,\beta,\,\beta',\,\gamma\,;\,x,\,y\,).$$

Here the arguments of $x,\ y,\ 1-x,\ 1-y$ and $y-x$ of the factors $x^*,\ y^*,\ (1-x)^*,\ (1-y)^*$ and $(y-x)^*$ are assigned to be zero on the real region $0 < x < y < 1$.

4. Integral representations

Theorem 5.1 in [MN] assures that the function

$$\int_C t^{\beta+\beta'-\gamma}(t-x)^{-\beta}(t-y)^{-\beta'}(t-1)^{\gamma-\alpha-1}\,dt$$

satisfies the system E_1 for any $C \in H_1(T,\mathcal{L})$, where $T = \mathbb{C}\backslash\{0,\,1\,x,\,y\}$ and $\mathcal{L}$ is the locally constant sheaf defined by $u(t) = t^{\beta+\beta'-\gamma}(t-x)^{-\beta}(t-y)^{-\beta'}(t-1)^{\gamma-\alpha-1}$. Using this, we give integral representaions of each member of the fundamental sets of solutions given in Section 2.

4.1 Integrals for F_1

In this subsection, we consider the integrals each of which is expressed by F_1. They are listed as $w_1, \ldots, w_{10}$ in the Appedix.

Suppose that X and Y are complex numbers satisfying $|X| < 1$ and $|Y| < 1$. Then, by using the binomial theorem

$$(1-X)^{-\lambda} = \sum_{i \geq 0} \frac{(\lambda)_i}{i!} X^i, \quad |X| < 1,$$

we have

$$\begin{aligned}
&\int_0^1 t^{\lambda_0}(1-t)^{\lambda_1}(1-Xt)^{\lambda_2}(1-Yt)^{\lambda_3}\, dt \\
&= \sum_{m_2, m_3 \geq 0} \frac{(-\lambda_2)_{m_2}(-\lambda_3)_{m_3}}{m_2!\, m_3!} X^{m_2} Y^{m_3} \int_0^1 t^{\lambda_0+m_2+m_3}(1-t)^{\lambda_1}\, dt \\
&= B(1+\lambda_0,\, 1+\lambda_1) \sum_{m_2, m_3 \geq 0} \frac{(-\lambda_2)_{m_2}(-\lambda_3)_{m_3}}{m_2!\, m_3!} \frac{(1+\lambda_0)_{m_2+m_3}}{(2+\lambda_{01})_{m_2+m_3}} X^{m_2} Y^{m_3} \\
&= B(1+\lambda_0,\, 1+\lambda_1)\, F_1(1+\lambda_0,\, -\lambda_2,\, -\lambda_3,\, 2+\lambda_{01}\,;\, X,\, Y),
\end{aligned}$$

where each argument of the factors of the integrand is fixed to be zero when X and Y are real numbers satisfying $-1 < X < 1$ and $-1 < Y < 1$. The conditions $\mathrm{Re}\,(\lambda_0) > -1$ and $\mathrm{Re}\,(\lambda_1) > -1$ for the existence of the integral can be relaxed into $\lambda_0,\, \lambda_1 \notin \mathbb{Z}_{<0}$ by analytic continuation on the parameters λ_0 and λ_1.

In this subsection, the function $u(t)$ is fixed to be

$$u(t) = t^{\mu_0}(t-x)^{\mu_x}(t-y)^{\mu_y}(t-1)^{\mu_1},$$

where

$$\mu_0 = \beta + \beta' - \gamma, \quad \mu_x = -\beta, \quad \mu_y = -\beta', \quad \mu_1 = \gamma - \alpha - 1,$$

and

$$\mu_\infty = 2 - \mu_{0xy1} = \alpha - 1$$

is also used.

When $|x| < 1$ and $|y| < 1$, the change of integration variable $t \mapsto 1/t$ implies

$$\begin{aligned}
&\int_{(1,\,\infty)} u_{(1,\,\infty)}(t)\, dt = \int_1^\infty t^{\mu_0}(t-x)^{\mu_x}(t-y)^{\mu_y}(t-1)^{\mu_1}\, dt \\
&= \int_1^0 \left(\frac{1}{t}\right)^{\mu_0} \left(\frac{1}{t} - x\right)^{\mu_x} \left(\frac{1}{t} - y\right)^{\mu_y} \left(\frac{1}{t} - 1\right)^{\mu_1} \frac{-1}{t^2}\, dt \\
&= \int_0^1 t^{\mu_\infty}(1-xt)^{\mu_x}(1-yt)^{\mu_y}(1-t)^{\mu_1}\, dt \\
&= B(1+\mu_\infty,\, 1+\mu_1)\, F_1(1+\mu_\infty,\, -\mu_x,\, -\mu_y,\, 2+\mu_{\infty 1}\,;\, x,\, y\,) \\
&= B(\alpha,\, \gamma-\alpha)\; F_1(\alpha,\, \beta,\, \beta',\, \gamma\,;\, x,\, y\,),
\end{aligned} \tag{4.1}$$

which corresponds to w_1 in Appendix.

When $|1-x|<1$ and $|1-y|<1$, the change of integration variable $t \mapsto (t-1)/t$ implies

$$\begin{aligned}
&\int_{(\infty,\,0)} u_{(\infty,\,0)}(t)\,dt = \int_{\infty}^{0} (-t)^{\mu_0}(x-t)^{\mu_x}(y-t)^{\mu_y}(1-t)^{\mu_1}\,dt \\
&= \int_0^1 \left(\frac{1-t}{t}\right)^{\mu_0}\left(\frac{1-(1-x)t}{t}\right)^{\mu_x}\left(\frac{1-(1-y)t}{t}\right)^{\mu_y}\left(\frac{1}{t}\right)^{\mu_1}\frac{1}{t^2}\,dt \\
&= \int_0^1 t^{\mu_\infty}(1-t)^{\mu_0}(1-(1-x)t)^{\mu_x}(1-(1-y)t)^{\mu_y}\,dt \\
&= B(1+\mu_\infty,\,1+\mu_0)\,F_1(1+\mu_\infty,\,-\mu_x,\,-\mu_y,\,2+\mu_{\infty 0}\,;\,1-x,\,1-y\,) \\
&= B(\alpha,\,1+\beta+\beta'-\gamma)\;F_1(\alpha,\,\beta,\,\beta',\,1+\alpha+\beta+\beta'-\gamma\,;\,1-x,\,1-y\,),
\end{aligned} \tag{4.2}$$

which corresponds to w_2 in Appendix.

When $|x|>1$ and $|y|>1$, for

$$\varepsilon_1 = \begin{cases} +1 & \text{if} \quad x>1 \\ -1 & \text{if} \quad x<-1 \end{cases} \qquad \text{and} \quad \varepsilon_2 = \begin{cases} +1 & \text{if} \quad y>1 \\ -1 & \text{if} \quad y<-1 \end{cases} ,$$

we have

$$\begin{aligned}
&\int_{(0,\,1)} u_{(0,\,1)}(t)\,dt = \int_0^1 t^{\mu_0}(\varepsilon_1(x-t))^{\mu_x}(\varepsilon_2(y-t))^{\mu_y}(1-t)^{\mu_1}\,dt \\
&= (\varepsilon_1 x)^{\mu_x}(\varepsilon_2 y)^{\mu_y}\int_0^1 t^{\mu_0}(1-t)^{\mu_1}\left(1-\frac{1}{x}\,t\right)^{\mu_x}\left(1-\frac{1}{y}\,t\right)^{\mu_y}\,dt \\
&= B(1+\mu_0,\,1+\mu_1)\,(\varepsilon_1 x)^{\mu_x}(\varepsilon_2 y)^{\mu_y}\,F_1(1+\mu_0,\,-\mu_x,\,-\mu_y,\,2+\mu_{01}\,;\,x^{-1},\,y^{-1}\,) \\
&= B(1+\beta+\beta'-\gamma,\,\gamma-\alpha)\,(\varepsilon_1 x)^{-\beta}(\varepsilon_2 y)^{-\beta'} \\
&\times\;F_1(1+\beta+\beta'-\gamma,\,\beta,\,\beta',\,1+\beta+\beta'-\alpha\,;\,x^{-1},\,y^{-1}\,),
\end{aligned} \tag{4.3}$$

which corresponds to w_3 in Appendix.

When $|x|<1$ and $|x/y|<1$, for

$$\varepsilon_1 = \begin{cases} +1 & \text{if} \quad x>0 \\ -1 & \text{if} \quad x<0 \end{cases} \qquad \text{and} \quad \varepsilon_2 = \begin{cases} +1 & \text{if} \quad y>|x| \\ -1 & \text{if} \quad y<-|x| \end{cases} ,$$

the change of integration variable $t \mapsto xt$ implies

$$
\begin{aligned}
&\int_{\varepsilon_1(0,\,x)} u_{\varepsilon_1(0,\,x)}(t)\,dt = \varepsilon_1 \int_0^x (\varepsilon_1 t)^{\mu_0} (\varepsilon_1(x-t))^{\mu_x} (\varepsilon_2(y-t))^{\mu_y} (1-t)^{\mu_1}\,dt \\
&= (\varepsilon_1 x)^{1+\mu_{0x}} (\varepsilon_2 y)^{\mu_y} \int_0^1 t^{\mu_0}(1-t)^{\mu_x} \left(1-\frac{x}{y}t\right)^{\mu_y} (1-xt)^{\mu_1}\,dt \\
&= B(1+\mu_0,\, 1+\mu_x)\,(\varepsilon_1 x)^{1+\mu_{0x}} (\varepsilon_2 y)^{\mu_y}\, F_1(1+\mu_0,\, -\mu_1,\, -\mu_y,\, 2+\mu_{0x}\,;\, x,\, x/y\,) \\
&= B(1+\beta+\beta'-\gamma,\, 1-\beta)\,(\varepsilon_1 x)^{1+\beta'-\gamma} (\varepsilon_2 y)^{-\beta'} \\
&\times F_1(1+\beta+\beta'-\gamma,\, 1+\alpha-\gamma,\, \beta',\, 2+\beta'-\gamma\,;\, x,\, x/y\,),
\end{aligned} \tag{4.4}
$$

which corresponds to w_4 in Appendix.

When $|y| < 1$ and $|y/x| < 1$, for

$$
\varepsilon_1 = \begin{cases} +1 & \text{if} \quad y > 0 \\ -1 & \text{if} \quad y < 0 \end{cases} \qquad \text{and} \quad \varepsilon_2 = \begin{cases} +1 & \text{if} \quad x > |y| \\ -1 & \text{if} \quad x < -|y| \end{cases} \quad ,
$$

the change of integration variable $t \mapsto yt$ implies

$$
\begin{aligned}
&\int_{\varepsilon_1(0,\,y)} u_{\varepsilon_1(0,\,y)}(t)\,dt = \varepsilon_1 \int_0^y (\varepsilon_1 t)^{\mu_0} (\varepsilon_2(x-t))^{\mu_x} (\varepsilon_1(y-t))^{\mu_y} (1-t)^{\mu_1}\,dt \\
&= B(1+\mu_0,\, 1+\mu_y)\,(\varepsilon_1 y)^{1+\mu_{0y}} (\varepsilon_2 x)^{\mu_x}\, F_1(1+\mu_0,\, -\mu_x,\, -\mu_1,\, 2+\mu_{0y}\,;\, y/x,\, y\,) \\
&= B(1+\beta+\beta'-\gamma,\, 1-\beta')\,(\varepsilon_1 y)^{1+\beta-\gamma} (\varepsilon_2 x)^{-\beta} \\
&\times F_1(1+\beta+\beta'-\gamma,\, \beta,\, 1+\alpha-\gamma,\, 2+\beta-\gamma\,;\, y/x,\, y\,),
\end{aligned} \tag{4.5}
$$

which corresponds to w_5 in Appendix.

When $|1-x| < 1$ and $|(1-x)/(1-y)| < 1$, for

$$
\varepsilon_1 = \begin{cases} +1 & \text{if} \quad x < 1 \\ -1 & \text{if} \quad x > 1 \end{cases} \qquad \text{and} \quad \varepsilon_2 = \begin{cases} +1 & \text{if} \quad y-1 < -|x-1| \\ -1 & \text{if} \quad y-1 > |x-1| \end{cases} \quad ,
$$

the change of integration variable $t \mapsto 1-(1-x)t$ implies

$$\int_{\varepsilon_1(x,\,1)} u_{\varepsilon_1(x,\,1)}(t)\,dt = \varepsilon_1 \int_x^1 t^{\mu_0}(\varepsilon_1(t-x))^{\mu_x}(\varepsilon_2(t-y))^{\mu_y}(\varepsilon_1(1-t))^{\mu_1}\,dt$$

$$= \varepsilon_1 \int_1^0 (1-(1-x)t)^{\mu_0}(\varepsilon_1(1-x)(1-t))^{\mu_x}(\varepsilon_2(1-y-(1-x)t))^{\mu_y}(\varepsilon_1(1-x)t)^{\mu_1}\,(x-1)\,dt$$

$$= (\varepsilon_1(1-x))^{1+\mu_{x1}}(\varepsilon_2(1-y))^{\mu_y} \int_0^1 t^{\mu_1}(1-t)^{\mu_x}(1-(1-x)t)^{\mu_0}\left(1-\frac{1-x}{1-y}\,t\right)^{\mu_y} dt$$

$$= B(1+\mu_x,\, 1+\mu_1)\,(\varepsilon_1(1-x))^{1+\mu_{x1}}(\varepsilon_2(1-y))^{\mu_y}$$

$$\times\ F_1\left(1+\mu_1,\, -\mu_0,\, -\mu_y,\, 2+\mu_{1x}\,;\, 1-x,\, \frac{1-x}{1-y}\right)$$

$$= B(1-\beta,\, \gamma-\alpha)\,(\varepsilon_1(1-x))^{\gamma-\alpha-\beta}(\varepsilon_2(1-y))^{-\beta'}$$

$$\times\ F_1\left(\gamma-\alpha,\, \gamma-\beta-\beta',\, \beta',\, 1+\gamma-\alpha-\beta\,;\, 1-x,\, \frac{1-x}{1-y}\right), \tag{4.6}$$

which corresponds to w_6 in Appendix.

When $|1-y|<1$ and $|(1-y)/(1-x)|<1$, for

$$\varepsilon_1 = \begin{cases} +1 & \text{if} \quad y<1 \\ -1 & \text{if} \quad y>1 \end{cases} \quad \text{and} \quad \varepsilon_2 = \begin{cases} +1 & \text{if} \quad x-1<-|y-1| \\ -1 & \text{if} \quad x-1>|y-1| \end{cases},$$

the change of integration variable $t \mapsto 1-(1-y)t$ implies

$$\int_{\varepsilon_1(y,\,1)} u_{\varepsilon_1(y,\,1)}(t)\,dt = \varepsilon_1 \int_y^1 t^{\mu_0}(\varepsilon_2(t-x))^{\mu_x}(\varepsilon_1(t-y))^{\mu_y}(\varepsilon_1(1-t))^{\mu_1}\,dt$$

$$= B(1+\mu_y,\, 1+\mu_1)\,(\varepsilon_1(1-y))^{1+\mu_{y1}}(\varepsilon_2(1-x))^{\mu_x}$$

$$\times\ F_1\left(1+\mu_1,\, -\mu_x,\, -\mu_0,\, 2+\mu_{1y}\,;\, \frac{1-y}{1-x},\, 1-y\right)$$

$$= B(1-\beta',\, \gamma-\alpha)\,(\varepsilon_1(1-y))^{\gamma-\alpha-\beta'}(\varepsilon_2(1-x))^{-\beta}$$

$$\times\ F_1\left(\gamma-\alpha,\, \beta,\, \gamma-\beta-\beta',\, 1+\gamma-\alpha-\beta'\,;\, \frac{1-y}{1-x},\, 1-y\right), \tag{4.7}$$

which corresponds to w_7 in Appendix.

When $|1/x|<1$ and $|y/x|<1$, for

$$\varepsilon = \begin{cases} +1 & \text{if} \quad x>0 \\ -1 & \text{if} \quad x<0 \end{cases},$$

the change of integration variable $t \mapsto x/t$ implies

$$\begin{aligned}
&\int_{\varepsilon(x,\,\infty)} u_{\varepsilon(x,\,\infty)}(t)\,dt = \varepsilon \int_x^\infty (\varepsilon t)^{\mu_0}(\varepsilon(t-x))^{\mu_x}(\varepsilon(t-y))^{\mu_y}(\varepsilon(t-1))^{\mu_1}\,dt \\
&= \varepsilon \int_1^0 \left(\frac{\varepsilon x}{t}\right)^{\mu_0} \left(\varepsilon\left(\frac{x}{t}-x\right)\right)^{\mu_x} \left(\varepsilon\left(\frac{x}{t}-y\right)\right)^{\mu_y} \left(\varepsilon\left(\frac{x}{t}-1\right)\right)^{\mu_1} \frac{-x}{t^2}\,dt \\
&= (\varepsilon x)^{1+\mu_{0xy1}} \int_0^1 t^{\mu_\infty}(1-t)^{\mu_x}\left(1-\frac{y}{x}\,t\right)^{\mu_y}\left(1-\frac{1}{x}\,t\right)^{\mu_1} dt \\
&= B(1+\mu_\infty,\, 1+\mu_x)\,(\varepsilon x)^{1+\mu_{0xy1}} \\
&\qquad \times F_1\left(1+\mu_\infty,\, -\mu_1,\, -\mu_y,\, 2+\mu_{\infty x}\,;\, \frac{1}{x},\, \frac{y}{x}\right) \\
&= B(\alpha,\, 1-\beta)\,(\varepsilon x)^{-\alpha}\, F_1\left(\alpha,\, 1+\alpha-\gamma,\, \beta',\, 1+\alpha-\beta\,;\, \frac{1}{x},\, \frac{y}{x}\right),
\end{aligned} \tag{4.8}$$

which corresponds to w_8 in Appendix.

When $|1/y| < 1$ and $|x/y| < 1$, for

$$\varepsilon = \begin{cases} +1 & \text{if} \quad y > 0 \\ -1 & \text{if} \quad y < 0 \end{cases},$$

the change of integration variable $t \mapsto y/t$ implies

$$\begin{aligned}
&\int_{\varepsilon(y,\,\infty)} u_{\varepsilon(y,\,\infty)}(t)\,dt = \varepsilon \int_y^\infty (\varepsilon t)^{\mu_0}(\varepsilon(t-x))^{\mu_x}(\varepsilon(t-y))^{\mu_y}(\varepsilon(t-1))^{\mu_1}\,dt \\
&= B(1+\mu_\infty,\, 1+\mu_y)\,(\varepsilon y)^{1+\mu_{0xy1}} \\
&\qquad \times F_1\left(1+\mu_\infty,\, -\mu_x,\, -\mu_1,\, 2+\mu_{\infty y}\,;\, \frac{x}{y},\, \frac{1}{y}\right) \\
&= B(\alpha,\, 1-\beta')\,(\varepsilon y)^{-\alpha}\, F_1\left(\alpha,\, \beta,\, 1+\alpha-\gamma,\, 1+\alpha-\beta'\,;\, \frac{x}{y},\, \frac{1}{y}\right),
\end{aligned} \tag{4.9}$$

which corresponds to w_9 in Appendix.

When $|(y-x)/y| < 1$ and $|(x-y)/(1-y)| < 1$, for

$$\varepsilon_1 = \begin{cases} +1 & \text{if} \quad x,\, y > 0 \\ -1 & \text{if} \quad x,\, y < 0 \end{cases}, \quad \varepsilon_2 = \begin{cases} +1 & \text{if} \quad x,\, y < 1 \\ -1 & \text{if} \quad x,\, y > 1 \end{cases}, \quad \varepsilon_3 = \begin{cases} +1 & \text{if} \quad x < y \\ -1 & \text{if} \quad x > y \end{cases},$$

the change of integration variable $t \mapsto y + (x-y)t$ implies

$$\int_{\varepsilon_3(x,y)} u_{\varepsilon_3(x,y)}(t)\,dt = \varepsilon_3 \int_x^y (\varepsilon_1 t)^{\mu_0}(\varepsilon_3(t-x))^{\mu_x}(\varepsilon_3(y-t))^{\mu_y}(\varepsilon_2(1-t))^{\mu_1}\,dt$$

$$= \varepsilon_3 \int_1^0 (\varepsilon_1(y+(x-y)t))^{\mu_0}(\varepsilon_3(y-x)t)^{\mu_x}(\varepsilon_3(y-x)(1-t))^{\mu_y}$$

$$\times\ (\varepsilon_2((1-y)-(x-y)t))^{\mu_1}\,(x-y)\,dt$$

$$= (\varepsilon_3(y-x))^{1+\mu_{xy}}(\varepsilon_1 y)^{\mu_0}(\varepsilon_2(1-y))^{1+\mu_1}$$

$$\times \int_0^1 t^{\mu_y}(1-t)^{\mu_x}\left(1-\frac{y-x}{y}\,t\right)^{\mu_0}\left(1-\frac{x-y}{1-y}\,t\right)^{\mu_1}dt$$

$$= B(1+\mu_x,\,1+\mu_y)\,(\varepsilon_3(y-x))^{1+\mu_{xy}}(\varepsilon_1 y)^{\mu_0}(\varepsilon_2(1-y))^{1+\mu_1}$$

$$\times\ F_1\left(1+\mu_y,\,-\mu_0,\,-\mu_1,\,2+\mu_{12}\,;\,\frac{y-x}{y},\,\frac{x-y}{1-y}\right)$$

$$= B(1-\beta,\,1-\beta')\,(\varepsilon_3(y-x))^{1-\beta-\beta'}(\varepsilon_1 y)^{\beta+\beta'-\gamma}(\varepsilon_2(1-y))^{\gamma-\alpha-1}$$

$$\times\ F_1\left(1-\beta',\,\gamma-\beta-\beta',\,1+\alpha-\gamma,\,2-\beta-\beta'\,;\,\frac{y-x}{y},\,\frac{x-y}{1-y}\right), \tag{4.10}$$

which corresponds to w_{10} in Appendix.

4.2 Integrals for G_2

In this subsection, we consider the integrals each of which is expressed by G_2. They are listed as $f_2^{(*,*)}$ or $f_2^{(*,*)*}$ in the Appedix.

