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TASI Lectures on random matrix universality in AdS/CFT

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These lectures review recent work on the connection between chaos in black hole physics, random matrix level statistics, and spacetime wormholes. The main goal of these lectures is to introduce the basic concepts of random matrix level statistics for chaotic systems, and describe some steps towards a bulk understanding of the random matrix level statistics of a black hole in AdS/CFT.

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1. Overview

The goal of these lectures is to introduce some basic expectations for the structure of the energy spectrum for black holes in AdS/CFT, based on the fact that black holes are chaotic, and give some partial explanations from gravity. This structure involves the *statistics* of the energy levels of large black holes in AdS, which are dual to high energy states in the CFT. For a chaotic system, the energy levels are expected to be erratic functions of the parameters of the system, and not analytically computable in practice. But for chaotic systems, there is an observed simplicity behind tha chaos: universal statistical behavior of the spacing between energy levels, essentially depending only on some discrete symmetries of the system.

In the first part of these lectures, we begin by discussing why we believe that black hole are chaotic. Then we discuss the expected universal statistics of energy levels for chaotic systems. Roughly, the energy levels of a chaotic system are expected to be distributed like the eigenvalues of a *random matrix*. We make this statement precise, and in the process study the level statistics of random matrices drawn from certain simple ensembles.

In order to more usefully apply these ideas to AdS/CFT, we introduce the spectral form factor $|Z(\beta + it)|^2$, an auxiliary tool for studying the spectrum, computed by analytically continuing two copies of the partition function. The universal aspects of level statstics implies a simple ramp-plateau structure in the averaged spectral form factor at late times.

We end the lectures by discussing the computation of the spectral form factor in gravity, with an emphasis on computations in Jackiw-Teitelboim (JT) gravity. We give a physical explanation of the behavior of the spectral form factor before the plateau, and show how the ramp can be explained by the contribution of a spacetime wormhole. Finally, we briefly address some outstanding questions.

2. Black holes as chaotic systems

Black holes are *chaotic* systems. Rather than trying to define precisely what I mean by chaotic, let's simply look at a couple features of black holes which are generally accepted as signatures of chaos.

2.1 Thermalization

Possibly the first characteristic that one may think of when discussing chaotic systems is *thermalization*. Take the system of interest and put it in thermal equilibrium, and then perturb the system in some way. A chaotic system should eventually return to equilibrium, up to small errors.[1] Of course, there are caveats; for example, if the system has conserved charges, perturbations which add some charge don't quite thermalize, but for simplicity we can restrict our attention to perturbations which do not add any conserved charge.

A simple diagnostic of thermalization is the thermal two-point function of some observable O, with the operators separated by some time t:

$$\langle O(t)O(0)\rangle_\beta \equiv \frac{1}{Z(\beta)}\mathrm{Tr}[e^{-\beta H}O(t)O(0)]. \tag{1}$$

[1]For now we can imagine working in the thermodynamic limit, in which these errors become arbitrarily small. But later we will back off from this limit, and study these errors.

Here $Z(\beta)$ is the partition function $\mathrm{Tr}[e^{-\beta H}]$. The physical interpretation of this correlator is that we begin by perturbing the thermal state with O, wait some time t, and then measure O. If the time t is long, and the system the system is chaotic, then we imagine that by time t the initial perturbation is essentially undetectable. Then the answer we get for the measurement should be the same as if we had not perturbed the system: the expectation value of O in the thermal state. Mathematically, this means that at late times, we expect that the two-point function factorizes:

$$\langle O(t)O(0)\rangle_\beta \approx \langle O\rangle_\beta^2, \qquad t \to \infty. \tag{2}$$

Now let's apply this to a black hole. In this context, we replace expectation values in the thermal state with expectation values in the black hole background, and we perturb with operators O which act outside the black hole horizon. A simple choice is to take O to be the s-wave component of a scalar field.

O then creates a wave which eventually falls into the black hole - after this has happened, if we meausure O we detect just the ordinary black hole background, without the perturbation.

Of course, one potential problem with this analogy between black holes and a thermalizing system is that often, a black hole without the perturbation is not quite in thermal equilibrium - black holes evaporate. However, the timescale of evaporation is long compared to time for the perturbation to be swallowed by the black hole. And furthermore, if we consider a large black hole in AdS (which we will focus on for the remainder of this note), the black hole is indeed in equilibrium.

If we choose O to be the s-wave of a scalar field, the two-point function decays in a simple way. But of course, for a general (uncharged) operator, this correlator will eventually decay, as the perturbation always eventually falls into the black hole. But along the way, the behavior may be more interesting. A general operator excites the "quasinormal modes" of the black hole. They are quasinormal because unlike ordinary normal modes, they have complex frequencies signalling the decaying behavior. A black hole with its quasinormal modes excited by a perturbation oscillates in some way, with the amplitude of the oscillations decaying. This process is sometimes referred to as "ringdown".

One fun thing about this ringdown process is that we have now observed it in nature. LIGO had now measured the gravitational waves emitted by black holes in some collisions, with other black holes or with neutron stars. One can view this as a process in which the black hole is hit with a pretty large perturbation. Even then, the black hole quickly returns to equilibrium,[2] with the gravitational waves that LIGO detected exhibiting the oscillations of the black hole as it is ringing down.

Finally, let's say a few more words about the case in which we consider a large black hole in AdS. This case was studied in detail by Horowitz and Hubeny [1], and much of what I said I learned from this paper. In this case, by AdS/CFT, the Hartle-Hawking state of the large two-sided Schwarzschild black hole, or the eternal black hole as I will often refer to it as, is dual to the thermal state of the boundary CFT[3]. Then the correlators we have been discussing can be understood

[2]In reality, the final black hole is rotating rapidly, but we can roughly view the black hole as in equilibrium among states with that angular momentum. We could have generalized our earlier definitions this way too.

[3]Or, as we will often use, the thermofield double state of two copies of the CFT

precisely as correlators 2, and the "thermalization" of a perturbation to a black hole is dual to genuine thermalization of a perturbation to the thermal state of the CFT.

2.2 Butterfly Effect

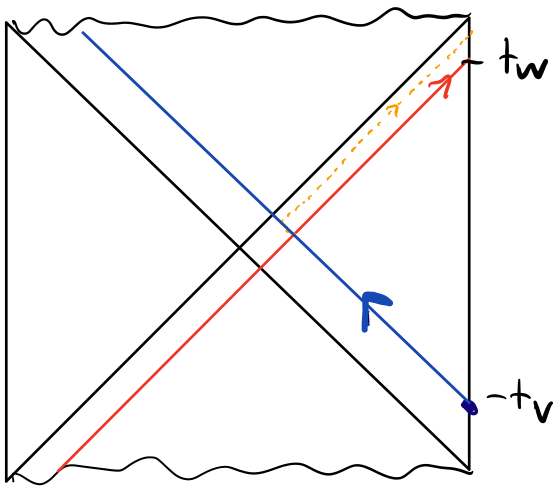
Black holes also exhibit the *butterfly effect*. By this I mean the following:

First, set up a finely tuned state. For example, a state in which all the air molecules in a room enter a single corner of the room at time zero. Then perturb the state sometime in the past, at time $-t_d$, for example by moving a couple of the air molecules. Then the butterfly effect means that if t_d is sufficiently long, a perturbation generically destroys the fine-tuning - the air molecules no longer congregate in the corner at time zero.

For an eternal black hole in AdS we can choose the fine tuned state to be the Hartle-Hawking state, describing an empty black hole, but with a particle W added that is tuned so that it leaves the black hole (or really, the white hole), and arrives at the boundar at time t_W. Then we can take this state and act with a perturbation V at the boundary at time $-t_V$.

What happens is that if time separation $t_W + t_V$ is sufficiently large, the perturbation created by V meets the outgoing particle W somewhere near the horizon, with a large relative boost - the perturbtion and W are moving rapidly with respect to one another. The strength of the gravitational interaction grows with their relative boost, and thus for large relative boost their interaction is strong and the perturbation strongly disturbs W's trajectory. For sufficiently large relative boost, W no longer leaves the black hole.

One can think about this process in multiple ways. One way is to think of the perturbation as "pulling" W back towards the black hole. Another is to think of the perturbation as just becoming part of the black hole, increasing its radius and thus trapping W closer to the new horizon. Another is to think of the perturbation as sourcing an Aichelburg-Sexl type shockwave geometry. When W passes through the shockwave, the effect is to shift the particle in a null direction, which delays the time at which W arrives at the boundary. Below we have pictured the initial trajector of W in red, and the shifted trajectory with the dotted orange line.



This phenomenon was first pointed out by 't Hooft and Dray for black holes [2], and was applied to AdS/CFT and connected to the butterfly effect by Shenker and Stanford [3, 4]. Shenker and Stanford used an out-of-time-ordered four-point function on the black hole background to study this effect. This correlator can also be thought of as a contribution to the correlator

$$\left\langle \left| [V(-t_V), W(t_W)] \right|^2 \right\rangle_{BH}. \tag{3}$$

That is, the expectation value of the square of the commutator between $W(t_W)$ and $V(t_V)$. If we take V and W to be two different single trace operators (simple fields), and the time separation $t_V + t_W$ is small, then this commutator is small, of order G_N. But for $t_V + t_W$ sufficiently large, the perturbation V has a large effect on the outgoing particle W, meaning that the commutator is large. Here it is important to look at the square of the commutator, as the expectation value of the commutator can have cancellations.

More precisely, the size of this correlator is set by G_N times the center of mass energy of the collision, which sets the strength of the gravitational interaction. This center of mass energy grows exponentially in the time separation, $E_{CM} \propto \exp[2\pi(t_V + t_W)/\beta]$, where β is the temperature of the black hole. Then roughly, the correlator behaves as

$$\left\langle \left| [V(-t_V), W(t_W)] \right|^2 \right\rangle_{BH} \sim G_N e^{\frac{2\pi}{\beta}(t_V + t_W)}, \tag{4}$$

which becomes of order one when $t_V + t_W = t_{sc} = \frac{\beta}{2\pi}\log(1/G_N)$. This is called the "scrambling time".

More precisely, if V and W are localized on the boundary, one must also wait for the perturbation to spread in space before having a large effect on W. This more general case was studied by Roberts, Stanford, and Susskind [5].

This exponential growth of this correlator signifies a sort of "maximal chaos" of black holes. This exponentially growing effect of an early perturbation on fine-tuning is characteristic of chaotic many-body systems in which sufficiently many of the degrees of freedom are directly coupled. A simple class is "k-local systems", such as the SYK model, in which the Hamiltonian is a sum of terms involving at most k degrees of freedom, but each degree of freedom is coupled to every other one.

As shown by Maldacena, Shenker, and Stanford [6], even for these all-to-all coupled systems with exponentially rapidly spreading perturbations, there is a strict bound on how quickly this spread can happen. Roughly, the exponent of growth of the correlator we have discussed cannot be larger than $2\pi/\beta$. Note that this is precisely the exponent that black holes exhibit - they saturate the chaos bound.

Here I have been very brief, and skipped over many details. I highly recommend that an interested reader studies all of the papers that I have mentioned.

2.3 Chaos and random matrices

So far, we have discussed how black holes thermalize, and exhibit the butterfly effect (with the maximal exponent!). Many views these facts as firmly establishing that black holes (and their duals in AdS/CFT) are chaotic systems. Rather than try to define what a chaotic system is, I will simply state characteristics that a chaotic system must have.

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So in this case, we will take our belief that black holes are chaotic to mean that we expect that black holes exhibit another characteristic of chaos, which has two sub-characteristics. We will roughly refer to this characteristic as "random matrix universality". A rough statement of this characteristic is as follows:

The Hamiltonian for a chaotic system "looks like" a *random matrix* for appropriate observables. Then

- The energy levels are distributed like the eigenvalues of a random matrix, and
- The energy eigenstates look like random vectors (Eigenstate Thermalization Hypothesis) [7]

The second part of the statement is known as the Eigenstate Thermalization Hypothesis, introduced by Srednicki. There is a rich literature on this subject, but in this note we will focus primarily on the random matrix statistics of the energy levels, and make only a few comments about the eigenstates.

Now, in order to make this rough statement of "random matrix universality" more precise, we must review some facts about random matrices.

3. Random Matrices

Before we begin, I point out that I learned much of the material from this section from the book "Quantum Signatures of Chaos", by Haake [8]. I highly recommend that the interested reader consult this book for more detail.

