

Lattice QED in external electromagnetic fields

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We study QED in external electromagnetic fields using methods developed for simulating lattice QCD. Our first project is to simulate QED in a constant (in space and time) external magnetic field on a euclidean space-time lattice using the Rational Hybrid Monte Carlo (RHMC) method. Observables we measure include the condensate $\langle \bar{\psi}\psi \rangle$ and the effective electron action after integrating out the fermion fields. We look for evidence that the combined effect of the magnetic field and the electron-positron attraction from QED produces a non-zero condensate in the limit of zero electron mass, a non-perturbative effect analogous to spontaneous chiral symmetry breaking. Very preliminary evidence is that such a condensate exists, at least for strong external magnetic fields and unphysically large electric charge. In addition, we are storing field configurations to measure the expected distortions and screenings of the coulomb field of a charged particle due to the vacuum polarization asymmetries produced by the magnetic field. We hope also to measure the dynamical contribution to the electron mass produced by the same mechanism that produces a finite condensate in the zero input mass limit.

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1. Introduction

Theoretical studies of electrons in external electromagnetic fields date from the early days of relativistic quantum mechanics [1, 2] and exhibit some of the first features of quantum electrodynamics [3] such as the Sauter-Schwinger effect – the production of electron-positron pairs in an external electric field. [See the review article by Dunne [4] for a good summary of the cases for which exact, closed form (simple integral or series) solutions are known]. However, such effects only become significant when the electric field E is of order $E_{cr} = m^2/e$ or larger where m is the electron mass and e is the electron charge. Similarly external magnetic fields only produce significant quantum effects when the magnetic field B is of order $B_{cr} = m^2/e$ or larger. Interest in these effects has recently been revived by new and planned facilities with extremely intense laser beams, whose interactions with charged particle beams can produce such strong fields [5, 6]. Additionally it has been realized that compact astronomical X-ray and γ -ray sources are probably neutron stars with such strong fields. For a review, see for example [7]. Finally it has been noted that in the next generation of electron/positron colliders, beam-beam interactions can produce electromagnetic fields orders of magnitude above E_{cr} or B_{cr} , where all conventional QED calculations break down [8].

We are interested in applying methods developed for simulating lattice QCD to lattice QED in strong external electromagnetic fields. For QED in external electric fields, the transition to euclidean space needed to apply lattice simulation methods results in a complex action which prevents the use of standard simulation methods. We are therefore starting with QED in external magnetic fields where the transition to euclidean space produces a real action bounded below where standard simulation methods apply.

We simulate lattice QED in a constant external magnetic field B using the RHMC method of Clark and Kennedy [9], with a non-compact gauge action and staggered fermions, using rational approximations to tune to a single electron. The coupling of the gauge fields to the electron is compact. Not only does this ensure that the action is gauge invariant, but it also allows an effectively constant magnetic field over the whole lattice. Note that the internal gauge(photon) field is in both the gauge action and the fermion action, while the external field is only in the fermion action.

Classically, charged particles in a constant magnetic field are restricted to helical orbits around magnetic field lines. Quantum mechanics restricts the motion of the charged particles in the plane transverse to the magnetic field to a discrete set of evenly spaced levels – the Landau levels – while leaving the motion in the direction of the magnetic field continuous. For a discussion of Landau levels in relativistic quantum mechanics see for example [10]. For strong magnetic fields, only low lying Landau levels are populated resulting in the effective dimensional reduction from $3 + 1$ dimensions to $1 + 1$ dimensions.

In quantum mechanics the chiral condensate $\langle \bar{\psi}\psi \rangle$ is enhanced in the presence of the magnetic field, but still vanishes as $m \rightarrow 0$. Various approximate calculations suggest that the addition of QED causes $\langle \bar{\psi}\psi \rangle$ to develop a non-zero contribution proportional to $(eB)^{3/2}$ which does not vanish in this limit, and furthermore that the electron gains a non-zero dynamic mass proportional to \sqrt{eB} , [11–16, 18]. This non-perturbative effect is referred to as ‘magnetic catalysis’. For good review article with a more complete set of references see Miransky and Shovkovy [17].

In QED, one effect of the magnetic field is to introduce asymmetries in the vacuum polarization. These distort the coulomb field of any charged particle in such an external magnetic field and also induce at least partial screening [19–22]. On the lattice, such effects can be measured using Wilson loops.