Suppose that X and Y are complex numbers satisfying $|X| < 1$ and $|Y| < 1$. Then we have

$$\int_{(\{0,\,X\},\,1)} t^{\lambda_0}(1-t)^{\lambda_1}(t-X)^{\lambda_2}(1-Yt)^{\lambda_3}\,dt$$

$$= \int_{(\{0,\,X\},\,1)} t^{\lambda_{02}}(1-t)^{\lambda_1}\left(1-\frac{X}{t}\right)^{\lambda_2}(1-Yt)^{\lambda_3}\,dt$$

$$= \sum_{m_2,\,m_3\ge 0} \frac{(-\lambda_2)_{m_2}(-\lambda_3)_{m_3}}{m_2!\,m_3!}\,X^{m_2}Y^{m_3}\int_{(\{0,\,X\},\,1)} t^{\lambda_{02}-m_2+m_3}(1-t)^{\lambda_1}\,dt$$

$$= B(1+\lambda_{02},\,1+\lambda_1)\sum_{m_2,\,m_3\ge 0}\frac{(-\lambda_2)_{m_2}(-\lambda_3)_{m_3}}{m_2!\,m_3!}\,\frac{(1+\lambda_{02})_{-m_2+m_3}}{(2+\lambda_{012})_{-m_2+m_3}}X^{m_2}Y^{m_3}$$

$$= B(1+\lambda_{02},\,1+\lambda_1)$$

$$\times \sum_{m_2,\,m_3\ge 0}\frac{(-\lambda_2)_{m_2}(-\lambda_3)_{m_3}}{m_2!\,m_3!}(1+\lambda_{02})_{-m_2+m_3}(-1-\lambda_{012})_{m_2-m_3}(-X)^{m_2}(-Y)^{m_3}$$

$$= B(1+\lambda_{02},\,1+\lambda_1)\,G_2(-\lambda_2,\,-\lambda_3,\,1+\lambda_{02},\,-1-\lambda_{012}\,;\,-X,\,-Y),$$

where

$$(\{0,\ X\},\ 1) = \frac{1}{d_{\lambda_{02}}} S(\{0,\ X\}\,;\ 1-\delta\,) + [\,1-\delta,\ 1\,)$$

for δ satisfying $|X| < 1-\delta < 1$. Here we have also used the equality

$$\begin{aligned}
&\int_{\mathrm{reg}\,(0,1)} t^{\lambda_0+m}(1-t)^{\lambda_1}\,dt = B(1+\lambda_0+m,\ 1+\lambda_1) \\
&= B(1+\lambda_0,\ 1+\lambda_1)\,\frac{(1+\lambda_0)_m}{(2+\lambda_{01})_m}
\end{aligned}$$

for $m \in \mathbb{Z}$ and $\lambda_0,\ \lambda_1 \notin \mathbb{Z}$ and the equality $(A)_m(1-A)_{-m} = (-)^m$ for $m \in \mathbb{Z}$.

In what follows, the symbol δ is used frequantly in the same meaning as above, and the function $u(t)$ is fixed to be

$$u(t) = (t-p)^{\mu_p}(t-q)^{\mu_q}(t-r)^{\mu_r}(t-s)^{\mu_s},$$

where

$$\{p,\ q,\ r,\ s\} = \{0,\ 1,\ x,\ y\}$$

and

$$\mu_0 = \beta+\beta'-\gamma,\quad \mu_x = -\beta,\quad \mu_y = -\beta',\quad \mu_1 = \gamma-\alpha-1.$$

In addition,

$$\mu_\infty = 2-\mu_{0xy1} = \alpha-1$$

is also used as before.

When $p < r,\ |(q-p)/(r-p)| < 1$ and $|(r-p)/(s-p)| < 1$, for

$$\varepsilon = \begin{cases} +1 & \text{if}\quad s > r \\ -1 & \text{if}\quad s < p \quad \text{and} \quad s < q \end{cases},$$

the change of integration variable $t \mapsto p + (r-p)t$ implies

$$\begin{aligned}
&\int_{(\{p,\,q\},\,r)} u_{r-\delta}(t)\,dt = \int_{(\{p,\,q\},\,r)} (t-p)^{\mu_p}(t-q)^{\mu_q}(r-t)^{\mu_r}(\varepsilon(s-t))^{\mu_s}\,dt \\
&= \int_{(\{0,\,(q-p)/(r-p)\},\,1)} ((r-p)t)^{\mu_p}(p-q+(r-p)t)^{\mu_q}((r-p)(1-t))^{\mu_r} \\
&\quad \times (\varepsilon(s-p-(r-p)t))^{\mu_s}\,dt \\
&= (r-p)^{1+\mu_{pqr}}(\varepsilon(s-p))^{\mu_s} B(1+\mu_{pq},\ 1+\mu_r) \\
&\quad \times G_2\Big(-\mu_q,\ -\mu_s,\ 1+\mu_{pq},\ -1-\mu_{pqr}\,;\ -\frac{q-p}{r-p},\ -\frac{r-p}{s-p}\Big).
\end{aligned} \tag{4.11}$$

The case $(p,\ q,\ r,\ s) = (0,\ x,\ y,\ 1)$ of (4.11) implies

$$\begin{aligned}
\int_{(\{0,\,x\},\,y)} u_{y-\delta}(t)\,dt &= y^{1+\mu_{0xy}}\,B(1+\mu_{0x},\ 1+\mu_y) \\
&\quad\times G_2\left(-\mu_x,\ -\mu_1,\ 1+\mu_{0x},\ -1-\mu_{0xy}\,;\ -\frac{x}{y},\ -y\right) \\
&= B(1+\beta'-\gamma,\ 1-\beta')\,y^{1-\gamma}\,G_2\left(\beta,\ 1+\alpha-\gamma,\ 1+\beta'-\gamma,\ \gamma-1\,;\ -\frac{x}{y},\ -y\right)
\end{aligned} \tag{4.12}$$

for $0 < |x| < y < 1$, which gives $f_2^{(0,\,0)\,x=0}$ for $y > 0$.

The case $(p,\ q,\ r,\ s) = (0,\ y,\ x,\ 1)$ of (4.11) implies

$$\begin{aligned}
\int_{(\{0,\,y\},\,x)} u_{x-\delta}(t)\,dt &= x^{1+\mu_{0xy}}\,B(1+\mu_{0y},\ 1+\mu_x) \\
&\quad\times G_2\left(-\mu_y,\ -\mu_1,\ 1+\mu_{0y},\ -1-\mu_{0xy}\,;\ -\frac{y}{x},\ -x\right) \\
&= B(1+\beta-\gamma,\ 1-\beta)\,x^{1-\gamma}\,G_2\left(\beta',\ 1+\alpha-\gamma,\ 1+\beta-\gamma,\ \gamma-1\,;\ -\frac{y}{x},\ -x\right) \\
&= B(1+\beta-\gamma,\ 1-\beta)\,x^{1-\gamma}\,G_2\left(1+\alpha-\gamma,\ \beta',\ \gamma-1,\ 1+\beta-\gamma\,;\ -x,\ -\frac{y}{x}\right)
\end{aligned} \tag{4.13}$$

for $0 < |y| < x < 1$, which gives $f_2^{(0,\,0)\,y=0}$ for $x > 0$.

The case $(p,\ q,\ r,\ s) = (y,\ x,\ 0,\ 1)$ of (4.11) implies

$$\begin{aligned}
\int_{(\{y,\,x\},\,0)} u_{0-\delta}(t)\,dt &= (-y)^{1+\mu_{0xy}}(1-y)^{\mu_1}\,B(1+\mu_0,\ 1+\mu_{xy}) \\
&\quad\times G_2\left(-\mu_x,\ -\mu_1,\ 1+\mu_{xy},\ -1-\mu_{0xy}\,;\ \frac{x-y}{y},\ \frac{y}{1-y}\right) \\
&= B(1+\beta+\beta'-\gamma,\ 1-\beta-\beta')\,(-y)^{1-\gamma}\,(1-y)^{-\beta} \\
&\quad\times G_2\left(\beta,\ 1+\alpha-\gamma,\ 1-\beta-\beta',\ \gamma-1\,;\ \frac{x-y}{y},\ \frac{y}{1-y}\right)
\end{aligned} \tag{4.14}$$

for $y < 0$ and $|x-y| < |y|$, which gives $f_2^{(0,\,0)\,x=y}$ for $y < 0$.

The case $(p,\ q,\ r,\ s) = (y,\ x,\ 1,\ 0)$ of (4.11) implies

$$\begin{aligned}
&\int_{(\{y,\,x\},\,1)} u_{1-\delta}(t)\,dt = (1-y)^{1+\mu_{xy1}} x^{\mu_0} B(1+\mu_{xy},\,1+\mu_1) \\
&\quad \times G_2\left(-\mu_x,\,-\mu_0,\,1+\mu_{xy},\,-1-\mu_{xy1}\,;\,\frac{y-x}{1-y},\,\frac{1-y}{y}\right) \\
&= B(1-\beta-\beta',\,\gamma-\alpha)\,(1-y)^{\gamma-\alpha-\beta-\beta'}\,y^{\beta+\beta'-\gamma} \\
&\quad \times G_2\left(\beta,\,\gamma-\beta-\beta',\,1-\beta-\beta',\,\alpha+\beta+\beta'-\gamma\,;\,\frac{y-x}{1-y},\,\frac{1-y}{y}\right)
\end{aligned} \tag{4.15}$$

for $1/2 < y < 1$ and $|x-y| < |1-y|$, which gives $f_2^{(1,\,1)\,x=y}$ for $1/2 < y < 1$.

The case $(p,\,q,\,r,\,s) = (0,\,x,\,1,\,y)$ of (4.11) implies

$$\begin{aligned}
&\int_{(\{x,\,0\},\,1)} u_{1-\delta}(t)\,dt = (\varepsilon y)^{\mu_y} B(1+\mu_{0x},\,1+\mu_1) \\
&\quad \times G_2\left(-\mu_x,\,-\mu_y,\,1+\mu_{0x},\,-1-\mu_{0x1}\,;\,-x,\,-y^{-1}\right) \\
&= B(1+\beta'-\gamma,\,\gamma-\alpha)\,(\varepsilon y)^{-\beta'} \\
&\quad \times G_2\left(\beta,\,\beta',\,1+\beta'-\gamma,\,\alpha-\beta'\,;\,-x,\,-y^{-1}\right),
\end{aligned} \tag{4.16}$$

where

$$\varepsilon = \begin{cases} +1 & \text{if} \quad y > 1 \\ -1 & \text{if} \quad y < -1 \end{cases},$$

for $0 < |x| < 1 < |y|$, which gives $f_2^{(0,\,\infty)}$.

The case $(p,\,q,\,r,\,s) = (0,\,y,\,1,\,x)$ of (4.11) implies

$$\begin{aligned}
&\int_{(\{y,\,0\},\,1)} u_{1-\delta}(t)\,dt = (\varepsilon x)^{\mu_x} B(1+\mu_{0y},\,1+\mu_1) \\
&\quad \times G_2\left(-\mu_y,\,-\mu_x,\,1+\mu_{0y},\,-1-\mu_{0y1}\,;\,-y,\,-x^{-1}\right) \\
&= B(1+\beta-\gamma,\,\gamma-\alpha)\,(\varepsilon x)^{-\beta} \\
&\quad \times G_2\left(\beta',\,\beta,\,1+\beta-\gamma,\,\alpha-\beta\,;\,-y,\,-x^{-1}\right) \\
&= B(1+\beta-\gamma,\,\gamma-\alpha)\,(\varepsilon x)^{-\beta} \\
&\quad \times G_2\left(\beta,\,\beta',\,\alpha-\beta,\,1+\beta-\gamma\,;\,-x^{-1},\,-y\right),
\end{aligned} \tag{4.17}$$

where

$$\varepsilon = \begin{cases} +1 & \text{if} \quad x > 1 \\ -1 & \text{if} \quad x < -1 \end{cases},$$

for $0 < |y| < 1 < |x|$, which gives $f_2^{(\infty,\,0)}$.

When $r < p$, $|(q-p)/(r-p)| < 1$ and $|(r-p)/(s-p)| < 1$, for

$$\varepsilon = \begin{cases} +1 & \text{if} \quad s < r \\ -1 & \text{if} \quad s > p \quad \text{and} \quad s > q \end{cases},$$

the change of integration variable $t \mapsto p + (r-p)t$ implies

$$\begin{aligned}
&\int_{(r,\,\{q,\,p\})} u_{r+\delta}(t)\,dt = \int_{(r,\,\{q,\,p\})} (p-t)^{\mu_p}(q-t)^{\mu_q}(t-r)^{\mu_r}(\varepsilon(t-s))^{\mu_s}\,dt \\
&= \int_{(\{0,\,(q-p)/(r-p)\},\,1)} ((p-r)t)^{\mu_p}(q-p+(p-r)t)^{\mu_q}((p-r)(1-t))^{\mu_r} \\
&\quad \times (\varepsilon(p-s-(p-r)t))^{\mu_s}\,dt \\
&= (p-r)^{1+\mu_{pqr}}(\varepsilon(p-s))^{\mu_s} B(1+\mu_r,\,1+\mu_{pq}) \\
&\quad \times G_2\left(-\mu_q,\,-\mu_s,\,1+\mu_{pq},\,-1-\mu_{pqr}\,;\,-\frac{q-p}{r-p},\,-\frac{r-p}{s-p}\right).
\end{aligned} \tag{4.18}$$

The case $(p,\,q,\,r,\,s) = (0,\,x,\,y,\,1)$ of (4.18) implies

$$\begin{aligned}
&\int_{(y,\,\{x,\,0\})} u_{y+\delta}(t)\,dt = (-y)^{1+\mu_{0xy}}\,B(1+\mu_{0x},\,1+\mu_y) \\
&\qquad \times G_2\left(-\mu_x,\,-\mu_1,\,1+\mu_{0x},\,-1-\mu_{0xy}\,;\,-\frac{x}{y},\,-y\right) \\
&= B(1+\beta'-\gamma,\,1-\beta')(-y)^{1-\gamma} \\
&\qquad \times G_2\left(\beta,\,1+\alpha-\gamma,\,1+\beta'-\gamma,\,\gamma-1\,;\,-\frac{x}{y},\,-y\right)
\end{aligned} \tag{4.19}$$

for $-1 < y < 0$ and $|x| < |y|$, which gives $f_2^{(0,\,0)\,x=0}$ for $-1 < y < 0$.

The case $(p,\,q,\,r,\,s) = (0,\,y,\,x,\,1)$ of (4.18) implies

$$\begin{aligned}\int_{(x,\,\{y,\,0\})} u_{x+\delta}(t)\,dt &= (-x)^{1+\mu_{0xy}}\,B(1+\mu_{0y},\,1+\mu_x)\\ &\quad\times G_2\Big(-\mu_y,\,-\mu_1,\,1+\mu_{0y},\,-1-\mu_{0xy}\,;\,-\frac{y}{x},\,-x\Big)\\ &= B(1+\beta-\gamma,\,1-\beta)\,(-x)^{1-\gamma}\,G_2\Big(\beta',\,1+\alpha-\gamma,\,1+\beta-\gamma,\,\gamma-1\,;\,-\frac{y}{x},\,-x\Big)\\ &= B(1+\beta-\gamma,\,1-\beta)\,(-x)^{1-\gamma}\,G_2\Big(1+\alpha-\gamma,\,\beta',\,\gamma-1,\,1+\beta-\gamma\,;\,-x,\,-\frac{y}{x}\Big)\end{aligned} \tag{4.20}$$

for $|y|<|x|$ with $-1<x<0$, which gives $f_2^{(0,\,0)\,y=0}$ for $-1<x<0$.

The case $(p,\,q,\,r,\,s)=(y,\,x,\,0,\,1)$ of (4.18) implies

$$\begin{aligned}\int_{(0,\,\{x,\,y\})} u_{0+\delta}(t)\,dt &= y^{1+\mu_{0xy}}(1-y)^{\mu_1}B(1+\mu_0,\,1+\mu_{xy})\\ &\quad\times G_2\Big(-\mu_x,\,-\mu_1,\,1+\mu_{xy},\,-1-\mu_{0xy}\,;\,\frac{x-y}{y},\,\frac{y}{1-y}\Big)\\ &= B(1+\beta+\beta'-\gamma,\,1-\beta-\beta')\,y^{1-\gamma}\,(1-y)^{-\beta}\\ &\quad\times G_2\Big(\beta,\,1+\alpha-\gamma,\,1-\beta-\beta',\,\gamma-1\,;\,\frac{x-y}{y},\,\frac{y}{1-y}\Big)\end{aligned} \tag{4.21}$$

for $0<y<1/2$ and $|x-y|<|y|$, which gives $f_2^{(0,\,0)\,x=y}$ for $0<y<1/2$.

The case $(p,\,q,\,r,\,s)=(1,\,y,\,x,\,0)$ of (4.18) implies

$$\begin{aligned}\int_{(x,\,\{y,\,1\})} u_{x+\delta}(t)\,dt &= (1-x)^{1+\mu_{xy1}}B(1+\mu_x,\,1+\mu_{y1})\\ &\quad\times G_2\Big(-\mu_y,\,-\mu_0,\,1+\mu_{y1},\,-1-\mu_{xy1}\,;\,\frac{1-y}{x-1},\,x-1\Big)\\ &= B(1-\beta',\,1-\beta)\,(1-x)^{\gamma-\alpha-\beta-\beta'}\\ &\quad\times G_2\Big(\beta',\,\gamma-\beta-\beta',\,\gamma-\alpha-\beta',\,\alpha+\beta+\beta'-\gamma\,;\,\frac{1-y}{x-1},\,x-1\Big)\\ &= B(1-\beta',\,1-\beta)\,(1-x)^{\gamma-\alpha-\beta-\beta'}\\ &\quad\times G_2\Big(\gamma-\beta-\beta',\,\beta',\,\alpha+\beta+\beta'-\gamma,\,\gamma-\alpha-\beta'\,;\,x-1,\,\frac{1-y}{x-1}\Big)\end{aligned} \tag{4.22}$$

for $0<x<1$ and $|1-y|<|1-x|$, which gives $f_2^{(1,\,1)\,y=1}$ for $0<x<1$.

The case $(p,\,q,\,r,\,s)=(1,\,x,\,y,\,0)$ of (4.18) implies

$$\begin{aligned}
&\int_{(y,\,\{x,\,1\})} u_{y+\delta}(t)\,dt = (1-y)^{1+\mu_{xy1}} B(1+\mu_y,\, 1+\mu_{x1}) \\
&\quad \times G_2\left(-\mu_x,\, -\mu_0,\, 1+\mu_{x1},\, -1-\mu_{xy1}\,;\, \frac{1-x}{y-1},\, y-1\right) \\
&= B(1-\beta,\, 1-\beta')\,(1-y)^{\gamma-\alpha-\beta-\beta'} \\
&\quad \times G_2\left(\beta,\, \gamma-\beta-\beta',\, \gamma-\alpha-\beta,\, \alpha+\beta+\beta'-\gamma\,;\, \frac{1-x}{y-1},\, y-1\right)
\end{aligned} \tag{4.23}$$

for $0 < y < 1$ and $|1-x| < |1-y|$, which gives $f_2^{(1,\,1)\,x=1}$ for $0 < y < 1$.

The case $(p,\, q,\, r,\, s) = (1,\, x,\, 0,\, y)$ of (4.18) implies

$$\begin{aligned}
&\int_{(0,\,\{1,\,x\})} u_{0+\delta}(t)\,dt = (\varepsilon(1-y))^{\mu_y} B(1+\mu_0,\, 1+\mu_{1x}) \\
&\quad \times G_2\left(-\mu_x,\, -\mu_y,\, 1+\mu_{x1},\, -1-\mu_{0x1}\,;\, x-1,\, (y-1)^{-1}\right) \\
&= B(1+\beta+\beta'-\gamma,\, \gamma-\alpha-\beta)\,(\varepsilon(1-y))^{-\beta'} \\
&\quad \times G_2\left(\beta,\, \beta',\, \gamma-\alpha-\beta,\, \alpha-\beta'\,;\, x-1,\, (y-1)^{-1}\right),
\end{aligned} \tag{4.24}$$

where

$$\varepsilon = \begin{cases} +1 & \text{if} \quad y < 0 \\ -1 & \text{if} \quad y > 1 \quad \text{and} \quad y > x \end{cases},$$

for $|1-x| < 1 < |y-1|$, which gives $f_2^{(1,\,\infty)}$.

The case $(p,\, q,\, r,\, s) = (1,\, y,\, 0,\, x)$ of (4.18) implies

$$\begin{aligned}
&\int_{(0,\,\{1,\,y\})} u_{0+\delta}(t)\,dt = (\varepsilon(1-x))^{\mu_x} B(1+\mu_0,\, 1+\mu_{1y}) \\
&\quad \times G_2\left(-\mu_y,\, -\mu_x,\, 1+\mu_{y1},\, -1-\mu_{0y1}\,;\, y-1,\, (x-1)^{-1}\right) \\
&= B(1+\beta+\beta'-\gamma,\, \gamma-\alpha-\beta')\,(\varepsilon(1-x))^{-\beta} \\
&\quad \times G_2\left(\beta',\, \beta,\, \gamma-\alpha-\beta',\, \alpha-\beta\,;\, y-1,\, (x-1)^{-1}\right) \\
&= B(1+\beta+\beta'-\gamma,\, \gamma-\alpha-\beta')\,(\varepsilon(1-x))^{-\beta} \\
&\quad \times G_2\left(\beta,\, \beta,\, \alpha-\beta,\, \gamma-\alpha-\beta'\,;\, (x-1)^{-1},\, y-1\right),
\end{aligned} \tag{4.25}$$

where

$$\varepsilon = \begin{cases} +1 & \text{if} \quad x < 0 \\ -1 & \text{if} \quad x > 1 \quad \text{and} \quad x > y \end{cases},$$

for $|1-y| < 1 < |x-1|$, which gives $f_2^{(\infty,\,1)}$.