In this section, we will study some statistical aspects of a random $L \times L$ matrix H, where H is drawn from some ensemble. Here H is supposed to be thought of as the Hamiltonian of a system, with a finite number L of energy levels.[4] In practice, we are often interested in systems with an infinite number of energy levels, but we can often focus on an energy window with a finite number of energy levels, and so we can roughly think about H as the Hamiltonian within that window.

We also, for now, want to assume that H has no symmetries, besides certain antiunitary symmetries. We can allow H to be invariant under the antiunitary operation of time reversal, T, where T^2 can be either plus or minus one. The random matrices we will study then can be grouped into three symmetry classes; those without T symmetry, those with T symmetry and $T^2 = +1$, and those with T symmetry and $T^2 = -1$.[5] This is known as "Dyson's threefold way", or the Wigner-Dyson classification.

Of couse, we are often interested in systems which do in fact have additional symmetries, but in that case one can focus on states with a fixed charge. We can think of our matrices H as Hamiltonians in this charge sector.

Then to make this idea of "random matrix universality" more precise, we need a sense in which we can say that the Hamiltonian of a chaotic system (in an energy window, for a fixed charge), looks like a random matrix, drawn from some ensemble of matrices corresponding to one of the three symmetry classes we mentioned.

[4]Note that in statistical physics, L is often supposed to be a length - be sure not to think of it as one!

[5]There are additional classes of interest if one is studying supersymmetric systems, but we will keep it simple and not discuss these.

3.1 Symmetry classes and ensembles

Consider a Hermitian matrix H from one of our three symmetry classes.

A generic Hermitian matrix H with no T symmetry has no degeneracies. We may diagonalize it using a unitary matrix, to get a matrix of the form

$$H \to \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \dots & \\ & & & \lambda_L \end{pmatrix}. \tag{5}$$

A generic Hermitian matrix H *with* T symmetry, where $T^2 = +1$ is a generic real symmetric matrix. This can be diagonalized with an orthogonal matrix, or brought to the following form with a unitary matrix

$$H \to \begin{pmatrix} 0 & i\lambda_1 & & & \\ -i\lambda_1 & 0 & & & \\ & & 0 & i\lambda_2 & \\ & & -i\lambda_2 & 0 & \\ & & & & \dots \end{pmatrix}. \tag{6}$$

If L is even, there are $L/2$ blocks, and if L is odd, there is a zero on the last diagonal. The eigenvalues are all doubly degenerate.

Finally, a generic Hermitian matrix H with T symmetry where $T^2 = -1$ can always be brought to a similar form to the previous case, but where the blocks are real quaternions. We will mostly focus on the first two cases for simplicity.

Now, for each of these three symmetry classes, we can consider the ensemble of all matrices in that class, endowed with a distribution

$$P(H)dH. \tag{7}$$

We are interested in the statistics of the energy levels in these ensembles. The statistics depend of course on the precise distribution $P(H)dH$, but we will see that there are some somewhat universal features that depend weakly on the choice of distribution for a large class of distributions. These robust features are the ones relevant for our statement of "random matrix universality".

The simplest choices of distribution which illustrate these features correspond to taking the individual elements of the matrix to be independent Gaussian variables (up to symmetry constraints), all with the same variance. For example, in the no T symmetry case,

$$\begin{aligned} P(H)dH &\propto \exp\left[-\frac{L}{2\sigma^2}\sum_{i\le j}|H_{ij}|^2\right]\prod_i dH_{ii}\prod_{i<j} dH_{ij}dH^*_{ij} \\ &= \exp\left[-\frac{L}{2\sigma^2}\mathrm{Tr}[H^2]\right]\prod_i dH_{ii}\prod_{i<j} dH_{ij}dH^*_{ij}. \end{aligned} \tag{8}$$

This is known as the Gaussian Unitary Ensemble, or GUE, as probability measure is invariant under conjugation of H by a unitary matrix. We've also chosen to include an explicit factor of $1/L$ in the variance of the Gaussian, for reasons of convenience that will become mroe clear later.

This gaussian ensemble has generalizations to the other two symmetry classes. For T invariant systems with $T^2 = +1$, H is a real symmetric matrix and

$$P(H)dH \propto \exp\left[-\frac{L}{2\sigma^2}\mathrm{Tr}[H^2]\right] \prod_{i\leq j} dH_{ij}. \tag{9}$$

This is invariant under conjugation of H by an orthogonal matrix, hence this ensemble is referred to as the Gaussian Orthogonal Ensemble (GOE).

Finally, for H with $T^2 = -1$, the obvious Gaussian ensemble is invariant under conjugation by certain symplectic matrices, and is called the Gaussian Symplectic Ensemble (GSE).

From now on we will refer to the three symmetry classes by their representative Gaussian ensemble; for example, the "GUE symmetry class".

I'll also note at the moment that we can simply generalize each of these three ensembles by replacing the $\mathrm{Tr}[H^2]$ in the Gaussian with $\mathrm{Tr}[V(H)]$ for some more general function $V(H)$ (such that the distribution is still normalizable). These distributions are still unitarily (or orthogonally/symplectically) invariant, though the matrix elements are no longer independent variables.

3.2 Eigenvalue statistics of the GUE

From now on, we will mostly focus on the GUE symmetry class. Our statement of random matrix universality is that for a chaotic system with no time reversal symmetry, the energy levels are distributed "like" those of a random matrix.

We have not yet defined what we mean by "like" in that sentence, but he have just introduced a notion of a random matrix with the right symmetries: a random matrix drawn from the GUE (or similar unitarily invariant ensemble). This is the relevant notion of a random matrix for the statement of random matrix universality.

So now in order to sharpen our statement of random matrix universality, we need to understand how the eigenvalues of a GUE matrix are distributed. Then we will find the "universal" features relevant for studying chaotic, but not random, systems.

To initiate our study of the distribution of eigenvalues of a GUE random matrix, we decompose our matrices H as

$$H = U^\dagger \Lambda U, \qquad \Lambda = \mathrm{diag}(E_1, \dots E_L), \tag{10}$$

where U is a unitary matrix. Now we want to write our probability measure in terms of the eigenvalues E_i, and the elements of the unitary matrix

$$P(H)dH \to P(\{E_i\}, U) d\{E_i\} dU. \tag{11}$$

For the GUE, the H_{ij} are independent random variables, but the eigenvalues will *not* be independent.

To proceed, we first look at the Gaussian part of the measure. This is simple to write in terms of the $\{E_i\}$ and U,

$$P(H) \propto \exp\left[-\frac{L}{2\sigma^2}\mathrm{Tr}[H^2]\right] = \exp\left[-\frac{L}{2\sigma^2}\sum_{i=1}^{L} E_i^2\right]. \tag{12}$$

Note that if we considered instead a more general unitarily invariant ensemble characterized by a function $V(H)$, the exponent would involve the function of the eigenvalues $\sum_{i=1}^L V(E_i)$.

The nontrivial part is to write the part of the measure dH in terms of the eigenvalues and the unitary matrix. To do this, we think of the H_{ij} as coordinates on the space of matrices, wth a metric. Then dH is the volume form. From now on, risking some notational confusion, we will refer to this factor in the measure as $dVol(H)$, and reserve the notation dH for something else we will encounter in a moment.

This volume form can be written in terms of the metric in our chosen coordinates,

$$dVol(H) = \sqrt{|\det g|} \prod_{i,j} dH_{ij}, \tag{13}$$

where the metric is

$$\begin{aligned} ds^2 &= \sum_i dH_{ii}^2 + \sum_{i<j} dH_{ij} dH_{ij}^* \\ &= \mathrm{Tr}[\ dH dH] \end{aligned} \tag{14}$$

Here dH now means the matrix of one forms dH_{ij}, *not* the volume form.

Now we simply change coordinates to $\{E_i\}$ and U, and ask what the metric is. Since the metric is simple in terms of dH, let's write dH in our new coordinates.

$$dH = d(U^\dagger \Lambda U) = dU^\dagger \Lambda U + U^\dagger d\Lambda U + U^\dagger \Lambda dU. \tag{15}$$

Noting that we can also write the line element as $\mathrm{Tr}[(UdHU^\dagger)^2]$, we conjugate this expression to find

$$\begin{aligned} UdHU^\dagger &= d\Lambda + UdU^\dagger \Lambda + \Lambda dU dU^\dagger \\ &= d\Lambda + [\Lambda, dUU^\dagger], \end{aligned} \tag{16}$$

where to get to the second line we used the fact that $0 = d(UU^\dagger) = dUU^\dagger + UdU^\dagger$.

Squaring this and taking the trace to get the line element, we find

$$ds^2 = \mathrm{Tr}[(UdHU^\dagger)^2] = \mathrm{Tr}[d\Lambda^2] + \mathrm{Tr}\big[[\Lambda, dUU^\dagger]^2\big]. \tag{17}$$

Cross terms between $d\Lambda$ and the commutator terms vanished because $[\Lambda, d\Lambda] = 0$.

Now we use the fact that $\mathrm{Tr}[d\Lambda^2] = \sum_i dE_i^2$, and $[\Lambda, dUU^\dagger]_{ij} = (E_i - E_j)(dUU^\dagger)_{ij}$ to find

$$ds^2 = \sum_i dE_i^2 + \sum_{i<j} (E_i - E_j)^2 (dUU^\dagger)_{ij} (dUU^\dagger)_{ji} \tag{18}$$

We can think of these coordinates as a version of polar coordinates on the space of matrices, with the differences between eigenvalues analogous to radial coordinates. The area of a surface with fixed eigenvalues, swept out by the unitaries, goes to zero as two eigenvalues approach each other.

For the volume form, we simply take the determinant of this metric to find

$$dVol(H) = \prod_{i<j} (E_i - E_j)^2 \ \prod_i dE_i \ \times \ (dU)_{Haar} \tag{19}$$

where $(dU)_{Haar}$ is the volume form on the unitary group with the Haar measure, $ds_{Haar}^2 = \mathrm{Tr}[(dUU^\dagger)^2]$, the unique metric on $U(L)$ invariant under left and right multiplication by unitaries,

$U \to U'UU''$. So our volume form is indeed analogous to the flat space measure in polar coordinates, with the factor of the Haar measure playing the role of the factor of the sphere measure in ordinary polar coordinates.

Let's now focus on the factor of the volume form involving the eigenvalues ("radial coordinates"):

$$\prod_{i<j}(E_i - E_j)^2 \prod_i dE_i = \Delta(\{E_i\})^2 \prod_i dE_i. \tag{20}$$

Here we defined the "Vandermonde determinant" $\Delta(\{E_i\}) \equiv \prod_{i<j}(E_i - E_j)$.

As an exercise, I encourage the interested reader to repeat this computation for the GOE, where one finds

$$dVol(H) = |\Delta(\{E_i\})| \prod_i dE_i \ (dO)_{Haar} \tag{21}$$

where now the Vandermonde appears at the first power, and the measure includes a factor of the Haar measure for the orthogonal group. For the GSE, one finds something similar, with the Vandermonde appearing to the fourth power.

Now, going back to the GUE, let's again point out the fact that the volume form goes to zero as any two eigenvalues approach each other, similar to how the volume form in ordinary polar coordinates goes to zero at the origin. This tells us something very siginificant:

Eigenvalues of random matrices repel - they have vanishing probability of being degenerate (in the absence of any symmetries), and have arbitrarily small probability of being arbitrarily close

Note that this fact came from the piece $dVol(H)$ in the probability measure for H. That means that we can replace $P(H)$ with a wide variety of functions of H, and we will still find this (very short ranged) repulsion between eigenvalues. This appearence of the Vandermonde in the measure for the eigenvalues is the source of all the universal behavior in the statistics of eigenvalues of random matrices, and the essential differences between the symmetry classes is due to the different power at which the Vandermonde appears.

So what are the universal features of the eigenvalue statistics, determined by this Vandermonde? The overall "shape" of the distribution of eigenvalues will end up depending strongly on our particular choice of $P(H)$, which is intuitively clear. But certain statistics involving the *spacings between nearby eigenvalues* will not.