We simulate lattice QED in an external magnetic field at $\alpha = e^2/4\pi = 1/137$, close to its physical value, over a range of eB values with $m = 0.1$ and $m = 0.2$ on a 36^4 lattice. To try and measure the value of the condensate $\langle \bar{\psi}\psi \rangle$ in the $m = 0$ limit, we also simulate at a larger α value $\alpha = 1/5$ and relatively large $eB \approx 0.4848$ over a much wider range of masses. At this α we also perform simulations at $eB = 0$ for comparison. At the smallest masses, we plan larger lattice simulations needed to test for a non-zero condensate at $m = 0$. Very short exploratory runs on larger lattices suggest that the condensate does have a measurably large $m = 0$ limit. In all our simulations we store a configuration every 100 trajectories for further analysis.

2. Lattice QED in an external magnetic field

We simulate using the non-compact gauge action

$$S(A) = \frac{\beta}{2} \sum_{n,\mu < \nu} [A_\nu(n + \hat{\mu}) - A_\nu(n) - A_\mu(n + \hat{\nu}) + A_\mu(n)]^2$$

where n is summed over the lattice sites, and μ and ν run from 1 to 4 subject to the restriction. $\beta = 1/e^2$. The expectation value of an observable $O(A)$ is then

$$\langle O \rangle = \frac{1}{Z} \int_{-\infty}^{\infty} \Pi_{n,\mu} dA_\mu(n) e^{-S(A)} [\det \mathcal{M}(A + A_{ext})]^{1/8} O(A)$$

where $\mathcal{M} = M^\dagger M$, A is the dynamic photon field and A_{ext} is the external photon field while

$$M(A + A_{ext}) = \sum_{\mu} D_{\mu}(A + A_{ext}) + m$$

where the operator D_{μ} is defined by

$$\begin{aligned} [D_{\mu}(A + A_{ext})\psi](n) &= \frac{1}{2}\eta_{\mu}(n)\{e^{i(A_{\mu}(n)+A_{ext,\mu}(n))}\psi(n + \hat{\mu}) \\ &\quad - e^{-i(A_{\mu}(n-\hat{\mu})+A_{ext,\mu}(n-\hat{\mu}))}\psi(n - \hat{\mu})\} \end{aligned}$$

and η_{μ} are the staggered phases.

We use the RHMC simulation method of Clark and Kennedy using rational approximations to $\mathcal{M}^{-1/8}$ and $\mathcal{M}^{\pm 1/16}$. To account for the range of normal modes of the non-compact gauge action, we randomly vary the trajectory lengths over the range of periods of these modes [23]. A_{ext} are chosen in the symmetric gauge in the x-y plane so that the magnetic fields from each plaquette are in the +z-direction and have the value B modulo 2π . This requires $eB = 2\pi n/(n_1 n_2)$, where n_1 and n_2 are the lattice dimensions in the x and y directions, and n is an integer in the range $[0, n_1 n_2/2]$ [24].

One of the observables we calculate is the electron contribution to the effective gauge action per site $\frac{-1}{8V}\text{trace}[\ln(\mathcal{M})]$. For this we use a rational approximation to \ln following Kelisky and Rivlin [25], and a stochastic approximation to the trace.

3. Simulations and Results

We first compare the chiral condensate per site for free electrons in an external magnetic field, $\langle \bar{\psi}\psi \rangle = \text{trace}[M^{-1}(A_{ext})]/4V$, measured on a 36^4 lattice as a function of allowed eB values and compare it with the known continuum result:

$$\langle \bar{\psi}\psi \rangle = \langle \bar{\psi}\psi \rangle|_{eB=0} + \frac{meB}{4\pi^2} \int_0^\infty \frac{ds}{s} e^{-sm^2} \left[\coth(eBs) - \frac{1}{eBs} \right].$$

for the ‘safe’ mass values $m = 0.1$ and $m = 0.2$ ¹.

We next calculate the effective fermion action per site, $\mathcal{L}_f = -\frac{1}{4V} \ln\{\det[M(A_{ext})]\} = -\frac{1}{4V} \text{trace}\{\ln[M(A_{ext})]\}$, on the lattice with the same parameters, and compare it with the known continuum values

$$\mathcal{L}_f = \mathcal{L}_f|_{B=0} + \frac{(eB)^2}{24\pi^2} \int_0^\infty \frac{ds}{s} e^{-m^2s} + \frac{eB}{8\pi^2} \int_0^\infty \frac{ds}{s^2} e^{-m^2s} \left[\coth(eBs) - \frac{1}{eBs} - \frac{eBs}{3} \right].$$

Figures 1, 2 show the chiral condensate and effective fermion action as functions of eB , comparing the lattice results on a 36^4 lattice with the known continuum values. This indicates that we have acceptable agreement for $eB \lesssim 0.65$.