When $p,\ q < r,\ s$ and $|(q-p)/(s-p)| < 1$ and $|(s-r)/(q-r)| < 1$, the change of integration variable $t \mapsto (q-st)/(1-t)$ implies

$$\begin{aligned}
&\int_{(\infty,\,\{p,\,q\})} u_{\infty+\delta}(t)\,dt = \int_{(\infty,\,\{p,\,q\})} (p-t)^{\mu_p}(q-t)^{\mu_q}(r-t)^{\mu_r}(s-t)^{\mu_s}\,dt \\
&= \int_{(\{0,\,(p-q)/(p-s)\},\,1)} \left(\frac{p-q+(s-p)t}{1-t}\right)^{\mu_p} \left(\frac{(s-q)t}{1-t}\right)^{\mu_q} \\
&\quad \times \left(\frac{r-q+(s-r)t}{1-t}\right)^{\mu_r} \left(\frac{s-q}{1-t}\right)^{\mu_s} \frac{s-q}{(1-t)^2}\,dt \\
&= (s-q)^{1+\mu_{sq}}\,(s-p)^{\mu_p}\,(r-q)^{\mu_r} \\
&\quad \times \int_{(\{0,\,(p-q)/(p-s)\},\,1)} (1-t)^{\mu_\infty} \left(1+\frac{p-q}{(s-p)t}\right)^{\mu_p} t^{\mu_{pq}} \left(1+\frac{s-r}{r-q}\,t\right)^{\mu_r}\,dt \\
&= (s-q)^{1+\mu_{sq}}\,(s-p)^{\mu_p}\,(r-q)^{\mu_r}\,B(1+\mu_\infty,\,1+\mu_{pq}) \\
&\quad \times G_2\!\left(-\mu_p,\,-\mu_r,\,1+\mu_{pq},\,-1-\mu_{\infty pq}\,;\,-\frac{q-p}{s-p},\,-\frac{s-r}{q-r}\right).
\end{aligned} \tag{4.26}$$

The case $(p,\ q,\ r,\ s) = (y,\ x,\ 0,\ 1)$ of (4.26) implies

$$\begin{aligned}
&\int_{(\infty,\,\{y,\,x\})} u_{\infty+\delta}(t)\,dt = (1-x)^{1+\mu_{x1}}\,(1-y)^{\mu_y}\,(-x)^{\mu_0}\,B(1+\mu_\infty,\,1+\mu_{xy}) \\
&\quad \times G_2\!\left(-\mu_y,\,-\mu_0,\,1+\mu_{xy},\,-1-\mu_{xy\infty}\,;\,\frac{y-x}{1-y},\,-\frac{1}{x}\right) \\
&= B(\alpha,\,1-\beta-\beta')\,(1-x)^{\gamma-\alpha-\beta}\,(1-y)^{-\beta'}\,(-x)^{\beta+\beta'-\gamma} \\
&\quad \times G_2\!\left(\beta',\,\gamma-\beta-\beta',\,1-\beta-\beta',\,\beta+\beta'-\alpha\,;\,\frac{y-x}{1-y},\,-\frac{1}{x}\right) \\
&= B(\alpha,\,1-\beta-\beta')\,(1-x)^{\gamma-\alpha-\beta}\,(1-y)^{-\beta'}\,(-x)^{\beta+\beta'-\gamma} \\
&\quad \times G_2\!\left(\gamma-\beta-\beta',\,\beta',\,\beta+\beta'-\alpha,\,1-\beta-\beta'\,;\,-\frac{1}{x},\,\frac{y-x}{1-y}\right)
\end{aligned} \tag{4.27}$$

for $\infty < x < -1,\ \infty < y < 0$ and $|y-x| < |1-y|$, which gives $f_2^{(\infty,\,\infty)\,x=y}$ for $\infty < x < -1$.

The case $(p,\ q,\ r,\ s) = (x,\ 0,\ y,\ 1)$ of (4.26) implies

$$\begin{aligned}
&\int_{(\infty,\ \{0,\,x\})} u_{\infty+\delta}(t)\,dt = B(1+\mu_\infty,\ 1+\mu_{0x})\,(1-x)^{1+\mu_x}\,y^{\mu_y}\\
&\quad\times G_2\left(-\mu_x,\ -\mu_y,\ 1+\mu_{0x},\ -1-\mu_{\infty 0x}\,;\ \frac{x}{1-x},\ \frac{1-y}{y}\right)\\
&= B(\alpha,\ 1+\beta'-\gamma)\,(1-x)^{1-\beta}\,y^{-\beta'}\\
&\quad\times G_2\left(\beta,\ \beta',\ 1+\beta'-\gamma,\ \gamma-\alpha-\beta'\,;\ \frac{x}{1-x},\ \frac{1-y}{y}\right)
\end{aligned} \tag{4.28}$$

for $x < 1/2 < y$, which gives $f_2^{(0,\,1)}$.

The case $(p,\ q,\ r,\ s) = (y,\ 0,\ x,\ 1)$ of (4.26) implies

$$\begin{aligned}
&\int_{(\infty,\ \{0,\,y\})} u_{\infty+\delta}(t)\,dt = B(1+\mu_\infty,\ 1+\mu_{0y})\,(1-y)^{1+\mu_y}\,x^{\mu_x}\\
&\quad\times G_2\left(-\mu_y,\ -\mu_x,\ 1+\mu_{0y},\ -1-\mu_{\infty 0y}\,;\ \frac{y}{1-y},\ \frac{1-x}{x}\right)\\
&= B(\alpha,\ 1+\beta-\gamma)\,(1-y)^{1-\beta'}\,x^{-\beta}\\
&\quad\times G_2\left(\beta',\ \beta,\ 1+\beta-\gamma,\ \gamma-\alpha-\beta\,;\ \frac{y}{1-y},\ \frac{1-x}{x}\right)\\
&= B(\alpha,\ 1+\beta-\gamma)\,(1-y)^{1-\beta'}\,x^{-\beta}\\
&\quad\times G_2\left(\beta,\ \beta,\ \gamma-\alpha-\beta,\ 1+\beta-\gamma\,;\ \frac{1-x}{x},\ \frac{y}{1-y}\right)
\end{aligned} \tag{4.29}$$

for $y < 1/2 < x$, which gives $f_2^{(1,\,0)}$.

When $r,\ s < p,\ q$ and $|(q-p)/(s-p)| < 1$ and $|(s-r)/(q-r)| < 1$, the change of integration variable $t \mapsto (q-st)/(1-t)$ implies

$$\begin{aligned}
&\int_{(\{q,\,p\},\ \infty)} u_{\infty-\delta}(t)\,dt = \int_{(\{q,\,p\},\ \infty)} (t-p)^{\mu_p}(t-q)^{\mu_q}(t-r)^{\mu_r}(t-s)^{\mu_s}\,dt\\
&= \int_{(\{0,\,(p-q)/(p-s)\},\ 1)} \left(\frac{q-p+(p-s)t}{1-t}\right)^{\mu_p}\left(\frac{(q-s)t}{1-t}\right)^{\mu_q}\\
&\quad\times\left(\frac{q-r+(r-s)t}{1-t}\right)^{\mu_r}\left(\frac{q-s}{1-t}\right)^{\mu_s}\frac{q-s}{(1-t)^2}\,dt\\
&= (q-s)^{1+\mu_{sq}}\,(p-s)^{\mu_p}\,(q-r)^{\mu_r}
\end{aligned}$$

$$\times \int_{(\{0,\,(p-q)/(p-s)\},\,1)} (1-t)^{\mu_\infty} \left(1+\frac{p-q}{(s-p)t}\right)^{\mu_p} t^{\mu_{pq}} \left(1+\frac{s-r}{r-q}\,t\right)^{\mu_r} dt$$

$$= (q-s)^{1+\mu_{sq}} (p-s)^{\mu_p} (q-r)^{\mu_r} B(1+\mu_{pq},\, 1+\mu_\infty)$$

$$\times G_2\left(-\mu_p,\, -\mu_r,\, 1+\mu_{pq},\, -1-\mu_{\infty pq}\,;\, -\frac{q-p}{s-p},\, -\frac{s-r}{q-r}\right). \tag{4.30}$$

The case $(p,\, q,\, r,\, s) = (y,\, x,\, 0,\, 1)$ of (4.30) implies

$$\int_{(\{x,\,y\},\,\infty)} u_{\infty-\delta}(t)\,dt = (x-1)^{1+\mu_{1x}} (y-1)^{\mu_y} x^{\mu_0} B(1+\mu_{xy},\, 1+\mu_\infty)$$

$$\times G_2\left(-\mu_y,\, -\mu_0,\, 1+\mu_{yx},\, -1-\mu_{\infty yx}\,;\, \frac{y-x}{1-y},\, -\frac{1}{x}\right)$$

$$= B(1-\beta-\beta',\, \alpha)\,(x-1)^{\gamma-\alpha-\beta} (y-1)^{-\beta'} x^{\beta+\beta'-\gamma}$$

$$\times G_2\left(\beta',\, \gamma-\beta-\beta',\, 1-\beta-\beta',\, \beta+\beta'-\alpha\,;\, \frac{y-x}{1-y},\, -\frac{1}{x}\right)$$

$$= B(1-\beta-\beta',\, \alpha)\,(x-1)^{\gamma-\alpha-\beta} (y-1)^{-\beta'} x^{\beta+\beta'-\gamma}$$

$$\times G_2\left(\gamma-\beta-\beta',\, \beta',\, \beta+\beta'-\alpha,\, 1-\beta-\beta'\,;\, -\frac{1}{x},\, \frac{y-x}{1-y}\right) \tag{4.31}$$

for $1 < x < \infty$, $1 < y < \infty$ and $|x-y| < |y-1|$, which gives $f_2^{(\infty,\,\infty)\,x=y}$ for $1 < x,\, y$.

The case $(p,\, q,\, r,\, s) = (y,\, 1,\, x,\, 0)$ of (4.30) implies

$$\int_{(\{1,\,y\},\,\infty)} u_{\infty-\delta}(t)\,dt = B(1+\mu_{y1},\, 1+\mu_\infty)\, y^{\mu_y} (1-x)^{1+\mu_x}$$

$$\times G_2\left(-\mu_y,\, -\mu_x,\, 1+\mu_{y1},\, -1-\mu_{\infty y1}\,;\, \frac{1-y}{y},\, \frac{x}{1-x}\right)$$

$$= B(\gamma-\alpha-\beta',\, \alpha)\, y^{-\beta'} (1-x)^{1-\beta}$$

$$\times G_2\left(\beta',\, \beta,\, \gamma-\alpha-\beta',\, 1+\beta'-\gamma\,;\, \frac{1-y}{y},\, \frac{x}{1-x}\right)$$

$$= B(\gamma-\alpha-\beta',\, \alpha)\, y^{-\beta'} (1-x)^{1-\beta}$$

$$\times G_2\left(\beta,\, \beta',\, 1+\beta'-\gamma,\, \gamma-\alpha-\beta'\,;\, \frac{x}{1-x},\, \frac{1-y}{y}\right) \tag{4.32}$$

for $x < 1/2 < y$, which gives $f_2^{(0,\,1)}$.

The case $(p,\ q,\ r,\ s) = (x,\ 1,\ y,\ 0)$ of (4.30) implies

$$\begin{aligned}
&\int_{(\{1,\,x\},\,\infty)} u_{\infty-\delta}(t)\,dt = B(1+\mu_{x1},\ 1+\mu_\infty)\,x^{\mu_x}\,(1-y)^{1+\mu_y} \\
&\quad\times G_2\left(-\mu_x,\ -\mu_y,\ 1+\mu_{x1},\ -1-\mu_{\infty x1}\,;\ \frac{1-x}{x},\ \frac{y}{1-y}\right) \\
&= B(\gamma-\alpha-\beta,\ \alpha)\,x^{-\beta}\,(1-y)^{1-\beta'} \\
&\quad\times G_2\left(\beta,\ \beta',\ \gamma-\alpha-\beta,\ 1+\beta-\gamma,\ ;\ \frac{1-x}{x},\ \frac{y}{1-y}\right)
\end{aligned} \tag{4.33}$$

for $y < 1/2 < x$, which gives $f_2^{(1,\,0)}$.

When $q < s$ and $|(q-s)/(p-s)| < 1$ and $|(r-s)/(q-s)| < 1$, for

$$\varepsilon = \begin{cases} +1 & \text{if} \quad p < q \\ -1 & \text{if} \quad p > s \end{cases} ,$$

the change of integration variable $t \mapsto ((q-s)+st)/t$ implies

$$\begin{aligned}
&\int_{(\{\infty,\,p\},\,q)} u_{q-\delta}(t)\,dt = \int_{(\{\infty,\,p\},\,q)} (\varepsilon(t-p))^{\mu_p}(q-t)^{\mu_q}(r-t)^{\mu_r}(s-t)^{\mu_s}\,dt \\
&= \int_{(\{0,\,(q-s)/(p-s)\},\,1)} \left(\varepsilon\,\frac{q-s+(s-p)t}{t}\right)^{\mu_p} \left(\frac{(s-q)(1-t)}{t}\right)^{\mu_q} \\
&\quad\times \left(\frac{s-q-(s-r)t}{t}\right)^{\mu_r} \left(\frac{s-q}{t}\right)^{\mu_s} \frac{s-q}{t^2}\,dt \\
&= (s-q)^{1+\mu_{qrs}}\,(\varepsilon(s-p))^{\mu_p} \\
&\quad\times \int_{(\{0,\,(q-s)/(p-s)\},\,1)} t^{\mu_{\infty p}} \left(1+\frac{q-s}{(s-p)t}\right)^{\mu_p} (1-t)^{\mu_q} \left(1+\frac{s-r}{q-s}\,t\right)^{\mu_r}\,dt \\
&= (s-q)^{1+\mu_{qrs}}\,(\varepsilon(s-p))^{\mu_p}\,B(1+\mu_{\infty p},\ 1+\mu_q) \\
&\quad\times G_2\left(-\mu_p,\ -\mu_r,\ 1+\mu_{\infty p},\ -1-\mu_{\infty pq}\,;\ -\frac{q-s}{p-s},\ -\frac{r-s}{q-s}\right).
\end{aligned} \tag{4.34}$$

The case $(p,\ q,\ r,\ s) = (x,\ 0,\ y,\ 1)$ of (4.34) implies

$$\begin{aligned}
\int_{(\{\infty,\, x\},\, 0)} u_{0-\delta}(t)\, dt &= B(1+\mu_{\infty x},\ 1+\mu_0)\,(\varepsilon(1-x))^{\mu_x} \\
&\quad \times G_2\left(-\mu_x,\ -\mu_y,\ 1+\mu_{\infty x},\ -1-\mu_{\infty x 0}\,;\ \frac{1}{x-1},\ y-1\right) \\
&= B(\alpha-\beta,\ 1+\beta+\beta'-\gamma)\,(\varepsilon(1-x))^{-\beta} \\
&\quad \times G_2\left(\beta,\ \beta',\ \alpha-\beta,\ \gamma-\alpha-\beta',\ ;\ \frac{1}{x-1},\ y-1\right),
\end{aligned} \tag{4.35}$$

where

$$\varepsilon = \begin{cases} +1 & \text{if} \quad x<0 \\ -1 & \text{if} \quad x>1 \end{cases},$$

for $|y-1|<1<|x-1|$, which gives $f_2^{(\infty,\,1)}$.

The case $(p,\ q,\ r,\ s) = (y,\ 0,\ x,\ 1)$ of (4.34) implies

$$\begin{aligned}
\int_{(\{\infty,\, y\},\, 0)} u_{0-\delta}(t)\, dt &= B(1+\mu_{\infty y},\ 1+\mu_0)\,(\varepsilon(1-y))^{\mu_y} \\
&\quad \times G_2\left(-\mu_y,\ -\mu_x,\ 1+\mu_{\infty y},\ -1-\mu_{\infty y 0}\,;\ \frac{1}{y-1},\ x-1\right) \\
&= B(\alpha-\beta',\ 1+\beta+\beta'-\gamma)\,(\varepsilon(1-y))^{-\beta'} \\
&\quad \times G_2\left(\beta',\ \beta,\ \alpha-\beta',\ \gamma-\alpha-\beta,\ ;\ \frac{1}{y-1},\ x-1\right) \\
&= B(\alpha-\beta',\ 1+\beta+\beta'-\gamma)\,(\varepsilon(1-y))^{-\beta'} \\
&\quad \times G_2\left(\beta,\ \beta',\ \gamma-\alpha-\beta,\ \alpha-\beta'\,;\ x-1,\ \frac{1}{y-1}\right),
\end{aligned} \tag{4.36}$$

where

$$\varepsilon = \begin{cases} +1 & \text{if} \quad y<0 \\ -1 & \text{if} \quad y>1 \end{cases},$$

for $|x-1|<1<|y-1|$, which gives $f_2^{(1,\,\infty)}$.

The case $(p,\ q,\ r,\ s) = (x,\ y,\ 1,\ 0)$ of (4.34) implies

$$\begin{aligned}
&\int_{(\{\infty,\,x\},\,y)} u_{y-\delta}(t)\,dt = B(1+\mu_{\infty x},\ 1+\mu_y)\,(-y)^{1+\mu_{0y1}}\,(\varepsilon x)^{\mu_x}\\
&\quad\times G_2\left(-\mu_x,\ -\mu_1,\ 1+\mu_{\infty x},\ -1-\mu_{\infty xy}\,;\ -\frac{y}{x},\ -\frac{1}{y}\right)\\
&= B(\alpha-\beta,\ 1-\beta')\,(-y)^{\beta-\alpha}\,(\varepsilon x)^{-\beta}\\
&\quad\times G_2\left(\beta,\ 1+\alpha-\gamma,\ \alpha-\beta,\ \beta+\beta'-\gamma,\ ;\ -\frac{y}{x},\ -\frac{1}{y}\right),
\end{aligned} \tag{4.37}$$

where

$$\varepsilon = \begin{cases} +1 & \text{if} \quad x > 1 \\ -1 & \text{if} \quad x < y \end{cases},$$

for $y < -1$ and $|y| < |x|$, which gives $f_2^{(\infty,\,\infty)\,x=\infty}$ for $y < -1$.

The case $(p,\ q,\ r,\ s) = (y,\ x,\ 1,\ 0)$ of (4.34) implies

$$\begin{aligned}
&\int_{(\{\infty,\,y\},\,x)} u_{x-\delta}(t)\,dt = B(1+\mu_{\infty y},\ 1+\mu_x)\,(-x)^{1+\mu_{0x1}}\,(\varepsilon y)^{\mu_y}\\
&\quad\times G_2\left(-\mu_y,\ -\mu_1,\ 1+\mu_{\infty y},\ -1-\mu_{\infty yx}\,;\ -\frac{x}{y},\ -\frac{1}{x}\right)\\
&= B(\alpha-\beta',\ 1-\beta)\,(-x)^{\beta'-\alpha}\,(\varepsilon y)^{-\beta'}\\
&\quad\times G_2\left(\beta',\ 1+\alpha-\gamma,\ \alpha-\beta',\ \beta+\beta'-\gamma,\ ;\ -\frac{x}{y},\ -\frac{1}{x}\right)\\
&= B(\alpha-\beta',\ 1-\beta)\,(-x)^{\beta'-\alpha}\,(\varepsilon y)^{-\beta'}\\
&\quad\times G_2\left(1+\alpha-\gamma,\ \beta',\ \beta+\beta'-\gamma,\ \alpha-\beta'\,;\ -\frac{1}{x},\ -\frac{x}{y}\right),
\end{aligned} \tag{4.38}$$

where

$$\varepsilon = \begin{cases} +1 & \text{if} \quad y > 1 \\ -1 & \text{if} \quad y < x \end{cases},$$

for $x < -1$ and $|x| < |y|$, which gives $f_2^{(\infty,\,\infty)\,y=\infty}$ for $x < -1$.

When $s < q$ and $|(q-s)/(p-s)| < 1$ and $|(r-s)/(q-s)| < 1$, for

$$\varepsilon = \begin{cases} +1 & \text{if} \quad p > q \\ -1 & \text{if} \quad p < s \end{cases},$$

the change of integration variable $t \mapsto ((q-s)+st)/t$ implies

$$\begin{aligned}
&\int_{(q,\,\{p,\,\infty\})} u_{q+\delta}(t)\,dt = \int_{(q,\,\{p,\,\infty\})} (\varepsilon(p-t))^{\mu_p}(t-q)^{\mu_q}(t-r)^{\mu_r}(t-s)^{\mu_s}\,dt\\
&= \int_{(\{0,\,(q-s)/(p-s)\},\,1)} \left(\varepsilon\,\frac{s-q+(p-s)t}{t}\right)^{\mu_p}\left(\frac{(q-s)(1-t)}{t}\right)^{\mu_q}\\
&\quad\times\left(\frac{q-s-(r-s)t}{t}\right)^{\mu_r}\left(\frac{q-s}{t}\right)^{\mu_s}\frac{q-s}{t^2}\,dt\\
&= (q-s)^{1+\mu_{qrs}}\,(\varepsilon(p-s))^{\mu_p}\\
&\quad\times\int_{(\{0,\,(q-s)/(p-s)\},\,1)} t^{\mu_{\infty p}}\left(1+\frac{q-s}{(s-p)t}\right)^{\mu_p}(1-t)^{\mu_q}\left(1+\frac{s-r}{q-s}\,t\right)^{\mu_r}\,dt\\
&= (q-s)^{1+\mu_{qrs}}\,(\varepsilon(p-s))^{\mu_p}\,B(1+\mu_q,\,1+\mu_{\infty p})\\
&\quad\times G_2\!\left(-\mu_p,\,-\mu_r,\,1+\mu_{\infty p},\,-1-\mu_{\infty pq}\,;\,-\frac{q-s}{p-s},\,-\frac{r-s}{q-s}\right).
\end{aligned}\tag{4.39}$$

The case $(p,\,q,\,r,\,s)=(x,\,1,\,y,\,0)$ of (4.39) implies

$$\begin{aligned}
\int_{(1,\,\{x,\,\infty\})} u_{1+\delta}(t)\,dt &= B(1+\mu_1,\,1+\mu_{\infty x})\,(\varepsilon x)^{\mu_x}\\
&\quad\times G_2\!\left(-\mu_x,\,-\mu_y,\,1+\mu_{\infty x},\,-1-\mu_{\infty x1}\,;\,-\frac{1}{x},\,-y\right)\\
&= B(\gamma-\alpha,\,\alpha-\beta)\,(\varepsilon x)^{-\beta}\\
&\quad\times G_2\!\left(\beta,\,\beta',\,\alpha-\beta,\,1+\beta-\gamma,\;;\,-\frac{1}{x},\,-y\right),
\end{aligned}\tag{4.40}$$

where

$$\varepsilon=\begin{cases}+1 & \text{if}\quad x>1\\ -1 & \text{if}\quad x<0\end{cases},$$

for $|y|<1<|x|$, which gives $f_2^{(\infty,\,0)}$.

The case $(p,\,q,\,r,\,s)=(y,\,1,\,x,\,0)$ of (4.39) implies

$$\begin{aligned}
\int_{(1,\,\{y,\,\infty\})} u_{1+\delta}(t)\,dt &= B(1+\mu_1,\,1+\mu_{\infty y})\,(\varepsilon y)^{\mu_y}\\
&\quad\times G_2\!\left(-\mu_y,\,-\mu_x,\,1+\mu_{\infty y},\,-1-\mu_{\infty y1}\,;\,-\frac{1}{y},\,-x\right)\\
&= B(\gamma-\alpha,\,\alpha-\beta')\,(\varepsilon y)^{-\beta'}
\end{aligned}$$

$$\times\, G_2\Big(\beta',\, \beta,\, \alpha-\beta',\, 1+\beta'-\gamma,\ ;\, -\frac{1}{y},\, -x\Big)$$

$$= B(\gamma-\alpha,\, \alpha-\beta')\,(\varepsilon y)^{-\beta'}$$

$$\times\, G_2\Big(\beta,\, \beta',\, 1+\beta'-\gamma,\, \alpha-\beta'\,;\, -x,\, -\frac{1}{y}\Big), \tag{4.41}$$

where

$$\varepsilon = \begin{cases} +1 & \text{if} \quad y > 1 \\ -1 & \text{if} \quad y < 0 \end{cases},$$

for $|x| < 1 < |y|$, which gives $f_2^{(0,\,\infty)}$.