The simplest example is the distribution of nearest-neighbor spacings $|E_{i+1} - E_i|$, where we take the eigenvalues to be ordered $E_{i+1} \geq E_j$. To get an idea of how these spacings are distributed, let's take a look at the simplest case, $L = 2$, so that we only have two eigenvalues. Call the spacing $E_2 - E_1 = s$, and the sum $E_1 + E_2 = x$. Then

$$\begin{aligned} P(H) &\propto \exp\left[-\frac{L}{2\sigma^2}(E_1^2 + E_2^2)\right] \\ &= \exp\left[-\frac{2}{\sigma^2}(s^2 + x^2)\right] \end{aligned} \tag{22}$$

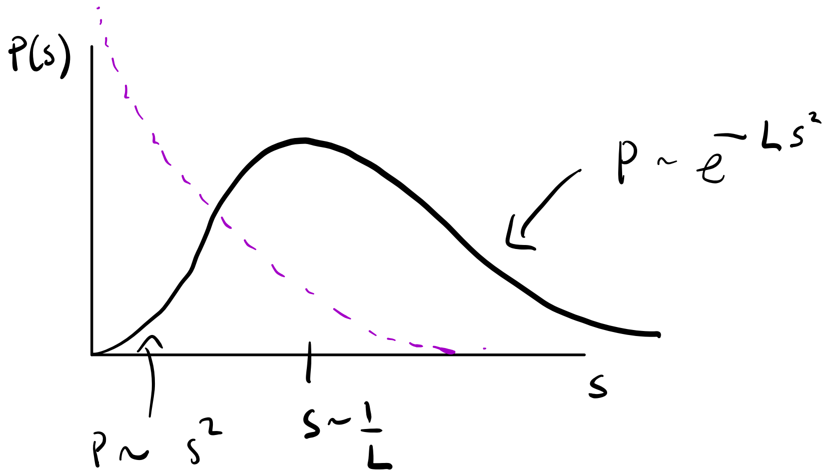
In the second line we used $L = 2$.

This gives $P(H)dH \propto \exp\left[-\frac{L}{\sigma^2}(s^2+x^2)\right]s^2 ds dx (dU)_{Haar}$. We can then integrate out x and U to find the measure for just s. Since the measure already factorizes, this means we just focus on the factor involving s, and find

$$P(s)ds \propto s^2 e^{-\frac{2}{\sigma^2}s^2} ds \tag{23}$$

The probability goes to zero like s^2, due to the Vandermonde determinant, and for large s decays as a Gaussian due to our choice of a Gaussian distribution.

It is much more difficult to study this problem for large L, but the same distribution holds approximately.



The probability of finding a nearest neighbor difference s goes to zero as s^2, due to the Vandermonde, and the probability of a large difference decays in a way depending on the choice of $P(H)$. In the GUE, it is Gaussian. The distribution has a peak at the mean nearest neighbor spacing, which is of order $1/L$ - this tells us that the L eigenvalues tend to be spread out over a range of order one (this is due to our explicit factor of L in Gaussian for $P(H)$).

Note that this is a very different distribution than what one would find if the eigenvalues were drawn independently. In this case, there would be no repulsion, and near-degeneracies would be likely. We have pictures the nearest neighbor distribution in this case with the purple dotted line.

This shape, describing $P(s)$, is called the "Wigner surmise", as it was surmised by Wigner to describe the spacings of energy levels of heavy nuclei (using the appropriate symmetry class). A histogram of the nearest neighbor spacings at high enough energies will start to approximate this shape. This is the context in which random matrix universality for chaotic systems was first discovered.

3.3 Statement of RMT universality

There is a lot of evidence, both experimental and numerical, that in *almost any* chaotic system, the distribution of nearest neighbor energy levels will closely follow the Wigner surmise, with appropriate generalizations for the GOE and GSE symmetry classes.[6]. To be slightly more precise, the Wigner surmise features a peak at the mean level spacing - we should then look at an energy window in which the mean level spacing in subsets of the window does not change very much. In different energy windows, the mean spacing may be different, but one still expects the nearest neighbor distributions to follow the Wigner surmise with that spacing.

[6]Of course, we must restrict our attention to a charge sector if there are symmetries.

This statement is the simplest case of **Random Matrix Universality** (or RMT universality, with RMT being random matrix theory). This sort of universality is remarkably universal, because the only thing it depends on is the symmetry class of the theory - One could look at two very different looking systems, in different dimensions, with totally different degrees of freedom, and if they are both chaotic and are in the same symmetry class, one expects to find the same shape of the distribution of nearest neighbor spacings.

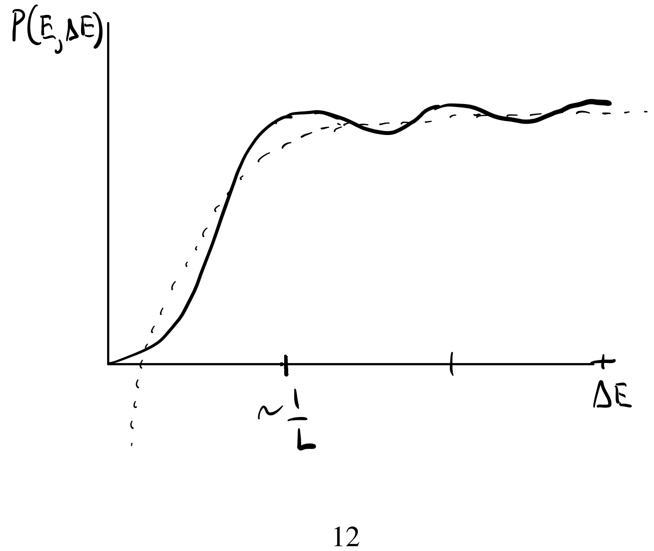
Now, let's take a moment to asky "why" the Hamiltonian of a definite system, which is *not* a random matrix, is supposed to have energy levels which are in some way distributed like those of a random matrix. I cannot give a satisfying answer to this question, as we do not have a general understanding of why RMT universality holds so well, but an intuitive picture to keep in mind is this: In order for a matrix H to have nearly degenerate eigenvalues, the matrix elements have to be finely tuned - this can be stated technically by saying that when we put a flat measure on the space of matrix elements, matrices with near-degeneracies have small measure. Then RMT universality is telling us something about how for chaotic systems, the Hamiltonian is not "finely tuned" in the sense needed to produce near degeneracies.

Note that in many chaotic systems of interest, the Hamiltonian can be highly structured, and feature no random numbers. Individual matrix elements of the Hamiltonian, in some simple basis, can be very simple, not random-looking things at all, described by simple functions of the parameters of the theory. But the eigenvalues can still end up being extremely complicated functions of these parameters. So the "fine tuning" of the matrix elements of the Hamiltonian we see for the nice systems that we often study is not the sort of fine tuning we mean when talking about eigenvalue repulsion.

Turning back to our statement of RMT universality, we now actually want to make a stronger statement than the statement we just gave about the nearest neighbor spacings. This stronger statement will involve the distribution of energy differences $|E_i - E_j|$, for E_i and E_j any two energy levels that are "sufficiently close". To make our statement, we introduce the "level pair correlator"

$$P(E, \Delta E) = \text{Probability of having energy levels } E_1, E_2 \text{ with } E = \frac{E_1 + E_2}{2} \text{ and } \Delta E = E_1 - E_2 \tag{24}$$

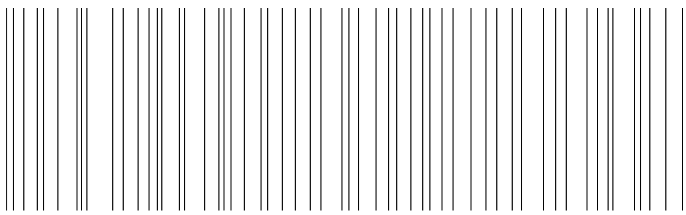
We will later compute this correlator in the GUE and find a function which looks like



This function has one parameter, which one can think of as the location of the first peak.

For small ΔE, the behavior is dominated by the distribution of the nearest neighbor spacings, which vanishes like ΔE^2, as we have seen. There is a peak at $\Delta E \sim 1/L$, representing the average nearest neighbor spacing. There are also peaks that are less pronounced at multiples of this spacing. If one takes this oscillating function and averages over the oscillations, as shown with the dotted lines, one finds a smooth behavior like $\sim -1/\Delta E^2$. This represents a long-ranged anticorrelation between eigenvalues - not only do nearest neighbors repel, but far away eigenvalues repel. This indicates a somewhat rigid distribution of the eigenvalues, so that they are nearly evenly spaced, and is often referred to as "spectral rigidity".

Below we have pictured a distribution of energy levels exemplifying this behavior - these are energy levels for a typical instance of the SYK model. Note the almost regular spacing.



So now we are prepared to make the strong statement of RMT universality. First we state it for the GUE symmetry class:

For a chaotic system in the GUE symmetry class, for energy levels in a window in which the average density of energy levels is approximately constant, the differences of energy levels are distributed according to the GUE expression for $P(E,\Delta E)$[7], with the appropriate mean level spacing inputted, up to energy differences $\Delta E \sim E_{Th}$, the "Thouless energy".

There are analogous statements for the GOE and GSE symmetry classes[8], using the GOE and GSE functions $P(E,\Delta E)$. We will also later find an alternative version of this statement which has some advantages.

Now, to understand this statment better, the next thing we will do is compute $P(E,\Delta E)$ in the GUE.

3.4 GUE level statistics

We begin our study of level statistics in the GUE by introducing the density of states

$$\rho(E) = \sum_{i=1}^{L} \delta(E - E_i). \tag{25}$$

It's also often useful to use a normalized density $\tilde{\rho}(E) = \frac{1}{L}\rho(E)$.

Then

$$\langle \tilde{\rho}(E) \rangle_H \equiv \int dVolH \; P(H) \; \tilde{\rho}(E), \tag{26}$$

where $P(H)$ is normalized so that $\langle 1 \rangle_H = 1$, is the (normalized) average density of energy levels, or in other words the probability of there being an energy level with energy E.

[7] For E chosen near the middle of the GUE spectrum

[8] As well as for the seven "Altland-Zirnbauer" ensembles relevant for SUSY syestems, which we have not discussed.

The unnormalized average density of states can be thought of in terms of the microcanonical entropy of the system,

$$\langle \rho(E) \rangle_H = e^{S(E)}. \tag{27}$$

Our goal is to understand the behavior of the pair correlation function, which we can express as $\langle \tilde{\rho}(E_1)\tilde{\rho}(E_2)\rangle_H$, but to get oriented it is useful to first compute the average density in the GUE.

The first step to computing the average density is to integrate out the unitaries in 26. Writing the measure in terms of the eigenvalues and unitaries using 19, we find

$$\begin{aligned}\langle \tilde{\rho}(\widehat{E})\rangle_H &\propto \int d\{E_i\}\, \Delta(\{E_i\})^2 e^{-\frac{L}{2}\sum_i E_i^2}\, \tilde{\rho}(\widehat{E}) \\ &= \int d\{E_i\}\, e^{-L^2 I_{eff}(\{E_i\})}\, \tilde{\rho}(\widehat{E}). \end{aligned} \tag{28}$$

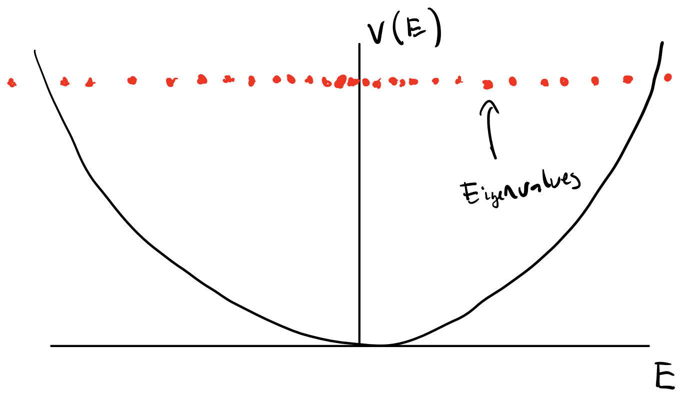
Here for later notational simplicity, we have chosen to look at the density at energy $\widehat{E}$, and chosen $\sigma = 1$.

In the second line, we introduced an effective action for the eigenvalues made from the Gaussian part of the measure as well as the Vandermonde,

$$I_{eff}(\{E_i\}) = -\frac{1}{2L^2}\sum_{i\neq j}\log|E_i - E_j|^2 + \frac{1}{2L}\sum_i E_i^2 \tag{29}$$

The effective action is normalized so that every sum over L eigenvalues is acompanied by a factor of $1/L$.

We can think of the eigenvalues as particles on a line with coordinates E_i subject to two forces in this effective action. One is a quadratic potential for the eigenvalues, $V(E) = \frac{1}{2}E^2$, and a repulsive force between eigenvalues coming from the Vandermonde.



This technique, of treating the matrix integral like it is describing a system of particles, is known as the "Coulomb gas" or "fluid" technique, and is due to Dyson.