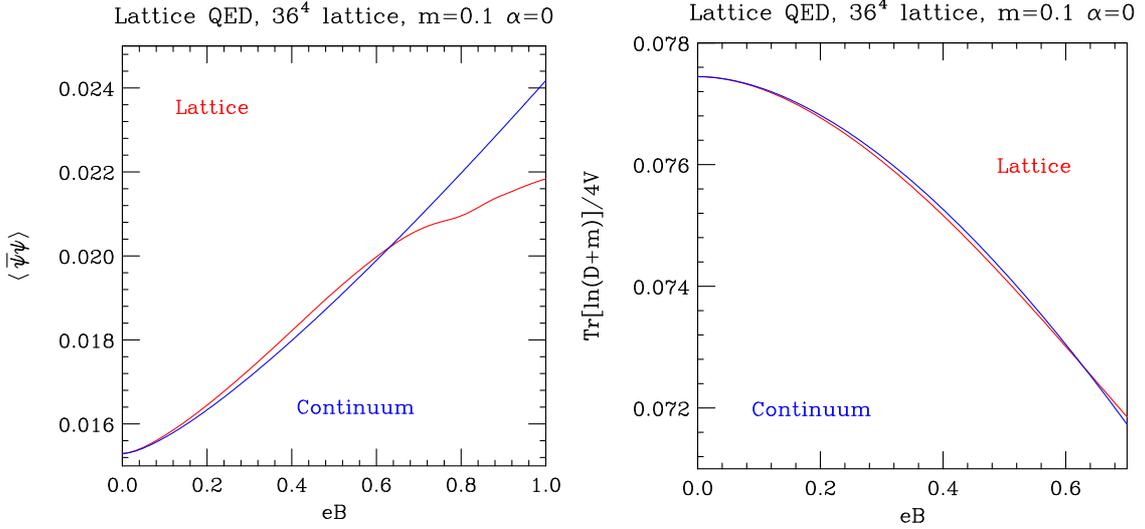


Figure 1: Free-electron $\langle \bar{\psi}\psi \rangle$ as functions of eB , **Figure 2:** Minus the effective action/site as functions of eB for a free electron, comparing the continuum and lattice results for $m = 0.1$.

We simulate QED on a 36^4 lattice with electron mass $m = 0.1$ and $m = 0.2$ in an external magnetic field B , with $\alpha = 1/137$, close to its physical value. Note that here the momentum cutoff is so low that the difference between bare and renormalized coupling and mass are at most a few percent and are neglected. We simulate for 12,500 trajectories for each m and B , storing configurations every 100 trajectories for future analysis. Figure 3 shows the condensates $\langle \bar{\psi}\psi \rangle$

¹A safe mass value is one for which $m \ll 1$ to keep discretization errors low, while $mL \gg 1$ (where L is the linear size of the lattice) to keep finite lattice size effects small.

as functions of eB for these runs compared with those for free electrons for $m = 0.1$. The most noticeable effect is that QED increases the values of these condensates. Figure 4 compares the effective fermion actions for QED with those for free electrons. Although the condensate decreases

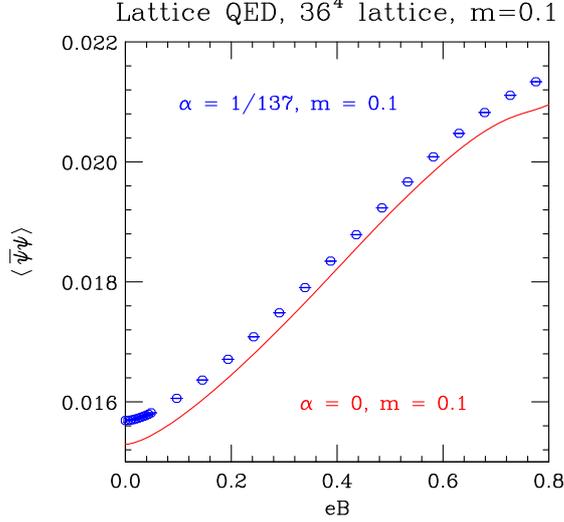


Figure 3: Electron $\langle \bar{\psi}\psi \rangle$ as functions of eB , comparing the lattice free and QED results for $m = 0.1$, $\alpha = 1/137$