The case $(p,\, q,\, r,\, s) = (x,\, y,\, 1,\, 0)$ of (4.39) implies

$$\int_{(y,\, \{x,\, \infty\})} u_{y+\delta}(t)\, dt = B(1+\mu_y,\, 1+\mu_{\infty x})\, y^{1+\mu_{0y1}}\, (\varepsilon x)^{\mu_x}$$

$$\times\, G_2\Big(-\mu_x,\, -\mu_1,\, 1+\mu_{\infty x},\, -1-\mu_{\infty xy}\,;\, -\frac{y}{x},\, -\frac{1}{y}\Big)$$

$$= B(1-\beta',\, \alpha-\beta)\,(\varepsilon x)^{-\beta}\, y^{\beta-\alpha}$$

$$\times\, G_2\Big(\beta,\, 1+\alpha-\gamma,\, \alpha-\beta,\, \beta+\beta'-\alpha,\ ;\, -\frac{y}{x},\, -\frac{1}{y}\Big), \tag{4.42}$$

where

$$\varepsilon = \begin{cases} +1 & \text{if} \quad x > y \\ -1 & \text{if} \quad x < -y \end{cases},$$

for $1 < y < |x|$, which gives $f_2^{(\infty,\,\infty)\, x=\infty}$ for $1 < y$.

The case $(p,\, q,\, r,\, s) = (y,\, x,\, 1,\, 0)$ of (4.39) implies

$$\int_{(x,\, \{y,\, \infty\})} u_{x+\delta}(t)\, dt = B(1+\mu_x,\, 1+\mu_{\infty y})\, x^{1+\mu_{0x1}}\, (\varepsilon y)^{\mu_y}$$

$$\times\, G_2\Big(-\mu_y,\, -\mu_1,\, 1+\mu_{\infty y},\, -1-\mu_{\infty yx}\,;\, -\frac{x}{y},\, -\frac{1}{x}\Big)$$

$$= B(1-\beta,\, \alpha-\beta')\,(\varepsilon y)^{-\beta'}\, x^{\beta'-\alpha}$$

$$\times\, G_2\Big(\beta',\, 1+\alpha-\gamma,\, \alpha-\beta',\, \beta+\beta'-\alpha,\ ;\, -\frac{x}{y},\, -\frac{1}{x}\Big)$$

$$= B(1-\beta,\, \alpha-\beta')\,(\varepsilon y)^{-\beta'}\, x^{\beta'-\alpha}$$

$$\times\, G_2\Big(1+\alpha-\gamma,\, \beta',\, \beta+\beta'-\alpha,\, \alpha-\beta'\,;\, -\frac{1}{x},\, -\frac{x}{y}\Big), \tag{4.43}$$

where

$$\varepsilon = \begin{cases} +1 & \text{if} \quad y > x \\ -1 & \text{if} \quad y < -x \end{cases},$$

for $1 < x < |y|$, which gives $f_2^{(\infty,\,\infty)\,y=\infty}$ for $1 < x$.

5. Derivation of the connection formulas

In this section, the function $u(t)$ is fixed to be

$$u(t) = (t-p)^{\mu_p}(t-q)^{\mu_q}(t-r)^{\mu_r}(t-s)^{\mu_s},$$

where

$$\{p,\ q,\ r,\ s\} = \{0,\ 1,\ x,\ y\}$$

and

$$\mu_0 = \beta + \beta' - \gamma, \quad \mu_x = -\beta, \quad \mu_y = -\beta', \quad \mu_1 = \gamma - \alpha - 1.$$

In addition,

$$\mu_\infty = 2 - \mu_{0xy1} = \alpha - 1$$

is also used as before. Let $\mathcal{L}$ be the locally constant sheaf defined by $u(t)$ on

$$T = \mathbb{P}^1(\mathbb{C})\backslash\{\,0,\ 1,\ x,\ y,\ \infty\,\}.$$

Suppose the genericity condition that

$$\mu_0, \quad \mu_x, \quad \mu_y, \quad \mu_1, \quad \mu_\infty \in \mathbb{C}\backslash\mathbb{Z}, \tag{5.1}$$

which implies that the rank of $H_1^{\mathrm{lf}}(T, \mathcal{L})$ or the rank of $H_1(T, \mathcal{L})$ turns out to be three ([Ch][KN]).

- When $c < a < x < y < b < c$ for $a,\ b,\ c,\ x,\ y \in P^1(\mathbb{R})$ with $\{\,a,\ b,\ c\,\} = \{\,0,\ 1,\ \infty\,\}$, and $\mu_{ax},\ \mu_{yb} \notin \mathbb{Z}$ in addition to (5.1), both of the sets

$$\big\{\,(a,\ x) \otimes u_{(a,\,x)}(t), \quad (\{a,\ x\},\ y) \otimes u_{y-\delta}(t), \quad (b,\ c) \otimes u_{(b,\,c)}(t)\,\big\}$$

and

$$\big\{\,(y,\ b) \otimes u_{(y,\,b)}(t), \quad (x,\ \{y,\ b\}) \otimes u_{x+\delta}(t), \quad (c,\ a) \otimes u_{(c,\,a)}(t)\,\big\}$$

give the bases of $H_1^{\mathrm{lf}}(T, \mathcal{L})$ and hence there exist the numbers $p_i,\ q_i,\ r_i$ such that

$$
\begin{aligned}
(a,\ x) &= p_1\,(y,\ b) + p_2\,(x,\ \{y,\ b\}) + p_3\,(c,\ a),\\
(\{a,\ x\},\ y) &= q_1(y,\ b) + q_2\,(x,\ \{y,\ b\}) + q_3\,(c,\ a),\\
(b,\ c) &= r_1\,(y,\ b) + r_2\,(x,\ \{y,\ b\}) + r_3\,(c,\ a).
\end{aligned}
$$

As for the intersection numbers, we have

$$
\begin{aligned}
&(y,\ b)^2 = \frac{-\langle\, e_{yb}\,\rangle}{\langle\, e_y\,\rangle\langle\, e_b\,\rangle}, \quad (y,\ b)\ \cdot\ (x,\ \{y,\ b\}) = 0, \quad (y,\ b)\ \cdot\ (c,\ a) = 0,\\
&(x,\ \{y,\ b\})\ \cdot\ (y,\ b) = 0, \quad (x,\ \{y,\ b\})^2 = \frac{-\langle\, e_{xyb}\,\rangle}{\langle\, e_x\,\rangle\langle\, e_{yb}\,\rangle}, \quad (x,\ \{y,\ b\})\ \cdot\ (c,\ a) = 0,\\
&(c,\ a)\ \cdot\ (y,\ b) = 0, \quad (c,\ a)\ \cdot\ (x,\ \{y,\ b\}) = 0, \quad (c,\ a)^2 = \frac{-\langle\, e_{ac}\,\rangle}{\langle\, e_c\,\rangle\langle\, e_a\,\rangle},
\end{aligned}
$$

and

$$
\begin{aligned}
&(a,\ x)\ \cdot\ (y,\ b) = 0, \quad (a,\ x)\ \cdot\ (x,\ \{y,\ b\}) = \frac{1}{\langle\, e_x\,\rangle}, \quad (a,\ x)\ \cdot\ (c,\ a) = \frac{1}{\langle\, e_a\,\rangle},\\
&(\{a,\ x\},\ y)\ \cdot\ (y,\ b) = \frac{1}{\langle\, e_y\,\rangle}, \quad (\{a,\ x\},\ y)\ \cdot\ (x,\ \{y,\ b\}) = \frac{-\langle\, e_{axyb}\,\rangle}{\langle\, e_{ax}\,\rangle\langle\, e_{yb}\,\rangle},\\
&(\{a,\ x\},\ y)\ \cdot\ (c,\ a) = \frac{1}{\langle\, e_{ax}\,\rangle}, \quad (b,\ c)\ \cdot\ (y,\ b) = \frac{1}{\langle\, e_b\,\rangle},\\
&(b,\ c)\ \cdot\ (x,\ \{y,\ b\}) = \frac{1}{\langle\, e_{yb}\,\rangle}, \quad (b,\ c)\ \cdot\ (c,\ a) = \frac{1}{\langle\, e_c\,\rangle},
\end{aligned}
$$

where D^2 designates $D \cdot D$ for brevity.

Hence we have

$$
\begin{aligned}
&p_1 = 0, \quad p_2 = \frac{-\langle\, e_{yb}\,\rangle}{\langle\, e_{xyb}\,\rangle}, \quad p_3 = \frac{-\langle\, e_c\,\rangle}{\langle\, e_{ca}\,\rangle},\\
&q_1 = \frac{-\langle\, e_b\,\rangle}{\langle\, e_{yb}\,\rangle}, \quad q_2 = \frac{\langle\, e_{axyb}\,\rangle\langle\, e_x\,\rangle}{\langle\, e_{ax}\,\rangle\langle\, e_{xyb}\,\rangle}, \quad q_3 = \frac{-\langle\, e_c\,\rangle\langle\, e_a\,\rangle}{\langle\, e_{ax}\,\rangle\langle\, e_{ca}\,\rangle},\\
&r_1 = \frac{-\langle\, e_y\,\rangle}{\langle\, e_{yb}\,\rangle}, \quad r_2 = \frac{-\langle\, e_x\,\rangle}{\langle\, e_{xyb}\,\rangle}, \quad r_3 = \frac{-\langle\, e_a\,\rangle}{\langle\, e_{ca}\,\rangle},
\end{aligned}
$$

and thus

$$(a,\ x) = \frac{-\langle\, e_{yb}\,\rangle}{\langle\, e_{xyb}\,\rangle}\,(x,\ \{y,\ b\}) + \frac{-\langle\, e_c\,\rangle}{\langle\, e_{ca}\,\rangle}\,(c,\ a), \tag{5.2}$$

$$(\{a,\ x\},\ y) = \frac{-\langle\, e_b\,\rangle}{\langle\, e_{yb}\,\rangle}(y,\ b) + \frac{\langle\, e_{axyb}\,\rangle\langle\, e_x\,\rangle}{\langle\, e_{ax}\,\rangle\langle\, e_{xyb}\,\rangle}\,(x,\ \{y,\ b\}) + \frac{-\langle\, e_c\,\rangle\langle\, e_a\,\rangle}{\langle\, e_{ax}\,\rangle\langle\, e_{ca}\,\rangle}\,(c,\ a), \tag{5.3}$$

$$(b,\ c) = \frac{-\langle\, e_y\,\rangle}{\langle\, e_{yb}\,\rangle}\,(y,\ b) + \frac{-\langle\, e_x\,\rangle}{\langle\, e_{xyb}\,\rangle}\,(x,\ \{y,\ b\}) + \frac{-\langle\, e_a\,\rangle}{\langle\, e_{ca}\,\rangle}\,(c,\ a). \tag{5.4}$$

Similarly we have

$$(y,\ b) = \frac{-\langle\, e_{ax}\,\rangle}{\langle\, e_{axy}\,\rangle}\,(\{a,\ x\},\ y) + \frac{-\langle\, e_c\,\rangle}{\langle\, e_{bc}\,\rangle}\,(b,\ c),$$

$$(x,\ \{y,\ b\}) = \frac{-\langle\, e_a\,\rangle}{\langle\, e_{ax}\,\rangle}(a,\ x) + \frac{-\langle\, e_{axyb}\,\rangle\langle\, e_y\,\rangle}{\langle\, e_{axy}\,\rangle\langle\, e_{yb}\,\rangle}\,(\{a,\ x\},\ y) + \frac{-\langle\, e_b\,\rangle\langle\, e_c\,\rangle}{\langle\, e_{yb}\,\rangle\langle\, e_{bc}\,\rangle}\,(b,\ c),$$

$$(c,\ a) = \frac{-\langle\, e_x\,\rangle}{\langle\, e_{ax}\,\rangle}\,(a,\ x) + \frac{-\langle\, e_y\,\rangle}{\langle\, e_{axy}\,\rangle}\,(\{a,\ x\},\ y) + \frac{-\langle\, e_b\,\rangle}{\langle\, e_{bc}\,\rangle}\,(b,\ c).$$

On the other hand, we have

$$\int_{(a,\,x)} u_{(a,\,x)}(t)\,dt = B(1+\mu_a,\ 1+\mu_x)\, f_1^{(a,\,a)\,x=a},$$

$$\int_{(\{a,\,x\},\,y)} u_{y-\delta}(t)\,dt = B(1+\mu_{ax},\ 1+\mu_y)\, f_2^{(a,\,a)\,x=a},$$

$$\int_{(b,\,c)} u_{(b,\,c)}(t)\,dt = B(1+\mu_b,\ 1+\mu_c)\, f_3^{(a,\,a)\,x=a},$$

and

$$\int_{(y,\,b)} u_{(y,\,b)}(t)\,dt = B(1+\mu_y,\ 1+\mu_b)\, f_1^{(b,\,b)\,y=b},$$

$$\int_{(x,\,\{y,\,b\})} u_{x+\delta}(t)\,dt = B(1+\mu_x,\ 1+\mu_{yb})\, f_2^{(b,\,b)\,y=b},$$

$$\int_{(c,\,a)} u_{(c,\,a)}(t)\,dt = B(1+\mu_c,\ 1+\mu_a)\, f_3^{(b,\,b)\,y=b}.$$

Therefore, we obtain

$$\begin{aligned}
f_1^{(a,\,a)\,x=a} &= \frac{-\langle\, e_{yb}\,\rangle}{\langle\, e_{xyb}\,\rangle}\frac{B(1+\mu_x,\,1+\mu_{yb})}{B(1+\mu_a,\,1+\mu_x)}\,f_2^{(b,\,b)\,y=b} + \frac{-\langle\, e_c\,\rangle}{\langle\, e_{ca}\,\rangle}\frac{B(1+\mu_c,\,1+\mu_a)}{B(1+\mu_a,\,1+\mu_x)}\,f_3^{(b,\,b)\,y=b} \\
&= \frac{\Gamma(-1-\mu_{xyb},\,2+\mu_{ax})}{\Gamma(-\mu_{yb},\,1+\mu_a)}\,f_2^{(b,\,b)\,y=b} + \frac{\Gamma(-1-\mu_{ca},\,2+\mu_{ax})}{\Gamma(-\mu_c,\,1+\mu_x)}\,f_3^{(b,\,b)\,y=b},
\end{aligned} \tag{5.5}$$

$$\begin{aligned}
f_2^{(a,\,a)\,x=a} &= \frac{-\langle\, e_b\,\rangle}{\langle\, e_{yb}\,\rangle}\frac{B(1+\mu_y,\,1+\mu_b)}{B(1+\mu_b,\,1+\mu_c)}\,f_1^{(b,\,b)\,y=b} \\
&+ \frac{\langle\, e_{axyb}\,\rangle\langle\, e_x\,\rangle}{\langle\, e_{ax}\,\rangle\langle\, e_{xyb}\,\rangle}\frac{B(1+\mu_x,\,1+\mu_{yb})}{B(1+\mu_b,\,1+\mu_c)}\,f_2^{(b,\,b)\,y=b} \\
&+ \frac{-\langle\, e_c\,\rangle\langle\, e_a\,\rangle}{\langle\, e_{ax}\,\rangle\langle\, e_{ca}\,\rangle}\frac{B(1+\mu_c,\,1+\mu_a)}{B(1+\mu_b,\,1+\mu_c)}\,f_3^{(b,\,b)\,y=b} \\
&= \frac{\Gamma(-1-\mu_{yb},\,2+\mu_{axy})}{\Gamma(-\mu_b,\,1+\mu_{ax})}\,f_1^{(b,\,b)\,y=b} \\
&+ \frac{\Gamma(-\mu_{ax},\,-1-\mu_{xyb},\,1+\mu_{yb},\,2+\mu_{axy})}{\Gamma(2+\mu_{axyb},\,-1-\mu_{axyb},\,-\mu_x,\,1+\mu_y)}\,f_2^{(b,\,b)\,y=b} \\
&+ \frac{\Gamma(-\mu_{ax},-1-\mu_{ca},\,2+\mu_{axy})}{\Gamma(-\mu_c,\,-\mu_a,\,1+\mu_y)}\,f_3^{(b,\,b)\,y=b},
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
f_3^{(a,\,a)\,x=a} &= \frac{-\langle\, e_y\,\rangle}{\langle\, e_{yb}\,\rangle}\frac{B(1+\mu_y,\,1+\mu_b)}{B(1+\mu_b,\,1+\mu_c)}f_1^{(b,\,b)\,y=b} + \frac{-\langle\, e_x\,\rangle}{\langle\, e_{xyb}\,\rangle}\frac{B(1+\mu_x,\,1+\mu_{yb})}{B(1+\mu_b,\,1+\mu_c)}\,f_2^{(b,\,b)\,y=b} \\
&+ \frac{-\langle\, e_a\,\rangle}{\langle\, e_{ca}\,\rangle}\frac{B(1+\mu_c,\,1+\mu_a)}{B(1+\mu_b,\,1+\mu_c)}\,f_3^{(b,\,b)\,y=b} \\
&= \frac{\Gamma(-1-\mu_{yb},\,2+\mu_{bc})}{\Gamma(-\mu_y,\,1+\mu_c)}\,f_1^{(b,\,b)\,y=b} + \frac{\Gamma(-1-\mu_{xyb},1+\mu_{yb},\,2+\mu_{bc})}{\Gamma(-\mu_x,\,1+\mu_b,\,1+\mu_c)}\,f_2^{(b,\,b)\,y=b} \\
&+ \frac{\Gamma(-1-\mu_{ca},\,2+\mu_{bc})}{\Gamma(-\mu_a,\,1+\mu_b)}\,f_3^{(b,\,b)\,y=b},
\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}
f_1^{(b,\,b)\,y=b} &= \frac{-\langle\, e_{ax}\,\rangle}{\langle\, e_{axy}\,\rangle}\frac{B(1+\mu_{ax},\,1+\mu_y)}{B(1+\mu_y,\,1+\mu_b)}\,f_2^{(a,\,a)\,x=a} + \frac{-\langle\, e_c\,\rangle}{\langle\, e_{bc}\,\rangle}\frac{B(1+\mu_b,\,1+\mu_c)}{B(1+\mu_y,\,1+\mu_b)}\,f_3^{(a,\,a)\,x=a} \\
&= \frac{\Gamma(-1-\mu_{axy},\,2+\mu_{yb})}{\Gamma(-\mu_{ax},\,1+\mu_b)}\,f_2^{(a,\,a)\,x=a} + \frac{\Gamma(-1-\mu_{cb},\,2+\mu_{yb})}{\Gamma(-\mu_c,\,1+\mu_y)}\,f_3^{(a,\,a)\,x=a},
\end{aligned} \tag{5.8}$$

$$\begin{aligned}
f_2^{(b,\,b)\,y=b} &= \frac{-\langle\, e_a\,\rangle}{\langle\, e_{ax}\,\rangle}\frac{B(1+\mu_a,\,1+\mu_x)}{B(1+\mu_x,\,1+\mu_{yb})}\,f_1^{(a,\,a)\,x=a} \\
&+ \frac{-\langle\, e_{axyb}\,\rangle\langle\, e_y\,\rangle}{\langle\, e_{axy}\,\rangle\langle\, e_{yb}\,\rangle}\frac{B(1+\mu_{ax},\,1+\mu_y)}{B(1+\mu_x,\,1+\mu_{yb})}\,f_2^{(a,\,a)\,x=a}
\end{aligned}$$

$$+\frac{-\langle e_b\rangle\langle e_c\rangle}{\langle e_{yb}\rangle\langle e_{bc}\rangle}\frac{B(1+\mu_b,\ 1+\mu_c)}{B(1+\mu_x,\ 1+\mu_{yb})}f_3^{(a,\,a)\,x=a}$$

$$=\frac{\Gamma(-1-\mu_{ax},\ 2+\mu_{bxy})}{\Gamma(-\mu_a,\ 1+\mu_{by})}f_1^{(a,\,a)\,x=a}$$

$$+\frac{\Gamma(-\mu_{by},\ -1-\mu_{axy},\ 1+\mu_{ax},\ 2+\mu_{bxy})}{\Gamma(2+\mu_{axyb},\ -1-\mu_{axyb},\ -\mu_y,\ 1+\mu_x)}f_2^{(a,\,a)\,x=a}$$

$$+\frac{\Gamma(-\mu_{yb}, -1-\mu_{bc},\ 2+\mu_{bxy})}{\Gamma(-\mu_c,\ -\mu_b,\ 1+\mu_x)}f_3^{(a,\,a)\,x=a}, \tag{5.9}$$

$$f_3^{(b,\,b)\,y=b}=\frac{-\langle e_x\rangle}{\langle e_{ax}\rangle}\frac{B(1+\mu_a,\ 1+\mu_x)}{B(1+\mu_c,\ 1+\mu_a)}f_1^{(a,\,a)\,x=a}$$

$$+\frac{-\langle e_y\rangle}{\langle e_{axy}\rangle}\frac{B(1+\mu_{ax},\ 1+\mu_y)}{B(1+\mu_c,\ 1+\mu_a)}f_2^{(a,\,a)\,x=a}$$

$$+\frac{-\langle e_b\rangle}{\langle e_{bc}\rangle}\frac{B(1+\mu_b,\ 1+\mu_c)}{B(1+\mu_c,\ 1+\mu_a)}f_3^{(a,\,a)\,x=a}$$

$$=\frac{\Gamma(-1-\mu_{ax},\ 2+\mu_{ac})}{\Gamma(-\mu_x,\ 1+\mu_c)}f_1^{(a,\,a)\,x=a}$$

$$+\frac{\Gamma(-1-\mu_{axy}, 1+\mu_{ax},\ 2+\mu_{ac})}{\Gamma(-\mu_y,\ 1+\mu_a,\ 1+\mu_c)}f_2^{(a,\,a)\,x=a}$$

$$+\frac{\Gamma(-1-\mu_{bc},\ 2+\mu_{ac})}{\Gamma(-\mu_b,\ 1+\mu_a)}f_3^{(a,\,a)\,x=a}. \tag{5.10}$$

In case $a=0$, $b=1$, $c=\infty$, the equalities (5.5), (5.6) and (5.7) combined with (4.1), (4.2), (4.4), (4.7), (4.12) and (4.22) imply the connection formulas (3.1), (3.2) and (3.3).