3.4.1 Saddle point evaluation

At large L, the integral 28 looks amenable to doing by saddle point, by finding the lowest action configuration of the eigenvalues.[9]

A useful technical step is to introduce the "resolvent",

$$R(\widehat{E}) \equiv \frac{1}{L}\sum_i \frac{1}{\widehat{E}-E_i} = \int dE \frac{\tilde{\rho}(E)}{\widehat{E}-E}. \tag{30}$$

The resolvent has poles at $\widehat{E}=E_i$ and satisfies

$$R(\widehat{E}+i\epsilon)-R(\widehat{E}-i\epsilon) = -2\pi i\tilde{\rho}(\widehat{E}), \tag{31}$$

where we used $\frac{1}{\widehat{E}\pm i\epsilon - E} = \mathcal{P}\frac{1}{\widehat{E}-E} \mp i\pi\delta(\widehat{E}-E)$.

Now we vary the effective action 29 with respect to an energy E_i and find the saddle point equation for each i

$$E_i = \frac{1}{L}\sum_{j\neq i}\frac{1}{E_i-E_j}. \tag{32}$$

A trick to solving this is to multiply by $\frac{1}{L}\frac{1}{\widehat{E}-E_i}$ and sum over i, to get

$$\frac{1}{L}\sum_i \frac{E_i}{\widehat{E}-E_i} = \frac{1}{L^2}\sum_{i\neq j}\frac{1}{\widehat{E}-E_i}\frac{1}{E_i-E_j}. \tag{33}$$

With some rearranging (Homework for an interested reader), this becomes

$$R(\widehat{E})^2 - \widehat{E}R(\widehat{E}) + 1 = \frac{1}{2L}\frac{d}{d\widehat{E}}R(\widehat{E}). \tag{34}$$

At large L we can naively drop the right hand side of this equation and find a simple quadratic equation for $R(E)$, which we solve to find

$$2R(\widehat{E}) = \widehat{E} - \sqrt{\widehat{E}^2-4}. \tag{35}$$

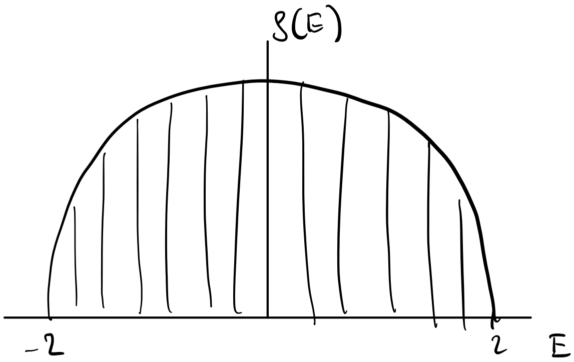
The sign of the square root was picked out by demanding that $R(\widehat{E}) \sim 1/\widehat{E}$ for large $\widehat{E}$, which is clear from its definition.

Earlier, we saw that the resolvent for a fixed set of eigenvalues is supposed to have poles at the location of the eigenvalues. Now we have tried to compute the resolvent for the saddle point configuration of the eigenvalues, and found a function that has a square root branch cut from $\widehat{E}=-2$ to 2. Essentially what has happened is that in dropping the RHS of 34, we aren't landing exactly on the true saddle point configuration of the eigenvalues, but instead some smeared out version. Smearing the poles in the resolvent leads to this branch cut.

Now, using 31, we find the (smeared) saddle point density

$$\tilde{\rho}_0(\widehat{E}) = \begin{cases} \frac{1}{2\pi}\sqrt{4-\widehat{E}^2}, & -2<\widehat{E}<2 \\ 0 & |\widehat{E}|>\sqrt{2} \end{cases} \tag{36}$$

[9]This is tricky, as the action is of order L^2 but there are there are L variables, so there are fluctuations around the saddle point with a small change in action which nontheless are very different than the saddle point configuration. However, using the saddle point is still useful, and once we change variables to the smooth density, this will no longer be an issue.



This is known as the "Wigner semicircle". I recommend verfying this on Mathematica, by generating a random GUE matrix and plotting a histogram of the eigenvalues.

Before we move on to the fluctuations around the saddle point, let's take a moment to note that the specific semicircular shape of the average density corresponds to our specific Gaussian distribution. If we had used a more general unitarily invariant measure, specified by a potential $V(E)$ for the eigenvalues, they would form an overall different shape. However, qualitatively, generically the same things would happen- by solving the saddle point equations and dropping the derivative of the resolvent, we would find a saddle point function for the resolvent with a square root branch cut, leading to a density of states that has a square root behavior near the edges.

3.4.2 Fluctuations

Now let's study the fluctuations around this saddle point for the density, which is necessary for computing the pair correlation function of the eigenvalues.

Motivated by the apperance of a smooth saddle point for the density of states, we rewrite the effective action 29 as

$$I_eff = -\int dEdE'\ \tilde{\rho}(E)\tilde{\rho}(E')\ \log|E-E'| + \frac{1}{2}\int dE\tilde{\rho}(E)E^2. \tag{37}$$

This is an exact rewriting if $\tilde{\rho}$ is a sum of delta functions, but now we want to view it as a smooth approximation to this sum of delta functions, given by the saddle point value plus flucutations, $\tilde{\rho}(E) = \tilde{\rho}_0(E) + \delta\tilde{\rho}(E)$.

Using the saddle point equations, we find[10]

$$I_eff = I_eff[\tilde{\rho}_0] - \int dEdE'\ \delta\tilde{\rho}(E)\delta\tilde{\rho}(E')\log|E-E'|. \tag{38}$$

Note that action is quadratic in the density and its fluctuations, and this would be the case even if we chose a non-quadratic potential. Also, the quadratic term in the fluctuations is independent of our choice of potential. This is a reflection of RMT universality- the saddle point depends on the choice of distribution, but the fluctuations, which describe the nearby correlations, are universal and come from the Vandermonde.

[10]We must be careful to take the principal part of the logarithm, so it does not include the $E_i = E_j$ singularity.

From here we go to fourier space, introducing $\delta\tilde{\rho}(s) = \int dE\ e^{iEs}\delta\tilde{\rho}(E)$, and find that the propagator is diagonalized

$$I_{eff} \supset \int ds\ \delta\tilde{\rho}(s)\delta\tilde{\rho}(-s)\frac{1}{|s|} \tag{39}$$

3.4.3 Putting things together

Now we can compute both $\langle\tilde{\rho}(E)\rangle_H$ and $\langle\tilde{\rho}(E_1)\tilde{\rho}(E_2)\rangle_H$ in the GUE, by using the saddle point plus quadratic fluctuations.

The density $\langle\tilde{\rho}(E)\rangle_H$ is given by the semicircle saddle point 36 to leading order, and small fluctuations do not give a correction.

The pair correlator $\langle\tilde{\rho}(E_1)\tilde{\rho}(E_2)\rangle_H$ has a contribution from the saddle point, where we replace each factor of $\tilde{\rho}$ with the saddle point $\tilde{\rho}_0$, plus a propagator contribution. The propagator contribution is suppressed by $1/L^2$ because the effective action is multiplied by L^2. We then find

$$\begin{aligned}
\langle\tilde{\rho}(E_1)\tilde{\rho}(E_2)\rangle_H \approx& \tilde{\rho}_0(E_1)\tilde{\rho}_0(E_2) + \int \frac{ds_1 ds_2}{4\pi^2} e^{is_1E_1+s_2E_2}\langle\delta\tilde{\rho}(s_1)\delta\tilde{\rho}(s_2)\rangle \\
=& \tilde{\rho}_0(E_1)\tilde{\rho}_0(E_2) - \frac{1}{4\pi^2L^2}\int ds e^{is(E_1-E_2)}|s| \\
=& \tilde{\rho}_0(E_1)\tilde{\rho}_0(E_2) - \frac{1}{2\pi L^2}\frac{1}{(E_1-E_2)^2}
\end{aligned} \tag{40}$$

Before we look at this answer more closely, we note that we did a perturbative computation in $1/L^2$. But nonperturbative effects can be important!

Computing nonperturbative corrections to this formula is a bit more involved, so here I will simply state the answer, first for $E_1 + E_2 = 0$, corresponding to energies at the center of the semicircle:

$$\langle\tilde{\rho}(E_1)\tilde{\rho}(E_2)\rangle_H = \frac{1}{\pi^2} - \frac{\sin^2\left(L(E_1-E_2)\right)}{L^2\pi^2(E_1-E_2)^2} + \frac{1}{L\pi}\delta(E_1-E_2). \tag{41}$$

This is often known as the "sine kernel formula". Here we used the fact that $\tilde{\rho}_0 \approx \frac{1}{\pi}$ in the middle of the spectrum.

Using $2\sin^2(x) = 1-\cos(2x)$, we recover our perturbative contributions, plus the delta function (which simply indicates that if there is a level at E_1, there is also that same level at $E_2 = E_1$), and a rapidly oscillating piece

$$\langle\tilde{\rho}(E_1)\tilde{\rho}(E_2)\rangle_H \supset \frac{1}{2\pi^2L^2}\frac{\cos\left(2L(E_1-E_2)\right)}{(E_1-E_2)^2}. \tag{42}$$

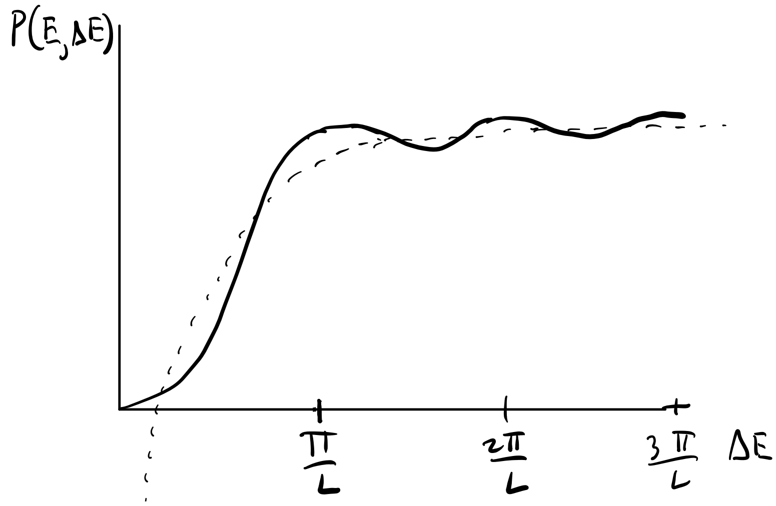
We can think about the cosine as the real part of $e^{i2L(E_1-E_2)}$, which is nonperturbative in $1/L$.

We also note that for small $|E_1 - E_2| \ll 1/L$, we can linearlize the sine in 41, and drop the delta function to find that the first term, from the average density, cancels off the second term. This means that the pair correlator goes to zero as eigenvalues approach each other. The frequency of the oscillating term is precisely tuned for this cancellation. Remembering that the unnormalized density is $\rho_0 = \frac{1}{\pi}$, we can generalize this formula to energies away from $E = 0$ (but not too close to the edge of the spectrum). Now using unnormalized densities, for small $\Delta E = E_1 - E_2$

$$\langle\rho(E_1)\rho(E_2)\rangle_H = \rho_0(E_1)^2 - \frac{\sin^2\left(\pi\rho_0(E)\Delta E\right)}{\pi^2\Delta E^2} + \rho_0(E)\delta(\Delta E), \qquad \Delta E \ll 1. \tag{43}$$

Remembering that ρ_0 is of order L, we see that the first term is of order L^2 while the second two terms are suppressed in $1/L$. The oscillations are tuned so that linearlizing the sine cancels the first term, and the peaks in the oscillations are spaced by mutliples of $1/\rho_0(E)$. As we can identify $1/\rho_0(E)$ as the mean level spacing, the peaks in the pair correlator can be thought of as telling us about the discrete spacings between eigenvalues. This is roughly why our perturbative computation could not see these nonperturbative oscillating pieces - we dropped the derivative of the resolvent in our saddle point computation by assuming it was small. For the smeared resolvent we ended up finding, this is true, but for the true resolvent, the discrete eigenvalues invalidate this approximation.

Finally, let's show a plot of the GUE pair correlator, again looking at $E = 0$.



Here we can see the features discussed earlier. The dotted line represents the perturbative contribution, $\sim -1/\Delta E^2$, giving the long-ranged "spectral rigidity". On top of this smooth long ranged piece, we have rapid oscillations indicating the underlying discrete spectrum.

3.5 Final remarks on RMT universality

The statement of RMT universality is that for a chaotic system with many energy levels, the sine kernel is a good approximation to the pair correlation function, up to some $\Delta E = E_{Thouless}$. Roughly, if one plots a histrogram of the energy differences, it will approximate the sine kernel.[11] This statement holds for systems in the GUE symmetry class, but there are simple generalizations to the other symmetry classes, using generalizations of the sine kernel formula.