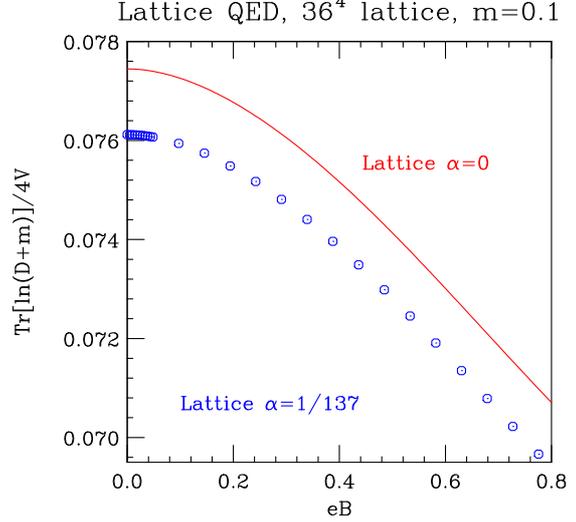


Figure 4: Minus the effective action/site as functions of eB , comparing the lattice free and QED results for $m = 0.1$, $\alpha = 1/137$.

with decreasing mass, and is clearly small in the limit $m \rightarrow 0$, these masses are not small enough to tell whether it vanishes in this limit over the whole range of eB as it does in the free electron case, or approaches a finite value (except at $eB = 0$) as approximate calculations suggest.

To determine whether magnetic catalysis does occur requires simulating at much smaller masses, where finite size effects are a potential problem. However, if the condensate does vanish in the $m = 0$ limit, the condensate is presumably a large momentum effect as it is for free electrons in an external magnetic field, and should be insensitive to finite size effects. If the condensate remains finite as $m \rightarrow 0$, this is due to the presence of near-zero modes. In this case, the condensate should show little finite lattice size dependence for larger masses, and increase with increasing lattice size for small masses. For large values of eB where the system is expected to be predominantly in the low lying Landau levels, the projection of the electron orbits on the plane orthogonal to the magnetic field is small (dimensional reduction). This suggests that to perform finite size analyses, one only needs to extend the lattice in the direction of the magnetic field and the time direction, while leaving the extent in the directions orthogonal to the external field fixed. This reduces the resources needed to perform a finite size analysis.

Since the zero mass condensate is still expected to be small, we need to do everything in our power to increase its value. This could be done by increasing the size of eB , however, the lattice restricts how far this can be increased. We are already planning on reducing m . This leaves increasing α as a method for increasing $\langle \bar{\psi}\psi \rangle$. For this reason, we are now simulating at $\alpha = 1/5$ to enhance the signal, and using electron masses as low as $m = 0.001$, still on a 36^4 lattice. We are simulating at $eB = 2\pi \times 100/36^2 \approx 0.4848$, reasonably large, but comfortably below $eB \approx 0.65$,

where lattice and continuum results are expected to diverge. In addition, we are simulating at $\alpha = 1/5$ and $eB = 0$ for comparison.

Figure 5 shows the condensates $\langle \bar{\psi}\psi \rangle$ as functions of mass m from our simulations at $\alpha = 1/5$ at $eB = 0$ and at $eB = 2\pi \times 100/36^2$ on a 36^4 lattice. While both appear to be approaching zero as $m \rightarrow 0$, the $eB = 0$ points do so relatively smoothly over the whole range of m values, while the points for non-zero eB and $m \geq 0.025$ appear to be headed for a non-zero value at $m = 0$. Below $m = 0.025$ the points at non-zero eB , curve more strongly, approaching something much closer to zero. This suggests that we need to perform a finite size analysis at these lower mass values. Very preliminary, short, exploratory simulations on $36^2 \times 72^2$ lattices at $\alpha = 1/5$ and $eB = 2\pi \times 100/36^2$ appear to indicate that, while the condensate for $m = 0.025$ shows little if any change from its value on a 36^4 lattice, those for smaller masses show significant increases. In fact, for the smallest mass ($m = 0.001$), the condensate on a $36^2 \times 64^2$ lattice appears to be roughly twice that on a 36^4 lattice, while that on a $36^2 \times 96^2$ lattice appears to be roughly thrice that on a 36^4 lattice. If these observations are ratified by simulations of significant length, we are seeing the first evidence for magnetic catalysis for QED in an external magnetic field from lattice QED simulations.

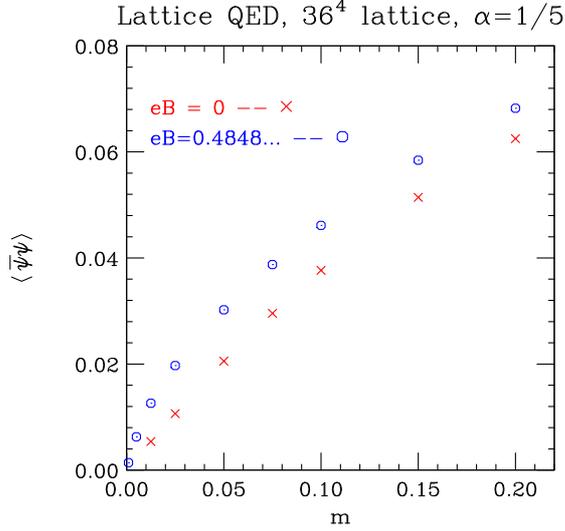


Figure 5: $\langle \bar{\psi}\psi \rangle$ as functions of m for $eB = 0$, lower set of points, and $eB = 2\pi \times 100/36^2$, upper set of points, for $\alpha = 1/5$ on a 36^4 lattice.