In case $a=\infty$, $b=0$, $c=1$, the equalities (5.8), (5.9) and (5.10) combined with (4.1), (4.3), (4.5), (4.8), (4.20) and (4.37) imply the connection formulas (3.19), (3.20) and (3.21).

- When $c<a<y<x<b<c$ for $a,\ b,\ c,\ x,\ y\in P^1(\mathbb{R})$ with $\{a,\ b,\ c\}=\{0,\ 1,\ \infty\}$, and $\mu_{ay},\ \mu_{xb}\notin\mathbb{Z}$ in addition to (5.1), by interchanging x and y (at the same time μ_x and μ_y) in the equalities from (5.5) to (5.10), we obtain

$$f_1^{(a,\,a)\,y=a}=\frac{\Gamma(-1-\mu_{xyb},\ 2+\mu_{ay})}{\Gamma(-\mu_{xb},\ 1+\mu_a)}f_2^{(b,\,b)\,x=b}+\frac{\Gamma(-1-\mu_{ca},\ 2+\mu_{ay})}{\Gamma(-\mu_c,\ 1+\mu_y)}f_3^{(b,\,b)\,x=b}, \tag{5.11}$$

$$f_2^{(a,\,a)\,y=a}=\frac{\Gamma(-1-\mu_{xb},\ 2+\mu_{axy})}{\Gamma(-\mu_b,\ 1+\mu_{ay})}f_1^{(b,\,b)\,x=b}$$

$$+\frac{\Gamma(-\mu_{ay},\ -1-\mu_{xyb},\ 1+\mu_{xb},\ 2+\mu_{axy})}{\Gamma(2+\mu_{axyb},\ -1-\mu_{axyb},\ -\mu_y,\ 1+\mu_x)}f_2^{(b,\,b)\,x=b}$$

$$+\frac{\Gamma(-\mu_{ay}, -1-\mu_{ca},\ 2+\mu_{axy})}{\Gamma(-\mu_c,\ -\mu_a,\ 1+\mu_x)}f_3^{(b,\,b)\,x=b}, \tag{5.12}$$

$$f_3^{(a,\,a)\,y=a} = \frac{\Gamma(-1-\mu_{xb},\, 2+\mu_{bc})}{\Gamma(-\mu_x,\, 1+\mu_c)}\, f_1^{(b,\,b)\,x=b} + \frac{\Gamma(-1-\mu_{xyb}, 1+\mu_{xb},\, 2+\mu_{bc})}{\Gamma(-\mu_y,\, 1+\mu_b,\, 1+\mu_c)}\, f_2^{(b,\,b)\,x=b}$$
$$+ \frac{\Gamma(-1-\mu_{ca},\, 2+\mu_{bc})}{\Gamma(-\mu_a,\, 1+\mu_b)}\, f_3^{(b,\,b)\,x=b}, \tag{5.13}$$

and

$$f_1^{(b,\,b)\,x=b} = \frac{\Gamma(-1-\mu_{axy},\, 2+\mu_{xb})}{\Gamma(-\mu_{ay},\, 1+\mu_b)}\, f_2^{(a,\,a)\,y=a} + \frac{\Gamma(-1-\mu_{cb},\, 2+\mu_{xb})}{\Gamma(-\mu_c,\, 1+\mu_y)}\, f_3^{(a,\,a)\,y=a}, \tag{5.14}$$

$$f_2^{(b,\,b)\,x=b} = \frac{\Gamma(-1-\mu_{ay},\, 2+\mu_{bxy})}{\Gamma(-\mu_a,\, 1+\mu_{bx})}\, f_1^{(a,\,a)\,y=a}$$
$$+ \frac{\Gamma(-\mu_{bx},\, -1-\mu_{axy},\, 1+\mu_{ay},\, 2+\mu_{bxy})}{\Gamma(2+\mu_{axyb},\, -1-\mu_{axyb},\, -\mu_x,\, 1+\mu_y)}\, f_2^{(a,\,a)\,y=a}$$
$$+ \frac{\Gamma(-\mu_{xb}, -1-\mu_{bc},\, 2+\mu_{bxy})}{\Gamma(-\mu_c,\, -\mu_b,\, 1+\mu_y)}\, f_3^{(a,\,a)\,y=a}, \tag{5.15}$$

$$f_3^{(b,\,b)\,x=b} = \frac{\Gamma(-1-\mu_{ay},\, 2+\mu_{ac})}{\Gamma(-\mu_y,\, 1+\mu_c)}\, f_1^{(a,\,a)\,y=a} + \frac{\Gamma(-1-\mu_{axy}, 1+\mu_{ay},\, 2+\mu_{ac})}{\Gamma(-\mu_x,\, 1+\mu_a,\, 1+\mu_c)}\, f_2^{(a,\,a)\,y=a}$$
$$+ \frac{\Gamma(-1-\mu_{bc},\, 2+\mu_{ac})}{\Gamma(-\mu_b,\, 1+\mu_a)}\, f_3^{(a,\,a)\,y=a} \tag{5.16}$$

where

$$\int_{(a,\,y)} u_{(a,\,y)}(t)\,dt = B(1+\mu_a,\, 1+\mu_y)\, f_1^{(a,\,a)\,y=a},$$
$$\int_{(\{a,\,y\},\,x)} u_{x-\delta}(t)\,dt = B(1+\mu_{ay},\, 1+\mu_x)\, f_2^{(a,\,a)\,y=a},$$
$$\int_{(b,\,c)} u_{(b,\,c)}(t)\,dt = B(1+\mu_b,\, 1+\mu_c)\, f_3^{(a,\,a)\,y=a},$$

and

$$\int_{(x,\,b)} u_{(x,\,b)}(t)\,dt = B(1+\mu_x,\, 1+\mu_b)\, f_1^{(b,\,b)\,x=b},$$
$$\int_{(y,\,\{x,\,b\})} u_{y+\delta}(t)\,dt = B(1+\mu_y,\, 1+\mu_{xb})\, f_2^{(b,\,b)\,x=b},$$
$$\int_{(c,\,a)} u_{(c,\,a)}(t)\,dt = B(1+\mu_c,\, 1+\mu_a)\, f_3^{(b,\,b)\,x=b}.$$

In case $a = 0,\ b = 1,\ c = \infty$, the equalities (5.11), (5.12) and (5.13) combined with (4.1), (4.2), (4.5), (4.6), (4.13) and (4.23) imply the connection formulas (3.4), (3.5) and (3.6).

In case $a = \infty,\ b = 0,\ c = 1$, the equalities (5.14), (5.15) and (5.16) combined with (4.1), (4.3), (4.4), (4.9), (4.19) and (4.38) imply the connection formulas (3.25), (3.26) and (3.27).

• When $c < a < x < y < b < c$ for $a,\ b,\ c,\ x,\ y \in P^1(\mathbb{R})$ with $\{\,a,\ b,\ c\,\} = \{\,0,\ 1,\ \infty\,\}$, and $\mu_{xy} \notin \mathbb{Z}$ in addition to (5.1), both of the sets

$$\left\{\,(x,\ y) \otimes u_{(x,\,y)}(t), \quad (a,\ \{x,\ y\}) \otimes u_{a+\delta}(t), \quad (b,\ c) \otimes u_{(b,\,c)}(t)\,\right\}$$

and

$$\left\{\,(x,\ y) \otimes u_{(x,\,y)}(t), \quad (\{x,\ y\},\ b) \otimes u_{b-\delta}(t), \quad (c,\ a) \otimes u_{(c,\,a)}(t)\,\right\}$$

give the bases of $H_1^{\mathrm{lf}}(T, \mathcal{L})$ and hence there exist the numbers $q_i,\ r_i$ such that

$$(a,\ \{x,\ y\}) = q_1\,(x,\ y) + q_2\,(\{x,\ y\},\ b) + q_3\,(c,\ a),$$

$$(b,\ c) = r_1\,(x,\ y) + r_2\,(\{x,\ y\},\ b) + r_3\,(c,\ a).$$

Since

$$(x,\ y)^2 = \frac{-\langle\, e_{xy}\,\rangle}{\langle\, e_x\,\rangle\langle\, e_y\,\rangle}, \quad (x,\ y)\cdot(\{x,\ y\},\ b) = 0, \quad (x,\ y)\cdot(c,\ a) = 0,$$

$$(\{x,\ y\},\ b)\cdot(x,\ y) = 0, \quad (\{x,\ y\},\ b)^2 = \frac{-\langle\, e_{xyb}\,\rangle}{\langle\, e_{xy}\,\rangle\langle\, e_b\,\rangle}, \quad (\{x,\ y\},\ b)\cdot(c,\ a) = 0,$$

$$(c,\ a)\cdot(x,\ y) = 0, \quad (c,\ a)\cdot(\{x,\ y\},\ b) = 0, \quad (c,\ a)^2 = \frac{-\langle\, e_{ca}\,\rangle}{\langle\, e_c\,\rangle\langle\, e_a\,\rangle},$$

and

$$(a,\ \{x,\ y\})\cdot(x,\ y) = 0, \quad (a,\ \{x,\ y\})\cdot(\{x,\ y\},\ b) = \frac{1}{\langle\, e_{xy}\,\rangle}, \quad (a,\ \{x,\ y\})\cdot(c,\ a) = \frac{1}{\langle\, e_a\,\rangle},$$

$$(b,\ c)\cdot(x,\ y) = 0, \quad (b,\ c)\cdot(\{x,\ y\},\ b) = \frac{1}{\langle\, e_b\,\rangle}, \quad (b,\ c)\cdot(c,\ a) = \frac{1}{\langle\, e_c\,\rangle},$$

we have

$$q_1 = 0, \quad q_2 = \frac{-\langle\, e_b\,\rangle}{\langle\, e_{xyb}\,\rangle}, \quad q_3 = \frac{-\langle\, e_c\,\rangle}{\langle\, e_{ac}\,\rangle},$$

$$r_1 = 0, \quad r_2 = \frac{-\langle\, e_{xy}\,\rangle}{\langle\, e_{xyb}\,\rangle}, \quad r_3 = \frac{-\langle\, e_a\,\rangle}{\langle\, e_{ca}\,\rangle},$$

and

$$(a,\ \{x,\ y\}) = \frac{-\langle\, e_b\,\rangle}{\langle\, e_{xyb}\,\rangle}\,(\{x,\ y\},\ b) + \frac{-\langle\, e_c\,\rangle}{\langle\, e_{ac}\,\rangle}\,(c,\ a), \tag{5.17}$$

$$(b,\ c) = \frac{-\langle\, e_{xy}\,\rangle}{\langle\, e_{xyb}\,\rangle}\,(\{x,\ y\},\ b) + \frac{-\langle\, e_a\,\rangle}{\langle\, e_{ca}\,\rangle}\,(c,\ a). \tag{5.18}$$

Similarly we have

$$(\{x,\ y\},\ b) = \frac{-\langle\, e_a \,\rangle}{\langle\, e_{axy} \,\rangle}\,(a,\ \{x,\ y\}) + \frac{-\langle\, e_c \,\rangle}{\langle\, e_{bc} \,\rangle}\,(b,\ c), \tag{5.19}$$

$$(c,\ a) = \frac{-\langle\, e_{xy} \,\rangle}{\langle\, e_{axy} \,\rangle}\,(a,\ \{x,\ y\}) + \frac{-\langle\, e_b \,\rangle}{\langle\, e_{bc} \,\rangle}\,(b,\ c). \tag{5.20}$$

On the other hand, we have

$$\int_{(x,\,y)} u_{(x,\,y)}(t)\,dt = B(1+\mu_x,\ 1+\mu_y)\, f_1^{(a,\,a)\,x=y},$$
$$\int_{(a,\,\{x,\,y\})} u_{a+\delta}(t)\,dt = B(1+\mu_a,\ 1+\mu_{xy})\, f_2^{(a,\,a)\,x=y},$$
$$\int_{(b,\,c)} u_{(b,\,c)}(t)\,dt = B(1+\mu_b,\ 1+\mu_c)\, f_3^{(a,\,a)\,x=y},$$

and

$$\int_{(x,\,y)} u_{(x,\,y)}(t)\,dt = B(1+\mu_x,\ 1+\mu_y)\, f_1^{(b,\,b)\,x=y},$$
$$\int_{(\{x,\,y\},\,b)} u_{b-\delta}(t)\,dt = B(1+\mu_{xy},\ 1+\mu_b)\, f_2^{(b,\,b)\,x=y},$$
$$\int_{(c,\,a)} u_{(c,\,a)}(t)\,dt = B(1+\mu_c,\ 1+\mu_a)\, f_3^{(b,\,b)\,x=y}.$$

Therefore, we obtain

$$f_1^{(a,\,a)\,x=y} = f_1^{(b,\,b)\,x=y}, \tag{5.21}$$

$$\begin{aligned} f_2^{(a,\,a)\,x=y} &= \frac{-\langle\, e_b \,\rangle}{\langle\, e_{xyb} \,\rangle}\frac{B(1+\mu_{xy},\ 1+\mu_b)}{B(1+\mu_a,\ 1+\mu_{xy})}\, f_2^{(b,\,b)\,x=y} + \frac{-\langle\, e_c \,\rangle}{\langle\, e_{ac} \,\rangle}\frac{B(1+\mu_c,\ 1+\mu_a)}{B(1+\mu_a,\ 1+\mu_{xy})}\, f_3^{(b,\,b)\,x=y} \\ &= \frac{\Gamma(-1-\mu_{xyb},\ 2+\mu_{axy})}{\Gamma(-\mu_b,\ 1+\mu_a)}\, f_2^{(b,\,b)\,x=y} + \frac{\Gamma(-1-\mu_{ac},\ 2+\mu_{axy})}{\Gamma(-\mu_c,\ 1+\mu_{xy})}\, f_3^{(b,\,b)\,x=y}, \end{aligned} \tag{5.22}$$

$$\begin{aligned} f_3^{(a,\,a)\,x=y} &= \frac{-\langle\, e_{xy} \,\rangle}{\langle\, e_{xyb} \,\rangle}\frac{B(1+\mu_{xy},\ 1+\mu_b)}{B(1+\mu_b,\ 1+\mu_c)}\, f_2^{(b,\,b)\,x=y} + \frac{-\langle\, e_a \,\rangle}{\langle\, e_{ac} \,\rangle}\frac{B(1+\mu_c,\ 1+\mu_a)}{B(1+\mu_b,\ 1+\mu_c)}\, f_3^{(b,\,b)\,x=y} \\ &= \frac{\Gamma(-1-\mu_{xyb},\ 2+\mu_{bc})}{\Gamma(-\mu_{xy},\ 1+\mu_c)}\, f_2^{(b,\,b)\,x=y} + \frac{\Gamma(-1-\mu_{ac},\ 2+\mu_{bc})}{\Gamma(-\mu_a,\ 1+\mu_b)}\, f_3^{(b,\,b)\,x=y} \end{aligned} \tag{5.23}$$

and

$$f_1^{(b,\,b)\,x=y} = f_1^{(a,\,a)\,x=y}, \tag{5.24}$$

$$\begin{aligned} f_2^{(b,\,b)\,x=y} &= \frac{-\langle\, e_a \,\rangle}{\langle\, e_{axy} \,\rangle} \frac{B(1+\mu_a,\, 1+\mu_{xy})}{B(1+\mu_{xy},\, 1+\mu_b)}\, f_2^{(a,\,a)\,x=y} + \frac{-\langle\, e_c \,\rangle}{\langle\, e_{bc} \,\rangle} \frac{B(1+\mu_b,\, 1+\mu_c)}{B(1+\mu_{xy},\, 1+\mu_b)}\, f_3^{(a,\,a)\,x=y} \\ &= \frac{\Gamma(-1-\mu_{axy},\, 2+\mu_{xyb})}{\Gamma(-\mu_a,\, 1+\mu_b)}\, f_2^{(a,\,a)\,x=y} + \frac{\Gamma(-1-\mu_{bc},\, 2+\mu_{xyb})}{\Gamma(-\mu_c,\, 1+\mu_{xy})}\, f_3^{(a,\,a)\,x=y}, \end{aligned} \tag{5.25}$$

$$\begin{aligned} f_3^{(b,\,b)\,x=y} &= \frac{-\langle\, e_{xy} \,\rangle}{\langle\, e_{axy} \,\rangle} \frac{B(1+\mu_a,\, 1+\mu_{xy})}{B(1+\mu_c,\, 1+\mu_a)}\, f_2^{(a,\,a)\,x=y} + \frac{-\langle\, e_b \,\rangle}{\langle\, e_{bc} \,\rangle} \frac{B(1+\mu_b,\, 1+\mu_c)}{B(1+\mu_c,\, 1+\mu_a)}\, f_3^{(a,\,a)\,x=y} \\ &= \frac{\Gamma(-1-\mu_{axy},\, 2+\mu_{ac})}{\Gamma(-\mu_{xy},\, 1+\mu_c)}\, f_2^{(a,\,a)\,x=y} + \frac{\Gamma(-1-\mu_{bc},\, 2+\mu_{ac})}{\Gamma(-\mu_b,\, 1+\mu_a)}\, f_3^{(a,\,a)\,x=y}. \end{aligned} \tag{5.26}$$

In case $a = 0,\ b = 1,\ c = \infty$, the equalities (5.21), (5.22) and (5.23) combined with (4.1), (4.2), (4.10), (4.15) and (4.21) imply the connection formulas (3.7), (3.8) and (3.9).

In case $a = \infty,\ b = 0,\ c = 1$, the equalities (5.24), (5.25) and (5.26) combined with (4.1), (4.3), (4.10), (4.14) and (4.27) imply the connection formulas (3.22), (3.23) and (3.24).

- When $b < c < a < x < y < b$ for $a,\ b,\ c,\ x,\ y \in P^1(\mathbb{R})$ with $\{\, a,\ b,\ c \,\} = \{\, 0,\ 1,\ \infty \,\}$, and $\mu_{ax} \notin \mathbb{Z}$ in addition to (5.1), both of the sets

$$\left\{\, (a,\ x) \otimes u_{(a,\,x)}(t), \quad (c,\ \{a,\ x\}) \otimes u_{c+\delta}(t), \quad (y,\ b) \otimes u_{(y,\,b)}(t) \,\right\}$$

and

$$\left\{\, (a,\ x) \otimes u_{(a,\,x)}(t), \quad (\{a,\ x\},\ y) \otimes u_{y-\delta}(t), \quad (b,\ c) \otimes u_{(b,\,c)}(t) \,\right\}$$

give the bases of $H_1^{\mathrm{lf}}(T, \mathcal{L})$ and their elements satisfy the equalities

$$\begin{aligned} (\{a,\ x\},\ y) &= \frac{-\langle\, e_c \,\rangle}{\langle\, e_{cax} \,\rangle}\, (c,\ \{a,\ x\}) + \frac{-\langle\, e_b \,\rangle}{\langle\, e_{yb} \,\rangle}\, (y,\ b), \\ (b,\ c) &= \frac{-\langle\, e_{ax} \,\rangle}{\langle\, e_{cax} \,\rangle}\, (c,\ \{a,\ x\}) + \frac{-\langle\, e_y \,\rangle}{\langle\, e_{yb} \,\rangle}\, (y,\ b), \end{aligned}$$

which follow from replacing $(c,\ a,\ x,\ y,\ b,\ c)$ of (5.17) and (5.18) by $(b,\ c,\ a,\ x,\ y,\ b)$ (at the same time $\mu_c,\ \mu_a,\ \ldots,\ \mu_c$ by $\mu_b,\ \mu_c,\ \ldots,\ \mu_b$).

On the other hand, we have

$$\int_{(a,\,x)} u_{(a,\,x)}(t)\,dt = B(1+\mu_a,\,1+\mu_x)\,f_1^{(a,\,a)\,x=a},$$

$$\int_{(\{a,\,x\},\,y)} u_{y-\delta}(t)\,dt = B(1+\mu_{ax},\,1+\mu_y)\,f_2^{(a,\,a)\,x=a},$$

$$\int_{(b,\,c)} u_{(b,\,c)}(t)\,dt = B(1+\mu_b,\,1+\mu_c)\,f_3^{(a,\,a)\,x=a},$$

and

$$\int_{(a,\,x)} u_{(a,\,x)}(t)\,dt = B(1+\mu_a,\,1+\mu_x)\,f_1^{(a,\,b)},$$

$$\int_{(c,\,\{a,\,x\})} u_{c+\delta}(t)\,dt = B(1+\mu_c,\,1+\mu_{ax})\,f_2^{(a,\,b)},$$

$$\int_{(y,\,b)} u_{(y,\,b)}(t)\,dt = B(1+\mu_y,\,1+\mu_b)\,f_3^{(a,\,b)}.$$

Therefore, we obtain

$$f_1^{(a,\,a)\,x=a} = f_1^{(a,\,b)}, \tag{5.27}$$

$$\begin{aligned} f_2^{(a,\,a)\,x=a} &= \frac{-\langle\, e_c\,\rangle}{\langle\, e_{cax}\,\rangle}\frac{B(1+\mu_{ax},\,1+\mu_c)}{B(1+\mu_{ax},\,1+\mu_y)}\,f_2^{(a,\,b)} + \frac{-\langle\, e_b\,\rangle}{\langle\, e_{yb}\,\rangle}\frac{B(1+\mu_y,\,1+\mu_b)}{B(1+\mu_{ax},\,1+\mu_y)}\,f_3^{(a,\,b)} \\ &= \frac{\Gamma(-1-\mu_{cax},\,2+\mu_{axy})}{\Gamma(-\mu_c,\,1+\mu_y)}\,f_2^{(a,\,b)} + \frac{\Gamma(-1-\mu_{yb},\,2+\mu_{axy})}{\Gamma(-\mu_b,\,1+\mu_{ax})}\,f_3^{(a,\,b)}, \end{aligned} \tag{5.28}$$

$$\begin{aligned} f_3^{(a,\,a)\,x=a} &= \frac{-\langle\, e_{ax}\,\rangle}{\langle\, e_{cax}\,\rangle}\frac{B(1+\mu_{ax},\,1+\mu_c)}{B(1+\mu_b,\,1+\mu_c)}\,f_2^{(a,\,b)} + \frac{-\langle\, e_y\,\rangle}{\langle\, e_{yb}\,\rangle}\frac{B(1+\mu_y,\,1+\mu_b)}{B(1+\mu_b,\,1+\mu_c)}\,f_3^{(a,\,b)} \\ &= \frac{\Gamma(-1-\mu_{cax},\,2+\mu_{bc})}{\Gamma(-\mu_{ax},\,1+\mu_b)}\,f_2^{(a,\,b)} + \frac{\Gamma(-1-\mu_{yb},\,2+\mu_{bc})}{\Gamma(-\mu_y,\,1+\mu_c)}\,f_3^{(a,\,b)}. \end{aligned} \tag{5.29}$$

In case $a=0,\ b=1,\ c=\infty$, the equalities (5.27), (5.28) and (5.29) combined with (4.1), (4.4), (4.6), (4.12) and (4.28) imply the connection formulas (3.10), (3.11) and (3.12).