This statement of RMT universality is often known as the Bohigas, Giannoni, Schmit (BGS) conjecture.

Though we do not have a full understanding of why the BGS conjecture, or versions of this conjecture, should be true, the are experimental studies, many numerical studies in simple systems

[11]To do this, one must take into account that the average density of levels changes, and thus the mean spacing, and frequency of oscillation in the sine kernel, changes too. Instead it is useful to plot a histogram of "unfolded" differences, $x_{ij} = \rho_0(E)|E_i - E_j|$.

like the SYK model, [9, 10], Flouqet systems [11], and periodic orbits [12, 13] supporting the conjecture.

4. Black holes and RMT level statistics

As we discussed at the beginning of the lecture, black holes are chaotic systems. This suggests that their energy levels should have RMT statistics!

Now, we must be careful in making this statement: many black holes do not correspond to energy eigenstates, as they are unstable and evaporate. But this statement can be made more precise by studying large black holes in AdS, which are dual to high energy states in the boundary CFT.[12]

What is the gravitational description of this RMT behavior of the enegy levels of these black holes? This is the main question we will try to address for the rest of these notes.

If one were infinitely powerful, they could directly compute the energy levels in the boudnary CFT and make the histograms we discussed, or do some sort of averaging to compute the average pair correlator. But for non-supersymmetric black holes, like the AdS Schwarzschild black hole, we have no way using the bulk to compute the energy levels, so this is not a useful direction.

Instead, rather than computing the energy levels, or even the pair correlator, directly, we will compute an auxiliary object called the "spectral form factor". This object can be used to rephrase the statement of RMT universality in a simple way.

4.1 The spectral form factor

For our purposes, we will define the spectral form factor, for a system of possibly infinitely many energy levels, as

$$\begin{aligned} SFF(t) &\equiv Z(\beta+it)Z(\beta-it) \\ &= \sum_{i,j} e^{-(\beta+it)E_i} e^{-(\beta-it)E_j}. \end{aligned} \tag{44}$$

Here $Z(\beta)$ is the spectral form factor, and in our applications we want to take β to be above the Hawking-page transition.

We can think of this as something like a Fourier transform and Laplace transform of two copies of the density of states $\rho(E_1)\rho(E_2)$ with respect to ΔE and E respectively

$$SFF(t) = \int dE d\Delta E \; \rho(E_1)\rho(E_2) e^{-2\beta E} e^{-it\Delta E}. \tag{45}$$

It is often useful to consider alternate versions of the spectral form factor, where instead of fixing the temperature, we fix the average energy E within some window, perhaps with a smooth window function like a Gaussian. For an infinitely sharp window, this is simply

$$\propto \int d\Delta E \; \rho(E_1)\rho(E_2)\big|_{E_1+E_2=2E} e^{-it\Delta E}, \tag{46}$$

[12] Perhaps by better understanding the gravitational origin of RMT statistics for large black holes in AdS, one could find a useful analog of RMT level statistics for evaporating black holes.

which of course is zero unless E is one of the energies of the system, but in pratice we always use a finite window for E containing many energy levels.

Now, to make contact with our statement of RMT universality, we can ask: "what is the spectral form factor, for H averaged over the GUE?"

To do this we can simply take our formula 45, average the product of densities, and use our GUE sine kernel formula. It's useful to first focus on fixed average energy E, leaving the E integral to the end (since we are averaging, the SFF is nonzero for any fixed E).

For simplicity, let's take $E = 0$, and so we find

$$SFF\big|_E(t) = \int d\Delta E\ e^{-i\Delta E t}\left(L^2\frac{1}{\pi} - \frac{1}{2\pi^2\Delta E^2} + \frac{\cos(2L\Delta E)}{2\pi^2\Delta E^2} + \frac{L}{\pi}\delta(\Delta E)\right) \tag{47}$$

The first contribution in the parentheses, from the product of two average semiciricle densities, gives a delta function in t. Really, to compute the full SFF, we should think about this contribution as

$$\int dE_1 e^{-(\beta+it)E_1}\rho_0(E_1) \times \int dE_2 e^{-(\beta-it)E_2}\rho_0(E_2) \tag{48}$$

This is just the modulus squared of $\langle Z(\beta+it)\rangle$, the analytically continued partition function of the GUE, averaged over the ensemble. This decays in t, and provides the factorized part of the average spectral form factor, which we refer to as the "slope".

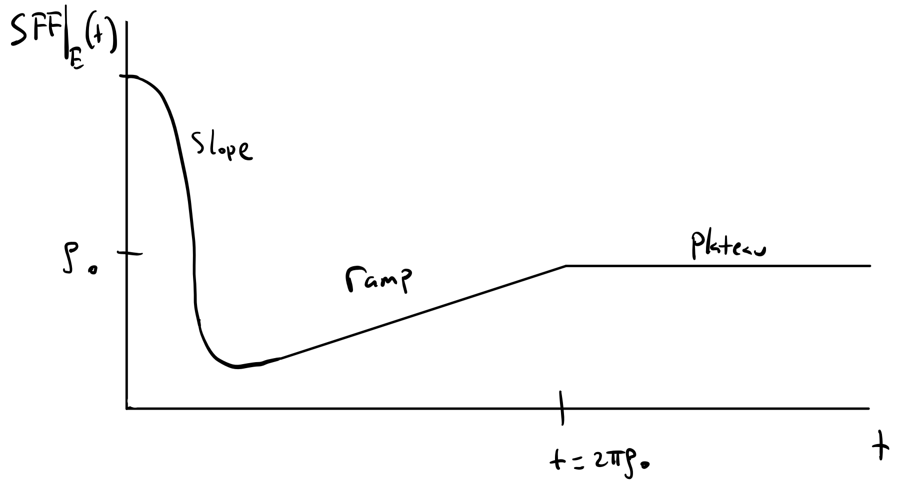
The more interesting contributions come from the connected part of the average, which give the remaining terms in the parentheses. These are the terms related to RMT universality.

Now looking at the second term in the parentheses in 47, we find

$$SFF\big|_E(t) \supset \frac{t}{2\pi} \tag{49}$$

This linear contribution to the spectral form factor is knows as the "ramp". It is independent of E, so doing the E integral with the Boltzmann weight is simple.

The last two terms, coming from nonperturbative effects, together are zero up until $t = 2\pi\rho_0(E) \sim L$. After that, they cancel the growing ramp so that, subtracting the factorized contribution, the SFF has a constant "plateau". The ramp grows like $t/2\pi$ until $t = 2\pi\rho_0(E)$, so the plateau has a height $\rho_0(E)$. The sudden transition at $t = 2\pi\rho(E)$ is due to the rapid oscillations in the cosine, with frequency $\rho(E)$.

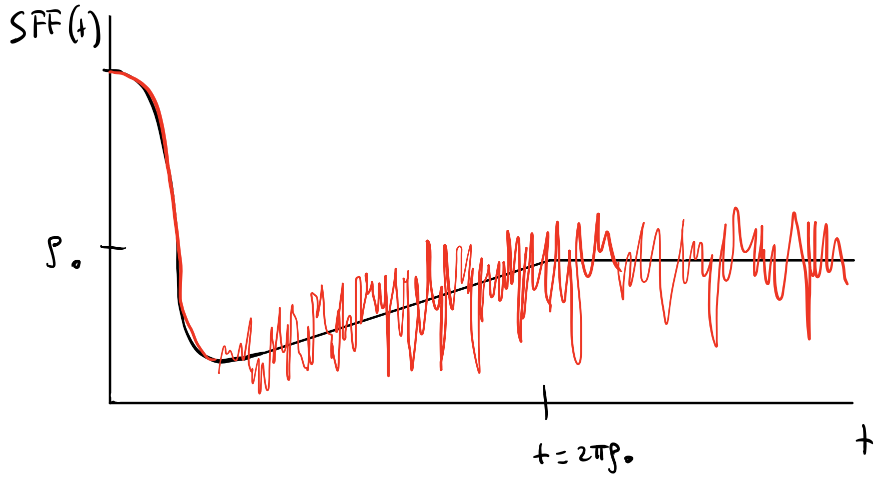


Here we have plotted a log-log plot of the spectral form factor, for some approximately fixed average energy E. As the plateau time depends on $\rho(E)$, which depends nontrivially on E, integrating over E smooths out the sharp ramp-plateau transition a bit. But for approximately fixed energy, we draw the transition as sharp. Note that the plateau and ramp are smaller than the $t = 0$ value of the spectral form factor by factors of $1/L$. In gravity, this factor becomes of order $1/\rho_0$, or e^{-S}, where S is the entropy. So the ramp and plateau are small nonperturbative effects, and are only visible because at late times the ordinary black hole contribution has decayed.

It is important to note that the behavior of the initial decaying slope depends on the shape of $\rho_0(E)$, and the energy window chosen for E. Roughly, through 48, the slope is like the square of the fourier transform of $\rho_0(E)$. For a smooth function, the fourier transform decays, and so the slope decays. But $\rho_0(E)$ for the GUE has sharp square-root behavior, at the edge of the spectrum. This leads to a slow power-law decay if the energy window includes the edge, as it would with our initial definition of the SFF 45.

Now, in order to use the SFF to make a statement about random matrix universality, we can as what the spectral form factor looks like for a fixed Hamiltonian. For example, what does it look like for a fixed draw from the GUE? I recommend that an interested reader try doing this themselves - Mathematica has some useful features for generating GUE matrices.

We have pictured a schematic log-log plot of this below, with $\beta = 0$



Here, at $t = 0$ the SFF is normalized to $Z(0)^2 = L^2$. It initially decays, before becoming small and oscillating erratically. The time at which this happens can be made short if we choose an energy window in the middle of the spectrum. The function oscillates rapidly around an average ramp and plateau. The oscillations depend sensitively on the specific choice of H, and thus can be averaged out to reveal the ramp and plateau. One could average over H drawn from the GUE, but one could also average over small time windows for a fixed H - at large L, the correlation time of the noise (how long in time one must go for the noise to be decorrelated), is short compared to the length of the ramp, so that one may average over time windows larger than the correlation time, but still have many windows along the ramp to average over.

Again, I recommend that an interested reader try this themselves. One can study the correlation time by computing things like $\langle SFF(t)SFF(t')\rangle$. This involves knowing higher order correlations of the density, but these are Gaussian variables with the pair correlation described by the sine kernel, so one just needs the sine kernel to compute this.

It is also useful to verify that the size of the noise is about the size of the average itself - that means the average ramp and plateau are not a good approximation to the noisy function!! I also recommend that the reader verify this ($\langle SFF(t)^2\rangle = 2\langle SFF(t)\rangle^2$) by computing the variance of the spectral form factor, using the Gaussian density correlators.

The fact that the ramp and plateau can be revealed upon time averaging (or perhaps by some other sort of averaging) even for a fixed system, not drawn from an ensemble, makes the spectral form factor a useful object for reformulating our statement of random matrix universality. Including some caveats about charge sectors and energy windows, our statement for the GUE symmetry class is:

For a chaotic system with many energy levels, the spectral form factor (in a fixed charge sector and normalized enegy window), averaged over sufficiently large time windows, has the ramp-plateau structure exhibited by the GUE, with a ramp of slope $1/2\pi$ ending at a plateau at time $2\pi\rho_0(E)$. Depending on ρ_0 or the choice of energy window, the ramp may extend at early times until the slope takes over, or until some t_{th} corresponding to the Thouless energy scale at which RMT universality becomes invalid.

This statement has generalizations for the GOE and GSE symmetry classes, with their corresponding ramp-plateau structures. In both those cases, the ramp is not linear at late times, and the "plateau" only approaches a constant at late times.

5. The ramp from a spacetime wormhole

In the last section of these notes, we will describe the computation of the spectral form factor for large black holes in AdS, and compute the ramp from a spacetime wormhole. At the end we will discuss remaining questions, such as the origin of the plateau, and the noise around the ramp and plateau.

Before we begin with the computation, let's set some expectations. First of all, to see the ramp in the spectral form factor, we need to do some sort of averaging. We will find, somewhat confusingly, that the gravitational path integral seems to give the ramp without averaging. We'll remark on that point later, but for now we will just compute and see what we get.

Second, in the GUE, the ramp came from a perturbative effect in the $1/L$ expansion, and the plateau came from a nonperturbative effect. More generally, we should think of L like $e^{S(E)}$, exponential in the entropy. In gravity, the entropy is like $\sim 1/G_N$, so L is exponentially large in $1/G_N$. That means that the $1/L$ expansion is an expansion in e^{-1/G_N}, so perturbative effects in $1/L$ are nonperturbative in $1/G_N$. Then the nonperturbative effects in $1/L$ are *doubly nonperturbative* in $1/G_N$!!