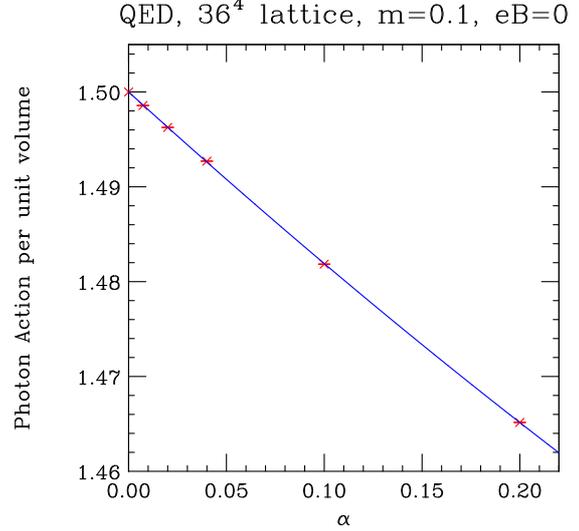


Figure 6: Gauge action/site at $eB = 0$, $m = 0.1$ on a 36^4 lattice as a function of α . The blue curve is a fit quadratic in α .

In figure 6 we plot the gauge(photon) action/site at $eB = 0$, $m = 0.1$ on a 36^4 lattice as a function of α . The curve is the quadratic fit $1.5 - 0.187499\alpha + 0.0661374\alpha^2$, which has χ^2/DOF of 11. Since the $O(\alpha)$ term dominates the QED contribution, it suggests that one can probably trust perturbation theory for this quantity. It should be checked against a perturbative calculation with lattice regularization, which would provide a check of lattice simulations.

4. Summary, Discussion and Conclusions

We simulate lattice QED in a constant external magnetic field B using the RHMC method. For electrons in such an external field without QED, we measure the chiral condensate and effective

action over the allowed range of eB ($|eB| \leq \pi$) and determine that the range of eB over which the lattice measurements agree with the known continuum values is $|eB| \lesssim 0.65$.

We perform simulations at the near-physical value of α , namely at $\alpha = 1/137$, with electron masses $m = 0.1$ and $m = 0.2$ on a 36^4 lattice, over the range of eB where lattice measurements are expected to be close to their continuum values. The condensate $\langle \bar{\psi}\psi \rangle$ is consistently larger than that without QED and increases with increasing eB . The effective action, which measures the response of the system to the applied magnetic field also lies above that without QED and has a similar eB dependence to that for the free electron in a magnetic field.

We also simulate at stronger coupling – $\alpha = 1/5$ – over a wider range of masses $0.001 \leq m \leq 0.2$ at $eB \sim 0.5$ and $eB = 0$, also on a 36^4 lattice, to determine whether having a (large) external magnetic field combined with the electron-positron attraction due to QED is sufficient to produce a non-zero condensate in the $m \rightarrow 0$ limit (magnetic catalysis). Since the smaller mass values are far below what is considered safe, we need to perform a finite size analysis to determine whether this occurs. Because strong magnetic fields restrict the extent of the electron wave-functions transverse to the external magnetic field we only need to increase the lattice extent in the direction of the magnetic field and that of time for this analysis. Very limited exploratory runs on such larger lattices appear to indicate that $\langle \bar{\psi}\psi \rangle$ does indeed remain finite in the zero mass limit.

We need to extend our simulations on larger lattices to test if $\langle \bar{\psi}\psi \rangle$ is non-vanishing in the zero mass limit for large eB (and α). If so, we should try to determine the eB and α dependence of this condensate.

We will use the configurations stored during these runs to determine the effect of the external magnetic field and QED on the coulomb field of a static point charge. In addition, we will measure the electron propagator on these configurations to measure the dynamic contribution to the electron mass.

Our next project will be to use lattice methods to study QED in an external electric field. This is a much more difficult task, since the external electric field makes the euclidean action complex. [It is the imaginary part of this action which describes the instability of the vacuum to decay, producing electron-positron pairs.] This means that standard simulation methods based on importance sampling will fail. Those methods which remain are less reliable.

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