- When $a<x<y<b<c<a$ for $a,\ b,\ c,\ x,\ y\in P^1(\mathbb{R})$ with $\{\,a,\ b,\ c\,\}=\{\,0,\ 1,\ \infty\,\}$, and $\mu_{yb}\notin\mathbb{Z}$ in addition to (5.1), both of the sets

$$\big\{\,(y,\,b)\otimes u_{(y,\,b)}(t),\quad (x,\,\{y,\,b\})\otimes u_{x+\delta}(t),\quad (c,\,a)\otimes u_{(c,\,a)}(t)\,\big\}$$

and

$$\big\{\,(a,\,x)\otimes u_{(a,\,x)}(t),\quad (\{y,\,b\},\,c)\otimes u_{c-\delta}(t),\quad (y,\,b)\otimes u_{(y,\,b)}(t)\,\big\}$$

give the bases of $H_1^{\mathrm{lf}}(T,\mathcal{L})$ and their elements satisfy the equalities

$$(x,\,\{y,\,b\}) = \frac{-\langle\, e_a\,\rangle}{\langle\, e_{ax}\,\rangle}(a,\,x) + \frac{-\langle\, e_c\,\rangle}{\langle\, e_{ybc}\,\rangle}(\{y,\,b\},\,c),$$

$$(c,\,a) = \frac{-\langle\, e_x\,\rangle}{\langle\, e_{ax}\,\rangle}(a,\,x) + \frac{-\langle\, e_{yb}\,\rangle}{\langle\, e_{ybc}\,\rangle}(\{y,\,b\},\,c),$$

which follow from replacing $(c,\,a,\,x,\,y,\,b,\,c)$ of (5.19) and (5.20) by $(a,\,x,\,y,\,b,\,c,\,a)$ (at the same time $\mu_c,\,\mu_a,\,\ldots,\,\mu_c$ by $\mu_a,\,\mu_x,\,\ldots,\,\mu_a$).

Since

$$\int_{(y,\,b)} u_{(y,\,b)}(t)\,dt = B(1+\mu_y,\,1+\mu_b)\,f_1^{(b,\,b)\,y=b},$$

$$\int_{(x,\,\{y,\,b\})} u_{x+\delta}(t)\,dt = B(1+\mu_x,\,1+\mu_{yb})\,f_2^{(b,\,b)\,y=b},$$

$$\int_{(c,\,a)} u_{(c,\,a)}(t)\,dt = B(1+\mu_c,\,1+\mu_a)\,f_3^{(b,\,b)\,y=b},$$

and

$$\int_{(a,\,x)} u_{(a,\,x)}(t)\,dt = B(1+\mu_a,\,1+\mu_x)\,f_1^{(a,\,b)},$$

$$\int_{(\{y,\,b\},\,c)} u_{c-\delta}(t)\,dt = B(1+\mu_{yb},\,1+\mu_c)\,f_2^{(a,\,b)},$$

$$\int_{(y,\,b)} u_{(y,\,b)}(t)\,dt = B(1+\mu_y,\,1+\mu_b)\,f_3^{(a,\,b)},$$

we obtain

$$f_1^{(b,\,b)\,y=b} = f_3^{(a,\,b)}, \tag{5.30}$$

$$\begin{aligned} f_2^{(b,\,b)\,y=b} &= \frac{-\langle\, e_a\,\rangle}{\langle\, e_{ax}\,\rangle}\frac{B(1+\mu_a,\,1+\mu_x)}{B(1+\mu_x,\,1+\mu_{yb})}\,f_1^{(a,\,b)} + \frac{-\langle\, e_c\,\rangle}{\langle\, e_{ybc}\,\rangle}\frac{B(1+\mu_{yb},\,1+\mu_c)}{B(1+\mu_x,\,1+\mu_{yb})}\,f_2^{(a,\,b)} \\ &= \frac{\Gamma(-1-\mu_{ax},\,2+\mu_{xyb})}{\Gamma(-\mu_a,\,1+\mu_{yb})}\,f_1^{(a,\,b)} + \frac{\Gamma(-1-\mu_{ybc},\,2+\mu_{xyb})}{\Gamma(-\mu_c,\,1+\mu_x)}\,f_2^{(a,\,b)}, \end{aligned} \tag{5.31}$$

$$\begin{aligned} f_3^{(b,\,b)\,y=b} &= \frac{-\langle\, e_x\,\rangle}{\langle\, e_{ax}\,\rangle}\frac{B(1+\mu_a,\,1+\mu_x)}{B(1+\mu_c,\,1+\mu_a)}\,f_1^{(a,\,b)} + \frac{-\langle\, e_{yb}\,\rangle}{\langle\, e_{ybc}\,\rangle}\frac{B(1+\mu_{yb},\,1+\mu_c)}{B(1+\mu_c,\,1+\mu_a)}\,f_2^{(a,\,b)} \\ &= \frac{\Gamma(-1-\mu_{ax},\,2+\mu_{ca})}{\Gamma(-\mu_x,\,1+\mu_c)}\,f_1^{(a,\,b)} + \frac{\Gamma(-1-\mu_{ybc},\,2+\mu_{ca})}{\Gamma(-\mu_{yb},\,1+\mu_a)}\,f_2^{(a,\,b)}. \end{aligned} \tag{5.32}$$

In case $a = 0,\ b = 1,\ c = \infty$, the equalities (5.30), (5.31) and (5.32) combined with (4.1), (4.5), (4.8), (4.17) and (4.20) imply the connection formulas (3.16), (3.17) and (3.18).

• When $c < a < x < y < b < c$ for $a,\ b,\ c,\ x,\ y \in P^1(\mathbb{R})$ with $\{\,a,\ b,\ c\,\} = \{\,0,\ 1,\ \infty\,\}$, and $\mu_{ax},\ \mu_{xy} \notin \mathbb{Z}$ in addition to (5.1), both of the sets

$$\big\{\,(a,\ x) \otimes u_{(a,\,x)}(t),\quad (\{a,\ x\},\ y) \otimes u_{y-\delta}(t),\quad (b,\ c) \otimes u_{(b,\,c)}(t)\,\big\}$$

and

$$\big\{\,(x,\ y) \otimes u_{(x,\,y)}(t),\quad (a,\ \{x,\ y\}) \otimes u_{a+\delta}(t),\quad (b,\ c) \otimes u_{(b,\,c)}(t)\,\big\}$$

give the bases of $H_1^{\mathrm{lf}}(T, \mathcal{L})$ and their elements satisfy the equalities

$$(a,\ x) = \frac{-\langle\, e_y \,\rangle}{\langle\, e_{xy} \,\rangle}\,(x,\ y) + (a,\ \{x,\ y\}),$$

$$(\{a,\ x\},\ y) = \frac{\langle\, e_{axy} \,\rangle\langle\, e_x \,\rangle}{\langle\, e_{ax} \,\rangle\langle\, e_{xy} \,\rangle}\,(x,\ y) + \frac{\langle\, e_a \,\rangle}{\langle\, e_{ax} \,\rangle}(a,\ \{x,\ y\}).$$

Since

$$\int_{(a,\,x)} u_{(a,\,x)}(t)\,dt = B(1+\mu_a,\ 1+\mu_x)\, f_1^{(a,\,a)\,x=a},$$

$$\int_{(\{a,\,x\},\,y)} u_{y-\delta}(t)\,dt = B(1+\mu_{ax},\ 1+\mu_y)\, f_2^{(a,\,a)\,x=a},$$

$$\int_{(b,\,c)} u_{(b,\,c)}(t)\,dt = B(1+\mu_b,\ 1+\mu_c)\, f_3^{(a,\,a)\,x=a},$$

and

$$\int_{(x,\,y)} u_{(x,\,y)}(t)\,dt = B(1+\mu_x,\ 1+\mu_y)\, f_1^{(a,\,a)\,x=y},$$

$$\int_{(a,\,\{x,\,y\})} u_{a+\delta}(t)\,dt = B(1+\mu_a,\ 1+\mu_{xy})\, f_2^{(a,\,a)\,x=y},$$

$$\int_{(b,\,c)} u_{(b,\,c)}(t)\,dt = B(1+\mu_b,\ 1+\mu_c)\, f_3^{(a,\,a)\,x=y},$$

we obtain

$$
\begin{aligned}
f_1^{(a,\,a)\,x=a} &= \frac{-\langle\, e_y \,\rangle}{\langle\, e_{xy} \,\rangle}\frac{B(1+\mu_x,\,1+\mu_y)}{B(1+\mu_a,\,1+\mu_x)}\, f_1^{((a,\,a)\,x=y} + \frac{B(1+\mu_a,\,1+\mu_{xy})}{B(1+\mu_a,\,1+\mu_x)}\, f_2^{(a,\,a)\,x=y} \\
&= \frac{\Gamma(-1-\mu_{xy},\,2+\mu_{ax})}{\Gamma(-\mu_y,\,1+\mu_a)}\, f_1^{(a,\,a)\,x=y} + \frac{\Gamma(1+\mu_{xy},\,2+\mu_{ax})}{\Gamma(2+\mu_{axy},\,1+\mu_x)}\, f_2^{(a,\,a)\,x=y}, \qquad (5.33)\\
f_2^{(a,\,a)\,x=a} &= \frac{\langle\, e_{axy} \,\rangle\langle\, e_x \,\rangle}{\langle\, e_{ax} \,\rangle\langle\, e_{xy} \,\rangle}\frac{B(1+\mu_x,\,1+\mu_y)}{B(1+\mu_{ax},\,1+\mu_y)}\, f_1^{(a,\,a)\,x=y} \\
&\quad + \frac{\langle\, e_a \,\rangle}{\langle\, e_{ax} \,\rangle}\frac{\Gamma(1+\mu_a,\,1+\mu_{xy})}{B(1+\mu_{ax},\,1+\mu_y)}\, f_2^{(a,\,a)\,x=y} \\
&= \frac{\Gamma(-\mu_{ax},\,-1-\mu_{xy})}{\Gamma(-1-\mu_{axy},\,-\mu_x)}\, f_1^{(a,\,a)\,x=y} + \frac{\Gamma(-\mu_{ax},\,1+\mu_{xy})}{\Gamma(-\mu_a,\,1+\mu_y)}\, f_2^{(a,\,a)\,x=y}, \qquad (5.34)\\
f_3^{(a,\,a)\,x=a} &= f_3^{(a,\,a)\,x=y}. \qquad (5.35)
\end{aligned}
$$

In case $a=0,\ b=1,\ c=\infty$, the equalities (5.33), (5.34) and (5.35) combined with (4.1), (4.4), (4.10), (4.12) and (4.21) imply the connection formulas (3.28), (3.29) and (3.30).

- When $a<x<b<y<c<a$ for $a,\ b,\ c,\ x,\ y \in P^1(\mathbb{R})$ with $\{\,a,\ b,\ c\,\} = \{\,0,\ 1,\ \infty\,\}$, and $\mu_{xb},\ \mu_{yc} \notin \mathbb{Z}$ in addition to (5.1), both of the sets

$$
\big\{\,(x,\ b)\otimes u_{(x,\,b)}(t),\quad (\{x,\ b\},\ y)\otimes u_{y-\delta}(t),\quad (c,\ a)\otimes u_{(c,\,a)}(t)\,\big\}
$$

and

$$
\big\{\,(y,\ c)\otimes u_{(y,\,c)}(t),\quad (b,\ \{y,\ c\})\otimes u_{b+\delta}(t),\quad (a,\ x)\otimes u_{(a,\,x)}(t)\,\big\}
$$

give the bases of $H_1^{\mathrm{lf}}(T,\mathcal{L})$ and their elements satisfy

$$
\begin{aligned}
(x,\ b) &= \frac{-\langle\, e_{yc} \,\rangle}{\langle\, e_{byc} \,\rangle}(b,\ \{y,\ c\}) + \frac{-\langle\, e_a \,\rangle}{\langle\, e_{ax} \,\rangle}(a,\ x),\\
(\{x,\ b\},\ y) &= \frac{-\langle\, e_c \,\rangle}{\langle\, e_{yc} \,\rangle}(y,\ c) + \frac{\langle\, e_{xbyc} \,\rangle\langle\, e_b \,\rangle}{\langle\, e_{xb} \,\rangle\langle\, e_{byc} \,\rangle}(b,\ \{y,\ c\}) + \frac{-\langle\, e_a \,\rangle\langle\, e_x \,\rangle}{\langle\, e_{ax} \,\rangle\langle\, e_{xb} \,\rangle}(a,\ x),\\
(c,\ a) &= \frac{-\langle\, e_y \,\rangle}{\langle\, e_{yc} \,\rangle}(y,\ c) + \frac{-\langle\, e_b \,\rangle}{\langle\, e_{byc} \,\rangle}(b,\ \{y,\ c\}) + \frac{-\langle\, e_x \,\rangle}{\langle\, e_{ax} \,\rangle}(a,\ x),
\end{aligned}
$$

which follow from replacing $(c,\ a,\ x,\ y,\ b,\ c)$ of (5.2), (5.3) and (5.4) by $(a,\ x,\ b,\ y,\ c,\ a)$ (at the same time $\mu_c,\ \mu_a,\ \ldots,\ \mu_c$ by $\mu_a,\ \mu_x,\ \ldots,\ \mu_a$).

Since

$$\int_{(x,\,b)} u_{(x,\,b)}(t)\,dt = B(1+\mu_x,\,1+\mu_b)\,f_1^{(b,\,b)\,x=b},$$

$$\int_{(\{x,\,b\},\,y)} u_{y-\delta}(t)\,dt = B(1+\mu_{xb},\,1+\mu_y)\,f_2^{(b,\,b)\,x=b},$$

$$\int_{(c,\,a)} u_{(c,\,a)}(t)\,dt = B(1+\mu_c,\,1+\mu_a)\,f_3^{(b,\,b)\,x=b}$$

and

$$\int_{(y,\,c)} u_{(y,\,c)}(t)\,dt = B(1+\mu_y,\,1+\mu_c)\,f_1^{(a,\,c)},$$

$$\int_{(b,\,\{y,\,c\})} u_{b+\delta}(t)\,dt = B(1+\mu_b,\,1+\mu_{yc})\,f_2^{(a,\,c)},$$

$$\int_{(a,\,x)} u_{(a,\,x)}(t)\,dt = B(1+\mu_a,\,1+\mu_x)\,f_3^{(a,\,c)},$$

we obtain

$$\begin{aligned}
f_1^{(b,\,b)\,x=b} &= \frac{-\langle\, e_{yc}\,\rangle}{\langle\, e_{byc}\,\rangle}\,\frac{B(1+\mu_b,\,1+\mu_{yc})}{B(1+\mu_x,\,1+\mu_b)}\,f_2^{(a,\,c)} + \frac{-\langle\, e_a\,\rangle}{\langle\, e_{ax}\,\rangle}\,\frac{B(1+\mu_a,\,1+\mu_x)}{B(1+\mu_x,\,1+\mu_b)}\,f_3^{(a,\,c)}\\
&= \frac{\Gamma(-1-\mu_{byc},\,2+\mu_{xb})}{\Gamma(-\mu_{yc},\,1+\mu_x)}\,f_2^{(a,\,c)} + \frac{\Gamma(-1-\mu_{ax},\,2+\mu_{xb})}{\Gamma(-\mu_a,\,1+\mu_b)}\,f_3^{(a,\,c)},
\end{aligned} \tag{5.36}$$

$$\begin{aligned}
f_2^{(b,\,b)\,x=b} &= \frac{-\langle\, e_c\,\rangle}{\langle\, e_{yc}\,\rangle}\,\frac{B(1+\mu_y,\,1+\mu_c)}{B(1+\mu_{xb},\,1+\mu_y)}\,f_1^{(a,\,c)} + \frac{\langle\, e_{xbyc}\,\rangle\langle\, e_b\,\rangle}{\langle\, e_{xb}\,\rangle\langle\, e_{byc}\,\rangle}\,\frac{B(1+\mu_b,\,1+\mu_{yc})}{B(1+\mu_{xb},\,1+\mu_y)}\,f_2^{(a,\,c)}\\
&\quad + \frac{-\langle\, e_a\,\rangle\langle\, e_x\,\rangle}{\langle\, e_{ax}\,\rangle\langle\, e_{xb}\,\rangle}\,\frac{\Gamma(1+\mu_a,\,1+\mu_x)}{B(1+\mu_{xb},\,1+\mu_y)}\,f_3^{(a,\,c)}\\
&= \frac{\Gamma(-1-\mu_{yc},\,2+\mu_{xby})}{\Gamma(-\mu_c,\,1+\mu_{xb})}\,f_1^{(a,\,c)} + \frac{\Gamma(-\mu_{xb},\,-1-\mu_{byc},\,1+\mu_{yc},\,2+\mu_{xby})}{\Gamma(2+\mu_{xbyc},\,-1-\mu_{xbyc},\,-\mu_b,\,1+\mu_y)}\,f_2^{(a,\,c)}\\
&\quad + \frac{\Gamma(-1-\mu_{ax},\,-\mu_{xb},\,2+\mu_{xby})}{\Gamma(-\mu_a,\,-\mu_x,\,1+\mu_y)}\,f_3^{(a,\,c)},
\end{aligned} \tag{5.37}$$

$$\begin{aligned}
f_3^{(b,\,b)\,x=b} &= \frac{-\langle\, e_y\,\rangle}{\langle\, e_{yc}\,\rangle}\,\frac{B(1+\mu_y,\,1+\mu_c)}{B(1+\mu_c,\,1+\mu_a)}\,f_1^{(a,\,c)} + \frac{-\langle\, e_b\,\rangle}{\langle\, e_{byc}\,\rangle}\,\frac{B(1+\mu_b,\,1+\mu_{yc})}{B(1+\mu_c,\,1+\mu_a)}\,f_2^{(a,\,c)}\\
&\quad + \frac{-\langle\, e_x\,\rangle}{\langle\, e_{ax}\,\rangle}\,\frac{B(1+\mu_a,\,1+\mu_x)}{B(1+\mu_c,\,1+\mu_a)}\,f_3^{(a,\,c)}\\
&= \frac{\Gamma(-1-\mu_{yc},\,2+\mu_{ca})}{\Gamma(-\mu_y,\,1+\mu_a)}\,f_1^{(a,\,c)} + \frac{\Gamma(-1-\mu_{byc},\,1+\mu_{yc},\,2+\mu_{ca})}{\Gamma(-\mu_b,\,1+\mu_c,\,1+\mu_a)}\,f_2^{(a,\,c)}\\
&\quad + \frac{\Gamma(-1-\mu_{ax},\,2+\mu_{ca})}{\Gamma(-\mu_x,\,1+\mu_c)}\,f_3^{(a,\,c)}.
\end{aligned} \tag{5.38}$$

In case $a = \infty,\ b = 0,\ c = 1$, the equalities (5.36), (5.37) and (5.38) combined with (4.1), (4.4), (4.7), (4.8), (4.12) and (4.25) imply the connection formulas (3.13), (3.14) and (3.15).

Appendix

The transformation formulas of the functions constituting the fundamental sets of solutions in Section 2 are listed. The functions w_j in the former part (the list for f_1^{**} and f_3^{**}) correspond to $z_j = z_{j+10m}$ $(1 \le j \le 10,\ 1 \le m \le 5)$ in the list from page 62 to page 64 in [AKdF], while some misprints are corrected. In this Appendix, the symbols $\varepsilon, \varepsilon_j$ denote the signature $\pm$ to load the factors like $(\varepsilon x)^*$ and $(\varepsilon(1-x))^*$ standardly on an appropriate real point.