We know how gravity can produce some nonperturbative effects; additional saddle points in the gravity path integral are a simple example. But doubly nonperturbative effects are more confusing, and thus the plateau is a bit more difficult to understand.

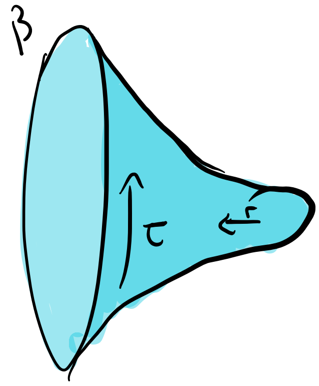
5.1 Computing partition functions in gravity

Part of why we are focusing on the spectral form factor is that it is relatively straightforward to study in gravity, as compared to the density correlator. Our definition of the spectral form factor 45 is the square of the analytically continued partition function, and the partition function is a simple object in the gravity path integral.

Let's now briefly review the computation of just an ordinary partition funcition $Z(\beta)$, in AdS/CFT.

In regular quantum mechanics, the partition function $\mathrm{Tr}[e^{-\beta H}]$ can be written as a path integral which is periodic in Euclidean time, $\tau \sim \tau + \beta$. But in bulk calculations in AdS/CFT, this means that we compute partition functions using the gravity path integral by fixing the metric on the asymptotic boundary to be periodic in Euclidean time, with a (renormalized) length β. Then we sum over all spacetimes with these boundary conditions, not making any restrictions on the interior of the spacetime.

For example, for temperatures above the Hawking Page transition, the Euclidean black hole dominates. Here we have drawn a picture of this geometry for 2d gravity, or for higher dimentional gravity, suppressing the angular directions. Note that the thermal circle has (renormalized) length β at the boundary, but is contractible in the middle.



The method we know how to use for computing the partition function using the gravity path integral is to sum over saddle points, and include small fluctuations around them.

To compute $Z(\beta + it)$, we can take the answer we compute for real β and analytically continue the result. If we do this for the black hole contribution, we find an initial decay [10]. Squaring this to find a contribution to the spectral form factor, this is part of the slope. One must be careful though, as the analytic continuation may lead to an exchange of dominance of saddle points; saddle points which are small in $Z(\beta)$ may give the leading contribution to $Z(\beta + it)$. This indeed happens in gravity. For example, the thermal AdS saddle point, which is subdominant in the partition function at high temperature, gives a non-decaying contribution to $Z(\beta + it)$. This is not part of the slope-ramp-plateau structure we expect, but this is because the thermal AdS contribution corresponds to contamination of the non-chaotic low-energy physics at large N in the boundary CFT - it is not physically significant for studying the black hole states, so we can ignore it. There are also other contributions which can be important, for example in 3d gravity, from other black hole saddles [14]. But these decay.

Not only do we have to worry about contributions from other saddle points, but at large t perturbation theory around a given saddle point may be difficult to control! (We still expect $Z(\beta + it)$ to decay though.)

To deal with these complications in computing the partition function, or spectral form factor, for these regimes of long time, we will focus on a particularly simple theory of two-dimensional dilaton gravity - Jackiw-Teitelboim (JT) gravity, where we have a lot of control. Then we will make statements about more general gravity.

5.2 Jackiw-Teitelboim (JT) gravity

In this section we will introduce JT gravity [15–20], in the particular case of negative cosmological constant ($\Lambda = -2$ for simplicity).

This is a theory of two-dimensional gravity, with a metric $g_{\mu\nu}$ and a dilaton ϕ. We can think of this theory as part of a dimensional reduction of a near-etremal four-dimensional black hole, with $\phi_{DR} = \phi_0 + \phi$ the area of the transverse two-sphere, where ϕ_0 sets the extremal entropy. The entropy of the black hole is proportional to the area of the horizon, so in the dimensional reduction the entropy is $S \propto \phi_0 + \phi_{horizon}$, involving the dilaton evaluated at the horizon. However, we will take this theory at face value, rather than thinking of it as coming from a dimensional reduction.

However, we will see that there is no analog of thermal AdS in this theory, the partition function is always dominated by the black hole, which can be understood physically from the dimensional reduction.

To compute the partition function for JT gravity, we do a path integral over two-dimensional Euclidean manifolds with an asymptotically AdS boundary of length β/ϵ, and with $\phi = 1/\epsilon$ at the boundary. We eventually take ϵ to zero, with counterterms in the action included to make the action finite.

The Euclidean action for JT gravity, evaluated on a manifold M has three pieces.

$$I = \underbrace{-\frac{S_0}{2\pi}\left[\frac{1}{2}\int_M \sqrt{g}R + \int_{\partial M}\sqrt{h}K\right]}_{\text{Topological term}=-S_0\chi(M)} \underbrace{-\frac{1}{2}\int_M \sqrt{g}\phi(R+2)}_{\text{Sets } R=-2} \underbrace{-\int_{\partial M}\sqrt{h}\phi(K-1)}_{\text{Gives action for boundary}}. \tag{50}$$

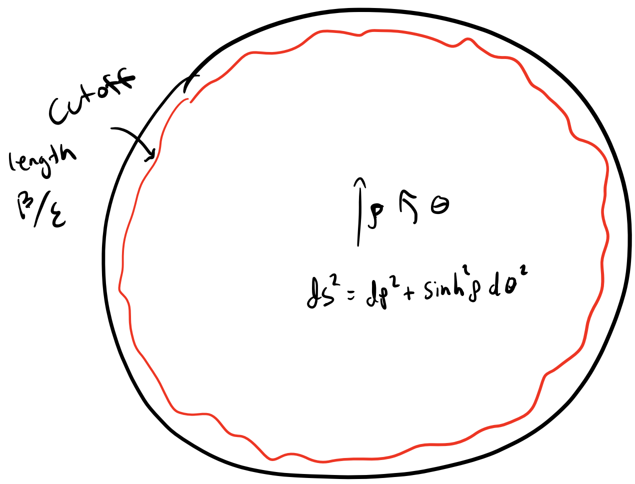
Here h is the induced metric on the boundary of the manifold, and K is the extrensic curvature of the boundary.

Note that the first term is the normal Einstein action in two-dimensions, with the GHY boundary term. This is just proportional to the Euler character of the manifold. S_0, the zero-temperature entropy, is proportional to ϕ_0 in the dimensional reduction, and is the coefficient of this topological term. Then contrbutions to the path integral are suppressed by their Euler character as $e^{S_0\chi}$, where χ is equal to one for the disk, but negative for surfaces with handles. The disk toplogy is the Euclidean black hole, and is weighted like e^{S_0}. One can see that this S_0 indeed contributes an β-independent additive term to the entropy $S(\beta)$.

The first term doesn't involve the dilaton, and is topological. The dilaton is necessary to make the theory nontrivial, but the simplicity of JT gravity comes from the second term. For our purposes, we define the path integral for JT gravity by integrating ϕ along an infinite contour parallel to the imaginary axis.[13] The integral over the dilaton the bulk then yields a delta function setting $R+2 = 0$ everywhere in the bulk. For a two-dimensional surface, this means that the curvature is everywhere constant - all surfaces that contribute to the path integral are pieces of the hyperbolic plane, or hyperbolic disk.

Finally, the last term gives all the interesting dynamics in JT gravity. Consider for example the contributions of surfaces of the disk topology. Thes are surfaces which are just pieces cut out of the hyperbolic disk, with some wiggly boundary curve, as pictured below

[13]Note that to do this we are forgetting about the origin of JT gravity from a dimensional reduction.



The boundary curve has length β/ϵ. In polar coordinates, the metric on the disk is

$$ds^2 = d\rho^2 + \sinh^2(\rho)d\theta^2. \tag{51}$$

Then we are cutting off the disk at a radius that is like $e^{\rho_c} \sim 1/\epsilon$. The rules for the path integral are that we now integrate over distinct manifolds, which correspond to distinc shapes of the boundary curve. The action for these manifolds depends on the extrinsic curvature of the boundary curve.

First, consider the saddle point contribution to this path integral. The saddle point, the Euclidean black hole, corresponds to taking a circular boundary, with constant extrinsic curvature. Note that this is the only saddle point- there is no version of therman AdS.

As we take the cutoff away, the extrinsic curvature of the circle approaches one. This is cancelled by the counterterm in the action, but we must also consider the first correction away from $K = 1$. This is going to zero like ϵ/β^2, but we are integrating the curvature over the whole boundary of length β/ϵ. The result is finite as $\epsilon \to 0$, and gives a contribution to the action $\propto -1/\beta$.

In the limit $\epsilon \to 0$, all fluctuations around the saddle point that contribute correspond to small fluctuations around the circular shape of the boundary. There are various approaches to doing the integral over fluctuations [10, 19, 21–24]. I highly recommend that an interested reader look at these papers.

The answer is somewhat remarkably one-loop exact. The reason for this has been explained by Stanford and Witten [21]. The contribution of the disk topology to the partition function is

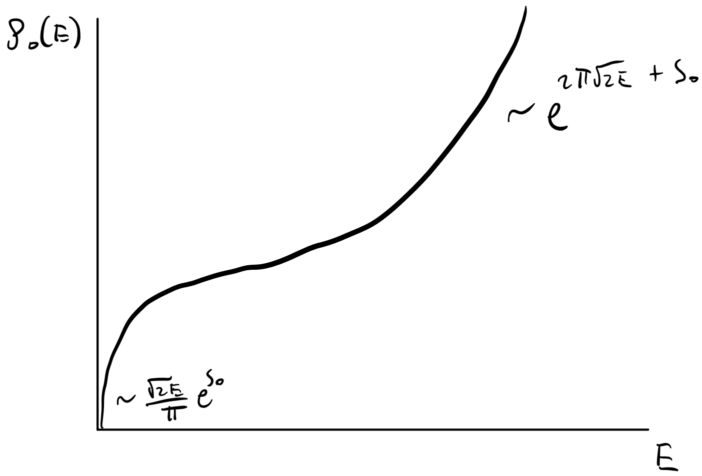
$$Z(\beta) \supset e^{S_0} \frac{1}{\sqrt{2\pi}\beta^{3/2}} e^{\frac{2\pi^2}{\beta}}. \tag{52}$$

Here the factor of e^{S_0} comes from the topological term, and the prefactor proportional to $1/\beta^{3/2}$ comes from the one-loop fluctuations.

Let's take a moment to think about the analytic continuation of this, $Z_{disk}(\beta+it)$, the square of which contributes the slope to the spectral form factor. For long times t, the exponent approaches zero, and this is dominated by the $1/(\beta+it)^{3/2} \sim 1/(it)^{3/2}$ from the one-loop fluctuations. This slow power law decay is similar to what one would find in the GUE, coming from a sharp square root edge in the spectrum.

In fact, at low energies, the JT spectrum is similar to the GUE spectrum (This has an explanation in the work [25] by myself, Shenker, and Stanford.) We can compute the leading approximation to the density of states by taking the inverse Laplace transform of the disk partition function, to find

$$\rho_0(E) = e^{S_0}\frac{1}{2\pi^2}\sinh(2\pi\sqrt{2E}). \tag{53}$$



At low energies, it behaves like $\propto \sqrt{E}$, similarly to the GUE semicircle near the edge. But at high energies, it grows exponentially like $e^{2\pi\sqrt{2E}}$, with the appropriate entropy for a near extremal 2d black hole.

5.3 The SFF in JT gravity

Now let's return to the spectral form factor, $|Z(\beta+it)|^2$. In order to compute this in JT gravity, we can first compute $Z(\beta_1)Z(\beta_2)$, and then continue $\beta_1 \to \beta + it$, $\beta_2 \to \beta - it$. To compute $Z(\beta_1)Z(\beta_2)$, we sum over spacetimes with two asymptotic boundaries of lengths β_1/ϵ and β_2/ϵ.

The spacetime that contributes with the least negative Euler character is two disconnected disks, giving a contribution which is a product of two copies of the partition function tht we just computed

$$Z(\beta_1)Z(\beta_2) \supset e^{S_0}\frac{1}{\sqrt{2\pi}\beta_1^{3/2}}e^{\frac{2\pi^2}{\beta_1}} \times e^{S_0}\frac{1}{\sqrt{2\pi}\beta_2^{3/2}}e^{\frac{2\pi^2}{\beta_2}}. \tag{54}$$

As we have seen, upon continuation this decays.

To find our non-decaying ramp, we will have to look at the contribution of other topologies.[14]

Before we go ahead and look at other topologies, let's first understand more physically *why* the spectral form factor decays at early times. To do this, it is useful to think of the spectral form factor in a somewhat more physical way, as the overlap of some states of multiple copies the black hole.

[14]Adding handles to the disk also gives decaying contributions, so we will end up looking at surfaces with both boundaries on one connected component of spacetime.

Consider the (unnormalized) thermofield double state of two copies of the boundary system in AdS/CFT

$$|TFD(\beta)\rangle \equiv \sum_i e^{-\beta E_i/2}|E_i\rangle_L |E_i\rangle_R^*, \tag{55}$$

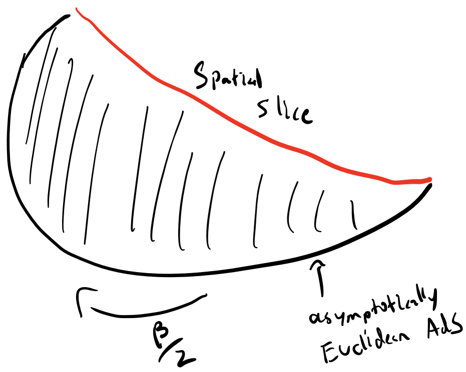
where the star denotes the complex conjugate state. In the path integral, this is prepared by Euclidean evolution over an interval of time $\beta/2$, with the two copies of the system corresponding to the two ends of the interval. This state satisfies $(H_L - H_R)|TFD(\beta)\rangle = 0$, so it is invariant under time evolution with $H_L - H_R$. And finally, the overlap of this state with itself is the partition function,

$$\langle TFD(\beta)|TFD(\beta)\rangle = Z(\beta), \tag{56}$$

which can be seen easily by using the path integral preparation of the state - gluing the two intervals of Euclidean evolution together creates a circle of Euclidean evolution of length β.

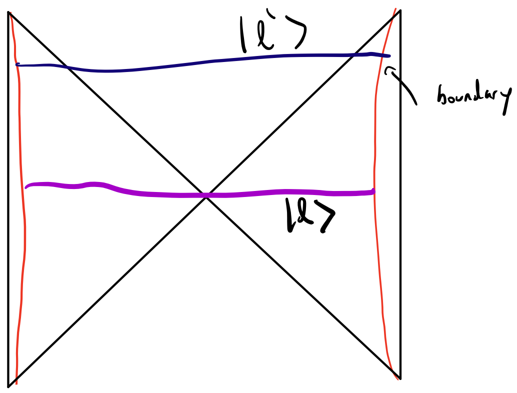
In the bulk, this state corresponds to the Hartle-Hawking state of the two-sided eternal black hole. The symmetry under evolution with $H_L - H_R$ corresponds to the Killing symmetry of the geometry, and evolution with $H_L + H_R$ corresponds to evolution in Killing time in opposite directions on each side of the black hole.

The Hartle Hawking state is prepared by the path integral which (in d=2, or suppressing angular directions in higher dimension) is an integral over a half-disk, with the circular portion of the boundary having length $\beta/2\epsilon$, representing the Euclidean evolution. The bulk portion of the boundary of the half-disk is the spatial slice on which the state is defined.



Again, we can see that the norm of this state is the partition function by using the path integral-gluing the two half-disks together along the bulk slice produces an integral over geometries with a single circular boundary of length β/ϵ.

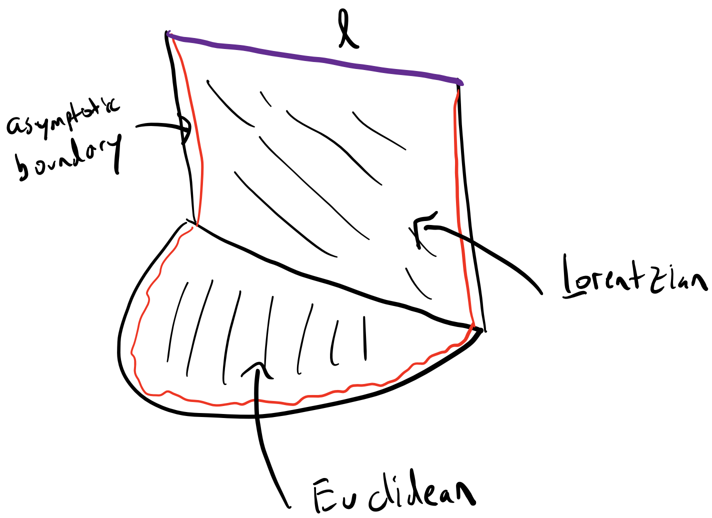
Let's now turn back to JT gravity to be more concrete. We are interested in studying the Hartle-Hawking state of the two-sided black hole in JT. A useful basis to think about is the basis of fixed length states [26, 27]; in JT gravity with two asymptotic boundaries, states $|\ell\rangle$ defined on spatial slices with a fixed (renormalized) length ℓ and zero extrinsic curvature form a complete basis.



So wavefunctions in this basis are functions of a length, $\psi(\ell)$, with the TFD state dual to the Hartle-Hawking state $\psi_{TFD,\beta}(\ell)$ (with a slight abuse of notation). This wavefunction is computed by a path integral over half-disks with a portion of the boundary of the disk being a bulk geodesic of renormalized length ℓ.[15]

To connect with the spectral form factor, we ask what happens if we evolve the TFD state with the Hamiltonian $H_L + H_R$, which evolves in Schwarzchild time "upwards in the Penrose diagram".

In JT gravity, we can imagine doing this evolution up to a time t, and compute the wavefunction in the ℓ basis, $\psi_{TFD,\beta}(\ell, t)$. This can be computed, for example, by a path integral with a portion which prepares the $t = 0$ state, and then a Lorentzian portion evolving up to time t, ending on a geodesic of length ℓ.



The renormalized length ℓ roughly measures the length of the Einstein Rosen bridge between the left and right asymptotic regions. Classically, this length grows linearly in t at large times [27], and doing the full computation one finds that this is true quantum mechanically in JT gravity as well [27].

[15] We could include corrections from other topologies, but they are negligible here.

Below we picture the modulus squared of the wavefunction $\psi_{TFD,\beta}(\ell,t)$ in JT gravity, for some short time t, and for a long time t. The wavefunction intitally has a peak around some order one ℓ, corresponding to the $t=0$ slice of the eternal black hole. But at longer times, the peak moves up to some $\ell \propto t$.

To see the relevance of this for the spectral form factor, we write the overlap of this time evolved TFD state with the initital state as

$$\langle TFD(\beta)|e^{-i(H_L+H_R)t/2}|TFD(\beta)\rangle = Z(\beta+it) \tag{57}$$

So $Z(\beta+it)$ computes the "return amplitude", the amplitude that the time evolved TFD returns to the intitial state. The SFF then compute the return probability.

Physically, this amplitude decays at early times because as the black hole is evolved in time, the Einstein Rosen bridge grows longer, and the wavefunction has small overlap with the $t=0$ state which has support only for short length Einstein rosen bridges [28].

Here in JT gravity, we can see this explicitly by computing the wavefunction [27], but in higher dimensions the same physics is there: the time evolved TFD state has a long wormhole, so the return amplitude is small.

5.4 The ramp from a spacetime wormhole in JT gravity

So now we have seen the physical origin of the decaying slope in the spectral form factor, by phrasing the spectral form factor as the return probability of the TFD state

$$SFF(t) = \left|\langle TFD(\beta)e^{-i(H_L+H_R)t/2}|TFD(\beta)\rangle\right|^2, \tag{58}$$

and seeing how the time-evolved TFD state develops a long wormhole. In order to find the non-decaying ramp in the spectral form factor, we must see a way for the TFD state to have a non-decaying amplitude to have a short length.

In the JT path integral, so far we have only considered the simplest topology, the disk topology. However, especially when considering the ramp, it is natural to look at other topologies.

First of all, the ramp only appears when computing with two copies of the system, $|Z(\beta+it)|^2$, and it does not appear in just $Z(\beta+it)$.[16] So then it is natural to look at spacetimes like the cylinder, which appear in the computation of the SFF, or $Z(\beta_1)Z(\beta_2)$, but not in the computation of an individual partition function.

Second, the ramp in $|Z(\beta+it)|^2$ is supposed to be exactly equal to $t/4\pi\beta$. In JT gravity, each contribution comes with a factor of $e^{S_0\chi}$, which is only independent of S_0 for $\chi=0$. This can happen if the topology is that of a cylinder, or a disk plus a disk with a handle. But if the disk with a handle grew, so that it could contribute at late times to the SFF, it would also give a growing contribution to $Z(\beta+it)$, which should not happen.

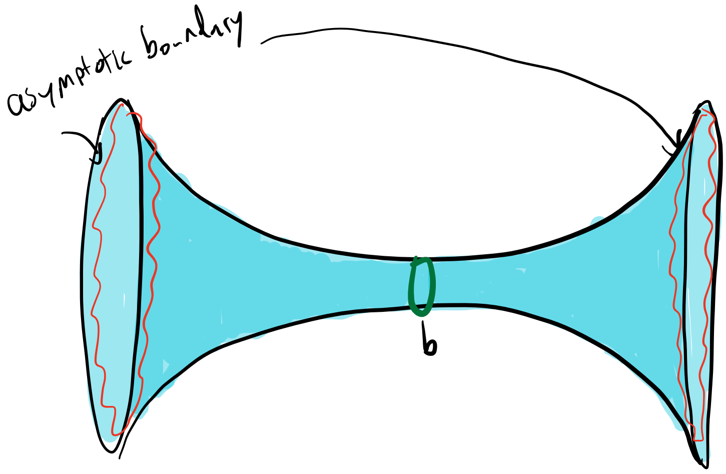
So we are motivated to look at the contribution of the cylinder topology to the SFF in JT gravity. Before we explain the physical interpretation of this contribution, we will show why it gives the ramp [25, 28, 29]

[16] Really, the ramp is describing the variance of the noisy contributions in $Z(\beta+it)$, but gravity is only giving us the smooth average contributions, and $Z(\beta+it)$ averages to zero at late times, with no hint of the ramp.

First, let's introduce the geometry of the cylinder with constant negative curvature and two asymptotically AdS boundaries. Some useful coordinates are

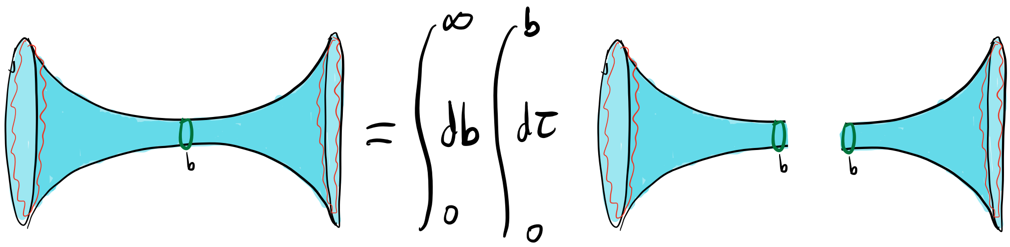
$$ds^2 = d\rho^2 + \frac{b^2}{(2\pi)^2}\cosh^2(\rho)d\theta^2 \tag{59}$$

Here b is a parameter of the geometry, and describes the length of a circular geodesic in the middle of the cylinder.



To compute the cylinder contribution to JT gravity, we allow the asymptotic boundaries to have some wiggly shapes, and integrate over these shapes. We also integrate over then length b of the neck of the cylinder.

It's useful to think of the cylinder as two "trumpets" glued together,



Here each trumpet is the path integral over cylinders with a circular geodesic of length b, and some asymptotic boundary of length β/ϵ or β'/ϵ. To find the contribution of each trumpet, we integrate over the shape of the wiggly asymptotic boundary. This again turns out to be one-loop exact [25]. The measure for gluing the trumpets can also be derived through careful consideration.

In gluing the trumpets together, we should not only integrate over the size b of the neck, but the relative twist length τ between the two trumpets. The twist length is basically the twist angle converted into a length, and is integrated from zero to b. Since the trumpets are independent of the twist, we find the integral for the cylinder contribution to $Z(\beta_1)Z(\beta_2)$

$$\int_0^\infty db\ b\ Z_{Tr}(\beta_1, b) Z_{Tr}(\beta_2, b), \tag{60}$$

where $Z_{Tr}(\beta, b)$ is the contribution of a trumpet.