- $f_3^{(0,0)}$:

$$\begin{aligned}
w_1 &= F_1(\alpha,\, \beta,\, \beta',\, \gamma\,;\, x,\, y\,)\\
&= (1-x)^{-\beta}(1-y)^{-\beta'} F_1\left(\gamma-\alpha,\, \beta,\, \beta',\, \gamma\,;\, \frac{x}{x-1},\, \frac{y}{y-1}\right)\\
&= (1-x)^{-\alpha} F_1(\alpha, \gamma-\beta-\beta', \beta', \gamma\,;\, \frac{x}{x-1}, \frac{x-y}{x-1})\\
&= (1-y)^{-\alpha} F_1\left(\alpha,\, \beta,\, \gamma-\beta-\beta',\, \gamma\,;\, \frac{y-x}{y-1},\, \frac{y}{y-1}\right)\\
&= (1-x)^{\gamma-\alpha-\beta}(1-y)^{-\beta'} F_1\left(\gamma-\alpha,\, \gamma-\beta-\beta',\, \beta',\, \gamma\,;\, x,\, \frac{y-x}{y-1}\right)\\
&= (1-x)^{-\beta}(1-y)^{\gamma-\alpha-\beta'} F_1\left(\gamma-\alpha,\, \beta,\, \gamma-\beta-\beta',\, \gamma\,;\, \frac{x-y}{x-1},\, y\right).
\end{aligned}$$

- $f_3^{(1,1)}$:

$$\begin{aligned}
w_2 &= F_1(\alpha,\, \beta,\, \beta',\, 1+\alpha+\beta+\beta'-\gamma\,;\, 1-x,\, 1-y\,)\\
&= x^{-\beta} y^{-\beta'} F_1\left(1+\beta+\beta'-\gamma,\, \beta,\, \beta',\, 1+\alpha+\beta+\beta'-\gamma\,;\, \frac{x-1}{x},\, \frac{y-1}{y}\right)\\
&= x^{-\alpha} F_1\left(\alpha,\, 1+\alpha-\gamma,\, \beta',\, 1+\alpha+\beta+\beta'-\gamma\,;\, \frac{x-1}{x},\, \frac{x-y}{x}\right)\\
&= y^{-\alpha} F_1\left(\alpha,\, \beta,\, 1+\alpha-\gamma,\, 1+\alpha+\beta+\beta'-\gamma\,;\, \frac{y-x}{y},\, \frac{y-1}{y}\right)\\
&= x^{1+\beta'-\gamma} y^{-\beta'}\\
&\times F_1\left(1+\beta+\beta'-\gamma,\, 1+\alpha-\gamma,\, \beta',\, 1+\alpha+\beta+\beta'-\gamma\,;\, 1-x,\, \frac{y-x}{y}\right)\\
&= x^{-\beta} y^{1+\beta-\gamma}\\
&\times F_1\left(1+\beta+\beta'-\gamma,\, \beta,\, 1+\alpha-\gamma,\, 1+\alpha+\beta+\beta'-\gamma\,;\, \frac{x-y}{x},\, 1-y\right).
\end{aligned}$$

- $f_3^{(\infty,\infty)}$:

$$
\begin{aligned}
w_3 &= (\varepsilon_1 x)^{-\beta}(\varepsilon_2 y)^{-\beta'}\, F_1\left(1+\beta+\beta'-\gamma,\, \beta,\, \beta',\, 1+\beta+\beta'-\alpha\,;\, \frac{1}{x},\, \frac{1}{y}\right) \\
&= (\varepsilon_1 x)^{-\beta}(\varepsilon_2 y)^{-\beta'}\left(1-\frac{1}{x}\right)^{-\beta}\left(1-\frac{1}{y}\right)^{-\beta'} \\
&\times F_1\left(\gamma-\alpha,\, \beta,\, \beta',\, 1+\beta+\beta'-\alpha\,;\, \frac{1}{1-x},\, \frac{1}{1-y}\right) \\
&= (\varepsilon_1 x)^{-\beta}(\varepsilon_2 y)^{-\beta'}\left(1-\frac{1}{x}\right)^{\gamma-\beta-\beta'-1} \\
&\times F_1\left(1+\beta+\beta'-\gamma,\, 1-\alpha,\, \beta',\, 1+\beta+\beta'-\alpha\,;\, \frac{1}{1-x},\, \frac{y-x}{y(1-x)}\right) \\
&= (\varepsilon_1 x)^{-\beta}(\varepsilon_2 y)^{-\beta'}\left(1-\frac{1}{y}\right)^{\gamma-\beta-\beta'-1} \\
&\times F_1\left(1+\beta+\beta'-\gamma,\, \beta,\, 1-\alpha,\, 1+\beta+\beta'-\alpha\,;\, \frac{x-y}{x(1-y)},\, \frac{1}{1-y}\right) \\
&= (\varepsilon_1 x)^{-\beta}(\varepsilon_2 y)^{-\beta'}\left(1-\frac{1}{x}\right)^{\gamma-\alpha-\beta}\left(1-\frac{1}{y}\right)^{-\beta'} \\
&\times F_1\left(\gamma-\alpha,\, 1-\alpha,\, \beta',\, 1+\beta+\beta'-\alpha\,;\, \frac{1}{x},\, \frac{x-y}{x(1-y)}\right) \\
&= (\varepsilon_1 x)^{-\beta}(\varepsilon_2 y)^{-\beta'}\left(1-\frac{1}{x}\right)^{-\beta}\left(1-\frac{1}{y}\right)^{\gamma-\alpha-\beta'} \\
&\times F_1\left(\gamma-\alpha,\, \beta,\, 1-\alpha,\, 1+\beta+\beta'-\alpha\,;\, \frac{y-x}{y(1-x)},\, \frac{1}{y}\right).
\end{aligned}
$$

- $f_1^{(0,0)\,x=0}$, $\quad f_1^{(0,1)}$, $\quad f_3^{(0,\infty)}$:

$$
\begin{aligned}
w_4 &= (\varepsilon_1 x)^{1+\beta'-\gamma}(\varepsilon_2 y)^{-\beta'} \\
&\times F_1\left(1+\beta+\beta'-\gamma,\, 1+\alpha-\gamma,\, \beta',\, 2+\beta'-\gamma\,;\, x,\, \frac{x}{y}\right) \\
&= (\varepsilon_1 x)^{1+\beta'-\gamma}(\varepsilon_2(y-x))^{-\beta'}(1-x)^{\gamma-\alpha-1} \\
&\times F_1\left(1-\beta,\, 1+\alpha-\gamma,\, \beta',\, 2+\beta'-\gamma\,;\, \frac{x}{x-1},\, \frac{x}{x-y}\right) \\
&= (\varepsilon_1 x)^{1+\beta'-\gamma}(\varepsilon_2(y-x))^{-\beta'}(1-x)^{\gamma-\alpha-\beta} \\
&\times F_1\left(1-\beta,\, 1-\alpha,\, \beta',\, 2+\beta'-\gamma\,;\, x,\, \frac{x(1-y)}{x-y}\right) \\
&= (\varepsilon_1 x)^{1+\beta'-\gamma}(\varepsilon_2 y)^{\beta-1}(\varepsilon_2(y-x))^{1-\beta-\beta'}(1-x)^{\gamma-\alpha-1}
\end{aligned}
$$

$$\times F_1\left(1-\beta,\, 1+\alpha-\gamma,\, 1-\alpha,\, 2+\beta'-\gamma\,;\, \frac{x(1-y)}{y(1-x)},\, \frac{x}{y}\right)$$
$$= (\varepsilon_1 x)^{1+\beta'-\gamma}(\varepsilon_2 y)^{-\beta'}(1-x)^{\gamma-\beta-\beta'-1}$$
$$\times F_1\left(1+\beta+\beta'-\gamma,\, 1-\alpha,\, \beta',\, 2+\beta'-\gamma\,;\, \frac{x}{x-1},\, \frac{x(1-y)}{y(1-x)}\right)$$
$$= (\varepsilon_1 x)^{1+\beta'-\gamma}(\varepsilon_2 y)^{1+\beta-\gamma}(\varepsilon_2(y-x))^{\gamma-\beta-\beta'-1}$$
$$\times F_1\left(1+\beta+\beta'-\gamma,\, 1+\alpha-\gamma,\, 1-\alpha,\, 2+\beta'-\gamma\,;\, \frac{x(1-y)}{x-y},\, \frac{x}{x-y}\right).$$

- $f_1^{(0,0)\,y=0},\quad f_1^{(1,0)},\quad f_3^{(\infty,0)}$:

$$w_5 = (\varepsilon_1 y)^{1+\beta-\gamma}(\varepsilon_2 x)^{-\beta}$$
$$F_1\left(1+\beta+\beta'-\gamma,\, \beta,\, 1+\alpha-\gamma,\, 2+\beta-\gamma\,;\, \frac{y}{x},\, y\right)$$
$$= (\varepsilon_1 y)^{1+\beta-\gamma}(\varepsilon_2(x-y))^{-\beta}(1-y)^{\gamma-\alpha-1}$$
$$\times F_1\left(1-\beta',\, \beta,\, 1+\alpha-\gamma,\, 2+\beta-\gamma\,;\, \frac{y}{y-x},\, \frac{y}{y-1}\right)$$
$$= (\varepsilon_1 y)^{1+\beta-\gamma}(\varepsilon_2(x-y))^{-\beta}(1-y)^{\gamma-\alpha-\beta'}$$
$$\times F_1\left(1-\beta',\, \beta,\, 1-\alpha,\, 2+\beta-\gamma\,;\, \frac{y(1-x)}{y-x},\, y\right)$$
$$= (\varepsilon_1 y)^{1+\beta-\gamma}(\varepsilon_2 x)^{\beta'-1}(\varepsilon_2(x-y))^{1-\beta-\beta'}(1-y)^{\gamma-\alpha-1}$$
$$\times F_1\left(1-\beta',\, 1-\alpha,\, 1+\alpha-\gamma,\, 2+\beta-\gamma\,;\, \frac{y}{x},\, \frac{y(1-x)}{x(1-y)}\right)$$
$$= (\varepsilon_1 y)^{1+\beta-\gamma}(\varepsilon_2 x)^{-\beta}(1-y)^{\gamma-\beta-\beta'-1}$$
$$\times F_1\left(1+\beta+\beta'-\gamma,\, \beta,\, 1-\alpha,\, 2+\beta-\gamma\,;\, \frac{y(1-x)}{x(1-y)},\, \frac{y}{y-1}\right)$$
$$= (\varepsilon_1 y)^{1+\beta-\gamma}(\varepsilon_2 x)^{1+\beta'-\gamma}(\varepsilon_2(x-y))^{\gamma-\beta-\beta'-1}$$
$$\times F_1\left(1+\beta+\beta'-\gamma,\, 1-\alpha,\, 1+\alpha-\gamma,\, 2+\beta-\gamma\,;\, \frac{y}{y-x},\, \frac{y(1-x)}{y-x}\right).$$

- $f_1^{(1,1)\,x=1},\quad f_1^{(1,\infty)},\quad f_3^{(1,0)}$:

$$w_6 = (\varepsilon_1(1-x))^{\gamma-\alpha-\beta}(\varepsilon_2(1-y))^{-\beta'}$$
$$\times F_1\left(\gamma-\alpha,\, \gamma-\beta-\beta',\, \beta',\, 1+\gamma-\alpha-\beta\,;\, 1-x,\, \frac{1-x}{1-y}\right)$$
$$= (\varepsilon_1(1-x))^{\gamma-\alpha-\beta}(\varepsilon_2(x-y))^{-\beta'}x^{\beta+\beta'-\gamma}$$
$$\times F_1\left(1-\beta,\, \gamma-\beta-\beta',\, \beta',\, 1+\gamma-\alpha-\beta\,;\, \frac{x-1}{x},\, \frac{x-1}{x-y}\right)$$

$$
\begin{aligned}
&= (\varepsilon_1(1-x))^{\gamma-\alpha-\beta}(\varepsilon_2(x-y))^{-\beta'}x^{1+\beta'-\gamma}\\
&\times F_1\left(1-\beta,\, 1-\alpha,\, \beta',\, 1+\gamma-\alpha-\beta\,;\, 1-x,\, \frac{y(1-x)}{y-x}\right)\\
&= (\varepsilon_1(1-x))^{\gamma-\alpha-\beta}(\varepsilon_2(1-y))^{\beta-1}(\varepsilon_2(x-y))^{1-\beta-\beta'}x^{\beta+\beta'-\gamma}\\
&\times F_1\left(1-\beta,\, \gamma-\beta-\beta',\, 1-\alpha,\, 1+\gamma-\alpha-\beta\,;\, \frac{y(1-x)}{x(1-y)},\, \frac{1-x}{1-y}\right)\\
&= (\varepsilon_1(1-x))^{\gamma-\alpha-\beta}(\varepsilon_2(1-y))^{-\beta'}x^{\alpha-\gamma}\\
&\times F_1\left(\gamma-\alpha,\, 1-\alpha,\, \beta',\, 1+\gamma-\alpha-\beta\,;\, \frac{x-1}{x},\, \frac{y(1-x)}{x(1-y)}\right)\\
&= (\varepsilon_1(1-x))^{\gamma-\alpha-\beta}(\varepsilon_2(1-y))^{\gamma-\alpha-\beta'}(\varepsilon_2(x-y))^{\alpha-\gamma}\\
&\times F_1\left(\gamma-\alpha,\, \gamma-\beta-\beta',\, 1-\alpha,\, 1+\gamma-\alpha-\beta\,;\, \frac{y(1-x)}{y-x},\, \frac{x-1}{x-y}\right).
\end{aligned}
$$

- $f_1^{(1,1)\,y=1}, \quad f_1^{(\infty,1)}, \quad f_3^{(0,1)}$:

$$
\begin{aligned}
w_7 &= (\varepsilon_1(1-y))^{\gamma-\alpha-\beta'}(\varepsilon_2(1-x))^{-\beta}\\
&\times F_1\left(\gamma-\alpha,\, \beta,\, \gamma-\beta-\beta',\, 1+\gamma-\alpha-\beta'\,;\, \frac{1-y}{1-x},\, 1-y\right)\\
&= (\varepsilon_1(1-y))^{\gamma-\alpha-\beta'}(\varepsilon_2(y-x))^{-\beta}y^{\beta+\beta'-\gamma}\\
&\times F_1\left(1-\beta',\, \beta,\, \gamma-\beta-\beta',\, 1+\gamma-\alpha-\beta'\,;\, \frac{y-1}{y-x},\, \frac{y-1}{y}\right)\\
&= (\varepsilon_1(1-y))^{\gamma-\alpha-\beta'}(\varepsilon_2(y-x))^{-\beta}y^{1+\beta-\gamma}\\
&\times F_1\left(1-\beta',\, \beta,\, 1-\alpha,\, 1+\gamma-\alpha-\beta'\,;\, \frac{x(1-y)}{x-y},\, 1-y\right)\\
&= (\varepsilon_1(1-y))^{\gamma-\alpha-\beta'}(\varepsilon_2(1-x))^{\beta'-1}(\varepsilon_2(y-x))^{1-\beta-\beta'}y^{\beta+\beta'-\gamma}\\
&\times F_1\left(1-\beta',\, 1-\alpha,\, \gamma-\beta-\beta',\, 1+\gamma-\alpha-\beta'\,;\, \frac{1-y}{1-x},\, \frac{x(1-y)}{y(1-x)}\right)\\
&= (\varepsilon_1(1-y))^{\gamma-\alpha-\beta'}(\varepsilon_2(1-x))^{-\beta}y^{\alpha-\gamma}\\
&\times F_1\left(\gamma-\alpha,\, \beta,\, 1-\alpha,\, 1+\gamma-\alpha-\beta'\,;\, \frac{x(1-y)}{y(1-x)},\, \frac{y-1}{y}\right)\\
&= (\varepsilon_1(1-y))^{\gamma-\alpha-\beta'}(\varepsilon_2(1-x))^{\gamma-\alpha-\beta}(\varepsilon_2(y-x))^{\alpha-\gamma}\\
&\times F_1\left(\gamma-\alpha,\, 1-\alpha,\, \gamma-\beta-\beta',\, 1+\gamma-\alpha-\beta'\,;\, \frac{y-1}{y-x},\, \frac{x(1-y)}{x-y}\right).
\end{aligned}
$$

- $f_1^{(\infty,\infty)\,x=\infty}, \quad f_1^{(\infty,0)}, \quad f_3^{(\infty,1)}$:

$$
\begin{aligned}
w_8 &= (\varepsilon\, x)^{-\alpha} F_1\left(\alpha,\ 1+\alpha-\gamma,\ \beta',\ 1+\alpha-\beta\,;\ \frac{1}{x},\ \frac{y}{x}\right) \\
&= (\varepsilon\, x)^{-\alpha}\left(1-\frac{1}{x}\right)^{\gamma-\alpha-1}\left(1-\frac{y}{x}\right)^{-\beta'} \\
&\times F_1\left(1-\beta,\ 1+\alpha-\gamma,\ \beta',\ 1+\alpha-\beta\,;\ \frac{1}{1-x},\ \frac{y}{y-x}\right) \\
&= (\varepsilon\, x)^{-\alpha}\left(1-\frac{1}{x}\right)^{-\alpha} \\
&\times F_1\left(\alpha,\ \gamma-\beta-\beta',\ \beta',\ 1+\alpha-\beta\,;\ \frac{1}{1-x},\ \frac{1-y}{1-x}\right) \\
&= (\varepsilon\, x)^{-\alpha}\left(1-\frac{y}{x}\right)^{-\alpha} \\
&\times F_1\left(\alpha,\ 1+\alpha-\gamma,\ \gamma-\beta-\beta',\ 1+\alpha-\beta\,;\ \frac{y-1}{y-x},\ \frac{y}{y-x}\right) \\
&= (\varepsilon\, x)^{-\alpha}\left(1-\frac{1}{x}\right)^{\gamma-\alpha-\beta}\left(1-\frac{y}{x}\right)^{-\beta'} \\
&\times F_1\left(1-\beta,\ \gamma-\beta-\beta',\ \beta',\ 1+\alpha-\beta\,;\ \frac{1}{x},\ \frac{y-1}{y-x}\right) \\
&= (\varepsilon\, x)^{-\alpha}\left(1-\frac{1}{x}\right)^{\gamma-\alpha-1}\left(1-\frac{y}{x}\right)^{1-\beta-\beta'} \\
&\times F_1\left(1-\beta,\ 1+\alpha-\gamma,\ \gamma-\beta-\beta',\ 1+\alpha-\beta\,;\ \frac{1-y}{1-x},\ \frac{y}{x}\right).
\end{aligned}
$$

- $f_1^{(\infty,\infty)\,y=\infty}, \quad f_1^{(0,\infty)}, \quad f_3^{(1,\infty)}$:

$$
\begin{aligned}
w_9 &= (\varepsilon\, y)^{-\alpha} F_1\left(\alpha,\ \beta,\ 1+\alpha-\gamma,\ 1+\alpha-\beta'\,;\ \frac{x}{y},\ \frac{1}{y}\right) \\
&= (\varepsilon\, y)^{-\alpha}\left(1-\frac{x}{y}\right)^{-\beta}\left(1-\frac{1}{y}\right)^{\gamma-\alpha-1} \\
&F_1\left(1-\beta',\ \beta,\ 1+\alpha-\gamma,\ 1+\alpha-\beta'\,;\ \frac{x}{x-y},\ \frac{1}{1-y}\right) \\
&= (\varepsilon\, y)^{-\alpha}\left(1-\frac{1}{y}\right)^{-\alpha} \\
&F_1\left(\alpha,\ \beta,\ \gamma-\beta-\beta',\ 1+\alpha-\beta'\,;\ \frac{1-x}{1-y},\ \frac{1}{1-y}\right) \\
&= (\varepsilon\, y)^{-\alpha}\left(1-\frac{x}{y}\right)^{-\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&F_1\left(\alpha,\, \gamma-\beta-\beta',\, 1+\alpha-\gamma,\, 1+\alpha-\beta'\,;\, \frac{x}{x-y},\, \frac{x-1}{x-y}\right)\\
&= (\varepsilon\, y)^{-\alpha}\left(1-\frac{1}{y}\right)^{\gamma-\alpha-\beta'}\left(1-\frac{x}{y}\right)^{-\beta}\\
&F_1\left(1-\beta',\, \beta,\, \gamma-\beta-\beta',\, 1+\alpha-\beta'\,;\, \frac{x-1}{x-y},\, \frac{1}{y}\right)\\
&= (\varepsilon\, y)^{-\alpha}\left(1-\frac{1}{y}\right)^{\gamma-\alpha-1}\left(1-\frac{x}{y}\right)^{1-\beta-\beta'}\\
&F_1\left(1-\beta',\, \gamma-\beta-\beta',\, 1+\alpha-\gamma,\, 1+\alpha-\beta'\,;\, \frac{x}{y},\, \frac{1-x}{1-y}\right).
\end{aligned}
$$

- $f_1^{(0,0)\,x=y}, \quad f_3^{(1,1)\,x=y}, \quad f_3^{(\infty,\infty)\,x=y}$:

$$
\begin{aligned}
w_{10} &= (\varepsilon_1 y)^{\beta+\beta'-\gamma}(\varepsilon_2(1-y))^{\gamma-\alpha-1}(\varepsilon_3(y-x))^{1-\beta-\beta'}\\
&\times F_1\left(1-\beta',\, \gamma-\beta-\beta',\, 1+\alpha-\gamma,\, 2-\beta-\beta'\,;\, \frac{y-x}{y},\, \frac{y-x}{y-1}\right)\\
&= (\varepsilon_1 x)^{\beta+\beta'-\gamma}(\varepsilon_2(1-x))^{\gamma-\alpha-1}(\varepsilon_3(y-x))^{1-\beta-\beta'}\\
&\times F_1\left(1-\beta,\, 1+\alpha-\gamma,\, \gamma-\beta-\beta',\, 2-\beta-\beta'\,;\, \frac{x-y}{x-1},\, \frac{x-y}{x}\right)\\
&= (\varepsilon_1 x)^{\beta'-1}(\varepsilon_1 y)^{1+\beta-\gamma}(\varepsilon_2(1-y))^{\gamma-\alpha-1}(\varepsilon_3(y-x))^{1-\beta-\beta'}\\
&\times F_1\left(1-\beta',\, 1-\alpha,\, 1+\alpha-\gamma,\, 2-\beta-\beta'\,;\, \frac{x-y}{x},\, \frac{x-y}{x(1-y)}\right)\\
&= (\varepsilon_1 x)^{1+\beta'-\gamma}(\varepsilon_1 y)^{\beta-1}(\varepsilon_2(1-x))^{\gamma-\alpha-1}(\varepsilon_3(y-x))^{1-\beta-\beta'}\\
&\times F_1\left(1-\beta,\, 1+\alpha-\gamma,\, 1-\alpha,\, 2-\beta-\beta'\,;\, \frac{y-x}{y(1-x)},\, \frac{y-x}{y}\right)\\
&= (\varepsilon_1 y)^{\beta+\beta'-\gamma}(\varepsilon_2(1-x))^{\beta'-1}(\varepsilon_2(1-y))^{\gamma-\alpha-\beta'}(\varepsilon_3(y-x))^{1-\beta-\beta'}\\
&\times F_1\left(1-\beta',\, \gamma-\beta-\beta',\, 1-\alpha,\, 2-\beta-\beta'\,;\, \frac{y-x}{y(1-x)},\, \frac{y-x}{1-x}\right)\\
&= (\varepsilon_1 x)^{\beta+\beta'-\gamma}(\varepsilon_2(1-x))^{\gamma-\alpha-\beta}(\varepsilon_2(1-y))^{\beta-1}(\varepsilon_3(y-x))^{1-\beta-\beta'}\\
&\times F_1\left(1-\beta,\, 1-\alpha,\, \gamma-\beta-\beta',\, 2-\beta-\beta'\,;\, \frac{x-y}{1-y}\, \frac{x-y}{x(1-y)}\right).
\end{aligned}
$$

- $f_2^{(0,0)\,x=0}$:

$$
\begin{aligned}
&(\varepsilon\, y)^{1-\gamma}\, G_2(\beta,\, 1+\alpha-\gamma,\, 1+\beta'-\gamma,\, \gamma-1\,;\, -x/y,\, -y)\\
&= (\varepsilon\,(y-x))^{1-\gamma}(1-x)^{\gamma-\alpha-1}\\
&\times G_2\left(\gamma-\beta-\beta',\, 1+\alpha-\gamma,\, 1+\beta'-\gamma,\, \gamma-1\,;\, \frac{x}{y-x},\, \frac{x-y}{1-x}\right)
\end{aligned}
$$

$$= (\varepsilon\, y)^{1-\gamma}(1-y)^{\gamma-\beta'-1}(1-x)^{-\beta}$$
$$\times G_2\left(\beta,\, 1-\alpha,\, 1+\beta'-\gamma,\, \gamma-1\,;\, \frac{x(y-1)}{y(1-x)},\, \frac{y}{1-y}\right)$$
$$= (\varepsilon\,(y-x))^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}(1-y)^{\gamma-\beta'-1}$$
$$\times\, G_2\left(\gamma-\beta-\beta',\, 1-\alpha,\, 1+\beta'-\gamma,\, \gamma-1\,;\, \frac{x(1-y)}{y-x},\, \frac{y-x}{1-y}\right).$$