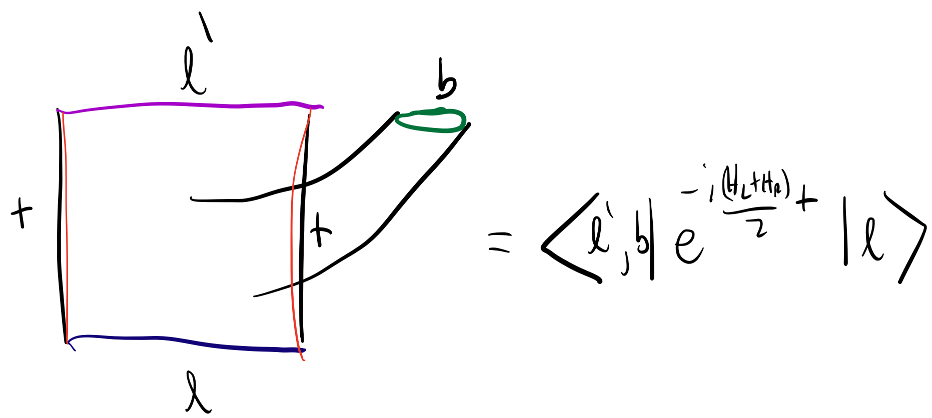
If one takes the formula for Z_{Tr} and does the integral ($Z_{Tr}(\beta, b) = \frac{1}{\sqrt{2\pi\beta}} e^{-b^2/2\beta}$ [25], and can be computed using similar methods to those used to compute the disk), one finds

$$Z(\beta_1)Z(\beta_2) \supset \frac{1}{2\pi}\frac{\sqrt{\beta_1\beta_2}}{\beta_1+\beta_2} \to \frac{t}{4\pi\beta}, \quad \beta_{1,2} \to \beta \pm it,\ t \gg \beta. \tag{61}$$

The factor of t has a clear interpretation in this case as coming from the relative twist between the two sides of the cylinder - roughly, when the boundary has length t, there are t ways to twist the left boundary with respect to the right boundary and produce a physically inequivalent geometry. This phenomenon of the linear growth of the ramp coming from a spontaneously broken relative time translation symmetry between two copies of the system has analogs in earlier derivations of the ramp for other systems [11, 12].

With this cylinder in hand, we can give a more physical interpretation of the ramp in the spectral form factor. The first step is to think of this cylindrical spacetime geometry as a "spacetime wormhole", which corresponds to the absorbtion and emmission of a closed baby universe. To see this baby universe, we consider cutting along circular spatial slices along the wormhole - each circular spatial slice is a closed universe, which we can think of as being traded between the two copies of the system in the SFF.

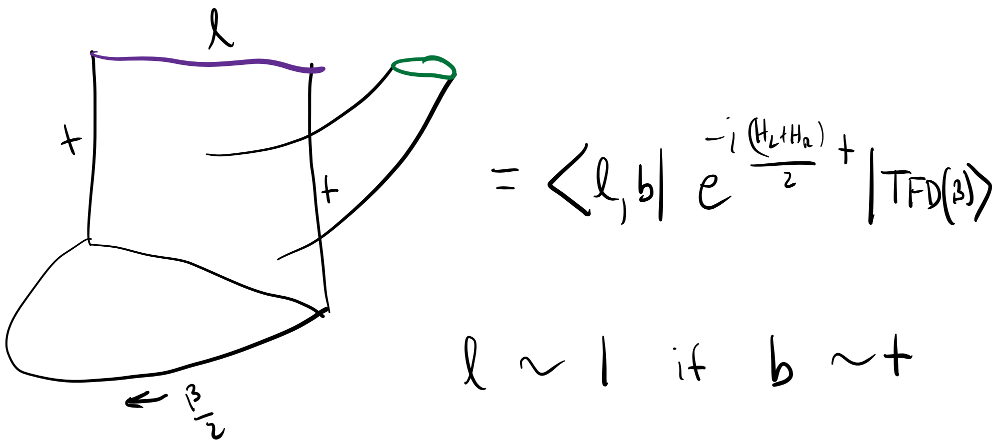
In JT, a basis for the states of a single closed universe is the length basis $|b\rangle$ - circles with fixed length b and zero extrinsic curvature. Path integrals with circular geodesic boundaries - "b" boundaries - compute amplitudes between states with external closed universes.



As an example, we've pictured the path integral which computes a transition amplitude between an initial state $|\ell\rangle$ of the two-sided black hole, and a final state $|\ell', b\rangle$, of the two-sided black hole and a closed universe. Beause of the topology of this spacetime, the amplitude to emit a baby universe is exponentially suppressed, $\sim e^{-S_0}$.

Now let's consider what happens if we let the TFD state evolve for a long time, but then emit a closed universe. The length of the Einstein Rosen bridge can become very long, of length $\ell \sim t$.

But it turns out, that if the black hole emits a closed universe of length $b \sim t$, the final length of the Einstein Rosen bridge can becomes of order one again - the closed universe "eats up" all the length of the Einstein Rosen bridge. Then the amplitude to return to the TFD state, plus the external closed universe, can be of order one, multiplied by the suppression factor $\sim e^{-S_0}$ from the small amplitude to emit the closed universe [28].



Note that the trumpet, pictured above, is precisely computing this amplitude, for the TFD state to evolve into the TFD state plus a closed universe,

$$\langle TFD(\beta), b| e^{-i(H_L+H_R)t/2} |TFD(\beta)\rangle. \tag{62}$$

When computing $Z(\beta + it)$, one does not see this effect because we do not allow any external closed universe states. But in the SFF, the two copies of the system can "trade" the baby universe. Since the systems are evolving in the opposite direction of time (SFF = $Z(\beta + it)Z(\beta - it$, with oppsite signs of time), they can both be viewed as "emitting" the closed universe that they are trading.

So the cylinder geometry in the SFF is non-decaying because it computes the amplitude for the emitted closed universe to eat up all the length of the Einstein rosen bridge, returning it to being short, it is exponentially small relative to the two disk contribution because of the penalty for changing topology, and it grows linearly in time because the closed universe which gives the dominant contribution has a length $b \sim t$, and the integral over the twist of the b circle in the cylinder then gives a factor of t.

5.5 The ramp in higher dimensions

Finally, we will make a couple comments about the ramp in higher dimensions. Two points which were not clear from our discussion in JT gravity are:

1. The cylinder contribution to $|Z(\beta + it)|^2$ does not have a saddle point,
2. In a certain application, the cylinder geometry can actually be thought of as a (slight complexification of a) quotient of the eternal black hole geometry.

This issues are important for studying the ramp in higher dimensions.

To see why the cylinder was not a saddle point, but that it is not a fundamental issue, consider the GUE computation of the SFF. For fixed average energy E, the ramp is equal to $t/2\pi$, independent of E. Then to compute the SFF, we integrate over E weighted by $e^{-2\beta E}$. This E integral has no saddle point.

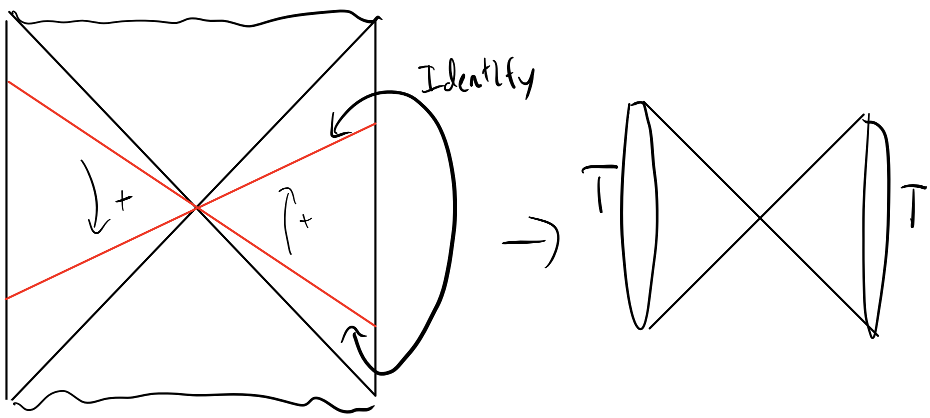
In JT gravity, we had enough control that we didn't need to rely on saddle points, (see [25] for a more comprehensive treatment of higher topologies in JT gravity), but in higher dimensions do not have much control. In order to find a quantity with a ramp where the cylinder is a saddle point, it is useful to fix the energy in a window in the spectral form factor, so the E integral has a saddle point, rather than use the Boltzmann weight

Consider the quantity $|Y_E(t)|^2$, with

$$Y_E(t) = \frac{1}{2\pi i}\int_{\gamma+i\mathbb{R}} d\beta\; e^{\beta E} Z(\beta+it) \tag{63}$$

We can also add a small width to the energy window by adding a Gaussian term in β. Then to compute $|Y_E(t)|^2$, we integrate over geometries with renormalized boundary lengths $\beta_L + it, \beta_R - it$, and also integrate over β_L, β_R. We can do the integrals over β_L and β_R by saddle point. It turns out that in JT, for this quantity, there is a saddle point from the cylinder contribution with $\beta_L = \beta_R = 0$ [29]. Then the cylinder has purely Lorentzian boundaries, with lengths $\pm iT$, and so we can thnk of the whole spacetime as purely Lorentzian (modulo some regularization which I will not discuss. See [29] for more.)

This spacetime is actually very simple, and exists in higher dimensions, not just AdS2: take the eternal black hole with energy E, and consider a quotient under killing time translation (with respect to $H_L - H_R$), $t \sim t+T$. This corresponds to identifying two spatial slices as pictured below, to form a "double cone" geometry, where the two boundaries are periodic with period T.



This double cone provides a candidate saddle point for the computation of the ramp, not just in JT gravity, but generally in AdS/CFT - the eternal black hole is always a solution, and thus the double cone always exists as well. Since we can form a similar solution from charged two-sided black holes, these saddle points should also be able to describe the SFF in nonzero charge sectors as well.

For the double cone, one can see that the action is always zero, because the bulk action and GHY boundary terms are both imaginary, but come with opposite signs on the two sides of the double cone, and cancel. It seems reasonable that the zero mode of relative rotations of the two boundaries with respect to each other gives a factor of t, leading to a linear ramp, but unlike in JT gravity, it is difficult to compute the coefficient.

5.6 An encourangement for further reading

In the lectures I actually gave at TASI, the last of the three lectures was focused on explaining work by myself, Stephen Shenker, and Douglas Stanford on the duality between JT gravity and a random matrix ensemble of boundary theories [25]. However, due partially to the fact that my lectures were prepared somewhat last-minute, and also to the fact that I could not find a way to explain that work much more concisely than is already done in the paper, my notes for the third lecture were essentially just the main sections of that paper. So rather than copy and paste from that paper to add to the end of these notes, I'll just encourage the interested reader to read the paper itself. Hopefully these notes have set the context well enough to prepare the reader, but I might also encourage the reader to consult some of the references on JT gravity mentioned in this paper first.

5.7 Final remarks

In these notes, we discussed our expectations for the statistics of the energy levels of black holes (specifically large black holes in AdS), coming from RMT universality of chaotic systems. We then translated these expectations into an expectation that the averaged spectral form factor has a ramp and plateau, and found a candidate explanation for the ramp in gravity. In JT gravity, one can be more careful and show that this is the correct explanation [25], but JT gravity has some peculiar features that are not shared with ordinary examples of AdS/CFT; in particular JT gravity is dual to an ensemble of Hamiltonians, not a specific system.

For an ordinary example of AdS/CFT, we only expect to find the ramp after doing some sort of averaging of the SFF. However, the gravity path integral seems like it might give us some sort of averaged contribution directly. As we mentioned, the size of the noise is of order the size of the average, so the average function is in no way a good approximation to the noisy function - gravity is not even giving us a good approximation to the answer. This problem is intimately related to the "factorization problem" [30, 31] associated with spacetime wormholes. Understanding the origin of the non-averaged noise in the SFF, and the factorization problem, is an ongoing area of research.

There are also remaining questions related to averaged aspects of the SFF. In particular, we have not given an explanation for the plateau. In JT gravity, one can give a sort of explanation [25], but it is unsatisfying in many ways. Additionally, if one studies systems with GOE or GSE statistics, the ramp and plateau should be corrected by additional terms. In versions of JT gravity, these seeem to be explained by contributions of non-orientable higher-topology surfaces [32], but it is not obvious how this generalizes beyond JT.

Of course, these are just a couple of the outstanding questions; I ensure any interested readers that there is much to be understood on this subject, and highly recommend taking a closer look at many of the papers referenced in these notes.

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