- $f_2^{(0,0)\, y=0}$:

$$(\varepsilon\, x)^{1-\gamma}\, G_2(1+\alpha-\gamma,\, \beta',\, \gamma-1,\, 1+\beta-\gamma\,;\, -x,\, -y/x\,)$$
$$= (\varepsilon\,(x-y))^{1-\gamma}(1-y)^{\gamma-\alpha-1}$$
$$\times G_2\left(1+\alpha-\gamma,\, \gamma-\beta-\beta',\, \gamma-1,\, 1+\beta-\gamma\,;\, \frac{y-x}{1-y},\, \frac{y}{x-y}\right)$$
$$= (\varepsilon\, x)^{1-\gamma}(1-x)^{\gamma-\beta-1}(1-y)^{-\beta'}$$
$$\times G_2\left(1-\alpha,\, \beta',\, \gamma-1,\, 1+\beta-\gamma\,;\, \frac{x}{1-x},\, \frac{y(x-1)}{x(1-y)}\right)$$
$$= (\varepsilon\,(x-y))^{1-\gamma}(1-x)^{\gamma-\beta-1}(1-y)^{\gamma-\alpha-\beta'}$$
$$\times\, G_2\left(1-\alpha,\, \gamma-\beta-\beta',\, \gamma-1,\, 1+\beta-\gamma\,;\, \frac{x-y}{1-x},\, \frac{y(1-x)}{x-y}\right).$$

- $f_2^{(0,0)\, x=y}$:

$$(\varepsilon\, x)^{1-\gamma}(1-x)^{\gamma-\alpha-1}\, G_2\left(1+\alpha-\gamma,\, \beta',\, \gamma-1,\, 1-\beta-\beta'\,;\, \frac{x}{1-x},\, \frac{y-x}{x}\right)$$
$$= (\varepsilon\, y)^{1-\gamma}(1-y)^{\gamma-\alpha-1}\, G_2\left(\beta,\, 1+\alpha-\gamma,\, 1-\beta-\beta',\, \gamma-1,\,;\, \frac{x-y}{y},\, \frac{y}{1-y}\right)$$
$$= (\varepsilon\, x)^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}(1-y)^{-\beta'}\, G_2\left(1-\alpha,\, \beta',\, \gamma-1,\, 1-\beta-\beta'\,;\, -x,\, \frac{x-y}{x(y-1)}\right)$$
$$= (\varepsilon\, y)^{1-\gamma}(1-x)^{-\beta}(1-y)^{\gamma-\alpha-\beta'}\, G_2\left(\beta,\, 1-\alpha,\, 1-\beta-\beta',\, \gamma-1\,;\, \frac{x-y}{y(1-x)},\, -y\right).$$

- $f_2^{(1,1)\, x=1}$:

$$(\varepsilon\,(1-y))^{\gamma-\alpha-\beta-\beta'}\, G_2\left(\beta,\, \gamma-\beta-\beta',\, \gamma-\alpha-\beta,\, \alpha+\beta+\beta'-\gamma\,;\, \frac{1-x}{y-1},\, y-1\right)$$
$$= (\varepsilon\,(x-y))^{\gamma-\alpha-\beta-\beta'}\, x^{\beta+\beta'-\gamma}$$
$$\times G_2\left(1+\alpha-\gamma,\, \gamma-\beta-\beta',\, \gamma-\alpha-\beta,\, \alpha+\beta+\beta'-\gamma\,;\, \frac{x-1}{y-x},\, \frac{y-x}{x}\right)$$

$$= (\varepsilon\,(1-y))^{\gamma-\alpha-\beta-\beta'}x^{-\beta}y^{\alpha+\beta-\gamma}$$
$$\times\, G_2\left(\beta,\, 1-\alpha,\, \gamma-\alpha-\beta,\, \alpha+\beta+\beta'-\gamma\,;\, \frac{y(1-x)}{x(y-1)},\, \frac{1-y}{y}\right)$$
$$= (\varepsilon\,(x-y))^{\gamma-\alpha-\beta-\beta'}x^{1+\beta'-\gamma}y^{\alpha+\beta-\gamma}$$
$$\times\, G_2\left(1+\alpha-\gamma,\, 1-\alpha,\, \gamma-\alpha-\beta,\, \alpha+\beta+\beta'-\gamma\,;\, \frac{y(x-1)}{y-x},\, \frac{x-y}{y}\right).$$

- $f_2^{(1,1)\,y=1}$:

$$(\varepsilon\,(1-x))^{\gamma-\alpha-\beta-\beta'}$$
$$\times\, G_2\left(\gamma-\beta-\beta',\, \beta',\, \alpha+\beta+\beta'-\gamma,\, \gamma-\alpha-\beta'\,;\, x-1,\, \frac{1-y}{x-1}\right)$$
$$= (\varepsilon\,(y-x))^{\gamma-\alpha-\beta-\beta'}y^{\beta+\beta'-\gamma}$$
$$\times\, G_2\left(\gamma-\beta-\beta',\, 1+\alpha-\gamma,\, \alpha+\beta+\beta'-\gamma,\, \gamma-\alpha-\beta'\,;\, \frac{x-y}{y},\, \frac{y-1}{x-y}\right)$$
$$= (\varepsilon\,(1-x))^{\gamma-\alpha-\beta-\beta'}x^{\alpha+\beta'-\gamma}y^{-\beta'}$$
$$\times\, G_2\left(1-\alpha,\, \beta',\, \alpha+\beta+\beta'-\gamma,\, \gamma-\alpha-\beta'\,;\, \frac{1-x}{x},\, \frac{x(1-y)}{y(x-1)}\right)$$
$$= (\varepsilon\,(y-x))^{\gamma-\alpha-\beta-\beta'}x^{\alpha+\beta'-\gamma}y^{1+\beta-\gamma}$$
$$\times\, G_2\left(1-\alpha,\, 1+\alpha-\gamma,\, \alpha+\beta+\beta'-\gamma,\, \gamma-\alpha-\beta'\,;\, \frac{y-x}{x},\, \frac{x(y-1)}{x-y}\right).$$

- $f_2^{(1,1)\,x=y}$:

$$(\varepsilon\,(1-x))^{\gamma-\alpha-\beta-\beta'}\,x^{\beta+\beta'-\gamma}$$
$$\times\, G_2\left(\gamma-\beta-\beta',\, \beta',\, \alpha+\beta+\beta'-\gamma,\, 1-\beta-\beta'\,;\, \frac{1-x}{x},\, \frac{x-y}{1-x}\right)$$
$$= (\varepsilon\,(1-y))^{\gamma-\alpha-\beta-\beta'}\,y^{\beta+\beta'-\gamma}$$
$$\times\, G_2\left(\beta,\, \gamma-\beta-\beta',\, 1-\beta-\beta',\, \alpha+\beta+\beta'-\gamma\,;\, \frac{y-x}{1-y},\, \frac{1-y}{y}\right)$$
$$= (\varepsilon\,(1-x))^{\gamma-\alpha-\beta-\beta'}\,x^{1+\beta'-\gamma}y^{-\beta'}$$
$$\times\, G_2\left(1-\alpha,\, \beta',\, \alpha+\beta+\beta'-\gamma,\, 1-\beta-\beta'\,;\, x-1,\, \frac{x-y}{y(1-x)}\right)$$
$$= (\varepsilon\,(1-y))^{\gamma-\alpha-\beta-\beta'}x^{-\beta}y^{1+\beta-\gamma}$$
$$\times\, G_2\left(\beta,\, 1-\alpha,\, 1-\beta-\beta',\, \alpha+\beta+\beta'-\gamma\,;\, \frac{y-x}{x(1-y)},\, y-1\right).$$

- $f_2^{(\infty,\infty)\,x=\infty}$:

$$
\begin{aligned}
&(\varepsilon_1 x)^{-\beta}(\varepsilon_2 y)^{\beta-\alpha}\, G_2\left(\beta,\, 1+\alpha-\gamma,\, \alpha-\beta,\, \beta+\beta'-\alpha\,;\, -\frac{y}{x},\, -\frac{1}{y}\right)\\
&= (\varepsilon_1 x)^{-\beta}(\varepsilon_2 y)^{\beta-\alpha}\left(1-\frac{1}{x}\right)^{\gamma-\alpha-1}\left(1-\frac{y}{x}\right)^{\alpha-\beta-\beta'}\\
&\times G_2\left(1+\alpha-\gamma,\, 1-\alpha,\, \beta+\beta'-\alpha,\, \alpha-\beta\,;\, \frac{x-y}{y(1-x)},\, \frac{y}{x-y}\right)\\
&= (\varepsilon_1 x)^{-\beta}(\varepsilon_2 y)^{\beta-\alpha}\left(1-\frac{1}{x}\right)^{-\beta}\left(1-\frac{1}{y}\right)^{\beta-\alpha}\\
&\times G_2\left(\beta,\, \gamma-\beta-\beta',\, \alpha-\beta,\, \beta+\beta'-\alpha\,;\, \frac{1-y}{x-1},\, \frac{1}{y-1}\right)\\
&= (\varepsilon_1 x)^{-\beta}(\varepsilon_2 y)^{\beta-\alpha}\left(1-\frac{1}{x}\right)^{\gamma-\alpha-\beta}\left(1-\frac{1}{y}\right)^{\beta-\alpha}\left(1-\frac{y}{x}\right)^{\alpha-\beta-\beta'}\\
&\times\, G_2\left(1-\alpha,\, \gamma-\beta-\beta',\alpha-\beta,\, \beta+\beta'-\alpha\,;\, \frac{y-1}{x-y},\, \frac{x-y}{x(y-1)}\right).
\end{aligned}
$$

- $f_2^{(\infty,\infty)\,y=\infty}$:

$$
\begin{aligned}
&(\varepsilon_1 x)^{\beta'-\alpha}(\varepsilon_2 y)^{-\beta'}\, G_2\left(1+\alpha-\gamma,\, \beta',\, \beta+\beta'-\alpha,\, \alpha-\beta'\,;\, -\frac{1}{x},\, -\frac{x}{y}\right)\\
&= (\varepsilon_1 x)^{\beta'-\alpha}(\varepsilon_2 y)^{-\beta'}\left(1-\frac{y}{x}\right)^{\alpha-\beta-\beta'}\left(1-\frac{1}{y}\right)^{\gamma-\alpha-1}\\
&\times G_2\left(1-\alpha,\, 1+\alpha-\gamma,\, \alpha-\beta',\, \beta+\beta'-\alpha\,;\, \frac{x}{y-x},\, \frac{y-x}{x(1-y)}\right)\\
&= (\varepsilon_1 x)^{\beta'-\alpha}(\varepsilon_2 y)^{-\beta'}\left(1-\frac{1}{x}\right)^{\beta'-\alpha}\left(1-\frac{1}{y}\right)^{-\beta'}\\
&\times G_2\left(\gamma-\beta-\beta',\, \beta',\, \beta+\beta'-\alpha,\, \alpha-\beta'\,;\, \frac{1}{x-1},\, \frac{1-x}{y-1}\right)\\
&= (\varepsilon_1 x)^{\beta'-\alpha}(\varepsilon_2 y)^{-\beta'}\left(1-\frac{1}{x}\right)^{\beta'-\alpha}\left(1-\frac{1}{y}\right)^{\gamma-\alpha-\beta'}\left(1-\frac{y}{x}\right)^{\alpha-\beta-\beta'}\\
&\times\, G_2\left(\gamma-\beta-\beta',\, 1-\alpha,\, \beta+\beta'-\alpha,\, \alpha-\beta'\,;\, \frac{y-x}{y(x-1)},\, \frac{x-1}{y-x}\right).
\end{aligned}
$$

- $f_2^{(\infty,\infty)\,x=y}$:

$$
\begin{aligned}
&(\varepsilon_1 x)^{\beta'-\alpha}(\varepsilon_2 y)^{-\beta'}\left(1-\frac{1}{x}\right)^{\gamma-\alpha-\beta}\left(1-\frac{1}{y}\right)^{-\beta'}\\
&\times G_2\left(\gamma-\beta-\beta',\, \beta',\, \beta+\beta'-\alpha,\, 1-\beta-\beta'\,;\, -\frac{1}{x},\, \frac{y-x}{1-y}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= (\varepsilon_1 x)^{-\beta} (\varepsilon_2 y)^{\beta-\alpha} \left(1-\frac{1}{x}\right)^{-\beta} \left(1-\frac{1}{y}\right)^{\gamma-\alpha-\beta'} \\
&\times G_2\left(\beta,\, \gamma-\beta-\beta',\, 1-\beta-\beta',\, \beta+\beta'-\alpha\,;\, \frac{y-x}{x-1},\, -\frac{1}{y}\right) \\
&= (\varepsilon_1 x)^{\beta'-\alpha} (\varepsilon_2 y)^{-\beta'} \left(1-\frac{1}{x}\right)^{\gamma-\alpha-1} \\
&\times\, G_2\left(1+\alpha-\gamma,\, \beta',\, \beta+\beta'-\alpha,\, 1-\beta-\beta'\,;\, \frac{1}{x-1},\, \frac{x-y}{y}\right) \\
&= (\varepsilon_1 x)^{-\beta} (\varepsilon_2 y)^{\beta-\alpha} \left(1-\frac{1}{y}\right)^{\gamma-\alpha-1} \\
&\times\, G_2\left(\beta,\, 1+\alpha-\gamma,\, 1-\beta-\beta',\, \beta+\beta'-\alpha\,;\, \frac{y-x}{x},\, \frac{1}{y-1}\right).
\end{aligned}
$$

- $f_2^{(0,1)}$:

$$
\begin{aligned}
&(1-x)^{-\beta} y^{-\beta'}\, G_2\left(\beta,\, \beta',\, 1+\beta'-\gamma,\, \gamma-\alpha-\beta'\,;\, \frac{x}{1-x},\, \frac{1-y}{y}\right) \\
&= (1-x)^{\gamma-\alpha-\beta} (y-x)^{-\beta'}\, G_2\left(\gamma-\beta-\beta',\, \beta',\, 1+\beta'-\gamma,\, \gamma-\alpha-\beta'\,;\, -x,\, \frac{y-1}{x-y}\right) \\
&= y^{1+\beta-\gamma} (y-x)^{-\beta}\, G_2\left(\beta,\, 1+\alpha-\gamma,\, 1+\beta'-\gamma,\, \gamma-\alpha-\beta'\,;\, \frac{x}{y-x},\, y-1\right) \\
&= y^{\beta+\beta'-\gamma} (1-x)^{\gamma-\alpha-1} (y-x)^{1-\beta-\beta'} \\
&\times\, G_2\left(\gamma-\beta-\beta',\, 1+\alpha-\gamma,\, 1+\beta'-\gamma,\, \gamma-\alpha-\beta'\,;\, -\frac{x}{y},\, \frac{1-y}{x-1}\right).
\end{aligned}
$$

- $f_2^{(1,0)}$:

$$
\begin{aligned}
&x^{-\beta} (1-y)^{-\beta'}\, G_2\left(\beta,\, \beta',\, \gamma-\alpha-\beta,\, 1+\beta-\gamma\,;\, \frac{1-x}{x},\, \frac{y}{1-y}\right) \\
&= (1-y)^{\gamma-\alpha-\beta'} (x-y)^{-\beta}\, G_2\left(\beta,\, \gamma-\beta-\beta',\, \gamma-\alpha-\beta,\, 1+\beta-\gamma\,;\, \frac{x-1}{y-x},\, -y\right) \\
&= x^{1+\beta'-\gamma} (x-y)^{-\beta'}\, G_2\left(1+\alpha-\gamma,\, \beta',\, \gamma-\alpha-\beta,\, 1+\beta-\gamma\,;\, x-1,\, \frac{y}{x-y}\right) \\
&= x^{\beta+\beta'-\gamma} (1-y)^{\gamma-\alpha-1} (x-y)^{1-\beta-\beta'} \\
&\times\, G_2\left(1+\alpha-\gamma,\, \gamma-\beta-\beta',\, \gamma-\alpha-\beta,\, 1+\beta-\gamma\,;\, \frac{1-x}{y-1},\, -\frac{y}{x}\right).
\end{aligned}
$$

- $f_2^{(0,\infty)}$:

$$
\begin{aligned}
&(\varepsilon\, y)^{-\beta'}\, G_2(\beta,\, \beta',\, 1+\beta'-\gamma,\, \alpha-\beta'\,;\, -x,\, -y^{-1}\,)\\
&= (\varepsilon\, y)^{-\beta'}\, (1-x)^{\beta'-\alpha}\left(1-\frac{x}{y}\right)^{-\beta'}\\
&\times G_2\left(\gamma-\beta-\beta',\, \beta',\, 1+\beta'-\gamma,\, \alpha-\beta'\,;\, \frac{x}{1-x},\, \frac{1-x}{x-y}\right)\\
&= (\varepsilon\, y)^{-\beta'}\left(1-\frac{1}{y}\right)^{\gamma-\beta'-1}\left(1-\frac{x}{y}\right)^{-\beta}\\
&\times G_2\left(\beta,\, 1-\alpha,\, 1+\beta'-\gamma,\, \alpha-\beta'\,;\, \frac{x(y-1)}{x-y},\, \frac{1}{y-1}\right)\\
&= (\varepsilon\, y)^{-\beta'}(1-x)^{\beta'-\alpha}\left(1-\frac{1}{y}\right)^{\gamma-\beta'-1}\left(1-\frac{x}{y}\right)^{1-\beta-\beta'}\\
&\times\, G_2\left(\gamma-\beta-\beta',\, 1-\alpha,\, 1+\beta'-\gamma,\, \alpha-\beta'\,;\, \frac{x(1-y)}{y(x-1)},\, \frac{1-x}{y-1}\right).
\end{aligned}
$$

- $f_2^{(\infty,0)}$:

$$
\begin{aligned}
&(\varepsilon\, x)^{-\beta}\, G_2(\beta,\, \beta',\, \alpha-\beta,\, 1+\beta-\gamma\,;\, -x^{-1},\, -y\,)\\
&= (\varepsilon\, x)^{-\beta}(1-y)^{\beta-\alpha}\left(1-\frac{y}{x}\right)^{-\beta}\\
&\times G_2\left(\beta,\, \gamma-\beta-\beta',\, \alpha-\beta,\, 1+\beta-\gamma\,;\, \frac{1-y}{y-x},\, \frac{y}{1-y}\right)\\
&= (\varepsilon\, x)^{-\beta}\left(1-\frac{1}{x}\right)^{\gamma-\beta-1}\left(1-\frac{y}{x}\right)^{-\beta'}\\
&\times G_2\left(1-\alpha,\, \beta',\, \alpha-\beta,\, 1+\beta-\gamma\,;\, \frac{1}{x-1},\, \frac{y(x-1)}{y-x}\right)\\
&= (\varepsilon\, x)^{-\beta}(1-y)^{\beta-\alpha}\left(1-\frac{1}{x}\right)^{\gamma-\beta-1}\left(1-\frac{y}{x}\right)^{1-\beta-\beta'}\\
&\times\, G_2\left(1-\alpha,\, \gamma-\beta-\beta',\, \alpha-\beta,\, 1+\beta-\gamma\,;\, \frac{1-y}{x-1},\, \frac{y(1-x)}{x(y-1)}\right).
\end{aligned}
$$

- $f_2^{(1,\infty)}$:

$$
\begin{aligned}
&(\varepsilon\, y)^{-\beta'}\left(1-\frac{1}{y}\right)^{-\beta'} G_2\left(\beta,\, \beta',\, \gamma-\alpha-\beta,\, \alpha-\beta'\,;\, x-1,\, (y-1)^{-1}\right)\\
&= (\varepsilon\, y)^{-\beta'}\, x^{\beta'-\alpha}\left(1-\frac{x}{y}\right)^{-\beta'}
\end{aligned}
$$

$$\times\, G_2\left(1+\alpha-\gamma,\ \beta',\ \gamma-\alpha-\beta,\ \alpha-\beta'\,;\ \frac{1-x}{x},\ \frac{x}{y-x}\right)$$

$$= (\varepsilon\, y)^{-\beta'}\left(1-\frac{1}{y}\right)^{\gamma-\alpha-\beta'}\left(1-\frac{x}{y}\right)^{-\beta}$$

$$\times\, G_2\left(\beta,\ 1-\alpha,\ \gamma-\alpha-\beta,\ \alpha-\beta'\,;\ \frac{y(1-x)}{x-y},\ -\frac{1}{y}\right)$$

$$= (\varepsilon\, y)^{-\beta'} x^{\beta'-\alpha}\left(1-\frac{1}{y}\right)^{\gamma-\alpha-1}\left(1-\frac{x}{y}\right)^{1-\beta-\beta'}$$

$$\times\, G_2\left(1+\alpha-\gamma,\ 1-\alpha,\ \gamma-\alpha-\beta,\ \alpha-\beta'\,;\ \frac{y(1-x)}{x(y-1)},\ -\frac{x}{y}\right).$$

- $f_2^{(\infty,1)}$:

$$(\varepsilon\, x)^{-\beta}\left(1-\frac{1}{x}\right)^{-\beta} G_2\left(\beta,\ \beta',\ \alpha-\beta,\ \gamma-\alpha-\beta'\,;\ (x-1)^{-1},\ y-1\right)$$

$$= (\varepsilon\, x)^{-\beta} y^{\beta-\alpha}\left(1-\frac{y}{x}\right)^{-\beta}$$

$$\times\, G_2\left(\beta,\ 1+\alpha-\gamma,\ \alpha-\beta,\ \gamma-\alpha-\beta'\,;\ \frac{y}{x-y},\ \frac{1-y}{y}\right)$$

$$= (\varepsilon\, x)^{-\beta}\left(1-\frac{1}{x}\right)^{\gamma-\alpha-\beta}\left(1-\frac{y}{x}\right)^{-\beta'}$$

$$\times\, G_2\left(1-\alpha,\ \beta',\ \alpha-\beta,\ \gamma-\alpha-\beta'\,;\ -\frac{1}{x},\ \frac{x(1-y)}{y-x}\right)$$

$$= (\varepsilon\, x)^{-\beta} y^{\beta-\alpha}\left(1-\frac{1}{x}\right)^{\gamma-\alpha-1}\left(1-\frac{y}{x}\right)^{1-\beta-\beta'}$$

$$\times\, G_2\left(1-\alpha,\ 1+\alpha-\gamma,\ \alpha-\beta,\ \gamma-\alpha-\beta'\,;\ -\frac{y}{x},\ \frac{x(1-y)}{y(x-1)}\right).$$

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