

The continuum limit of the modular discretization of AdS_2

Minos Axenides,^a Emmanuel Floratos^{b,a,*} and Stam Nicolis^c

^a*Institute of Nuclear and Particle Physics, NCSR “Demokritos”*

Aghia Paraskevi, GR-15310, Greece

^b*Physics Department, University of Athens, Zografou University Campus*

Athens, GR-15771, Greece

^c*Institut Denis Poisson, Université de Tours, Université d’Orléans, CNRS (UMR7013)*

Parc Grandmont, 37200 Tours, France

E-mail: axenides@inp.demokritos.gr, mflorato@phys.uoa.gr,

stam.nicolis@lmpt.univ-tours.fr

According to the ’t Hooft–Susskind holography, the black hole entropy, S_{BH} , is carried by the chaotic microscopic degrees of freedom, which live in the near horizon region and have a Hilbert space of states of finite dimension $d = \exp(S_{\text{BH}})$. In previous work we have proposed that the near horizon geometry, when the microscopic degrees of freedom can be resolved, can be described by the $\text{AdS}_2[\mathbb{Z}_N]$ discrete, finite and random geometry, where $N \propto S_{\text{BH}}$. What had remained as an open problem is how the smooth AdS_2 geometry can be recovered, in the limit when $N \rightarrow \infty$. In this contribution, we present the salient points of the solution to this problem, which involves embedding the discrete and finite $\text{AdS}_2[\mathbb{Z}_N]$ geometry in a family of finite geometries, $\text{AdS}_2^M[\mathbb{Z}_N]$, where M is another integer. This family can be constructed by an appropriate toroidal compactification and discretization of the ambient (2+1)-dimensional Minkowski space-time. In this construction N and M can be understood as “infrared” and “ultraviolet” cutoffs respectively. This construction allows us to obtain the continuum limit of the $\text{AdS}_2^M[\mathbb{Z}_N]$ discrete and finite geometry, by taking both N and M to infinity in a specific correlated way, following a reverse process: Firstly, by recovering the continuous, toroidally compactified, $\text{AdS}_2[\mathbb{Z}_N]$ geometry, by removing the ultraviolet cutoff; secondly, by removing the infrared cutoff, in a specific decompactification limit, while keeping the radius of AdS_2 finite. It is in this way that we recover the standard non-compact AdS_2 continuum space-time. This method can be applied directly to higher-dimensional AdS spacetimes.

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*Speaker

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1. Introduction

The present work, mathematically, belongs to the area of algebraic geometry over finite rings. However its relevance for physics stems from the proposal of using specific, discrete and finite arithmetic geometries, as toy models, in order to describe properties of quantum gravity in general and the structure of space-time, in particular, at distances of the order of the Planck scale (10^{-33} cm), where the notions of the metric and of the continuity of spacetime break down [1].

At Planck scale energies, quantum mechanics, as we know it from lower energy scales, implies that the notion of spacetime itself becomes ill-defined, through the appearance from the vacuum of real or virtual black holes of Planck length size [2].

Probing this scale by scattering experiments of any sort of particle-like objects, black holes will be produced and the strength of the gravitational interaction will be of $O(1)$, which leads to a breakdown of perturbative gravity and of the usual continuum spacetime description [3, 4].

The above remarks led some authors to consider the idea, that one has to abandon continuity of spacetime, locality of interactions and regularity of dynamics. Indeed there are recent arguments that quantization of gravity implies discretization and finiteness of space time [5, 6]. This is, indeed, an old idea, that was put forward, already way back, by the founders of quantum physics and gravity.

A few years ago the seminal paper [12] highlighted the relevance of the so-called “new black hole information paradox” [13], which finally lead to the conjectures that go under the label ER=EPR [14] and culminate in the so-called QM=GR correspondence [15].

These conjectures relate strongly the description of spacetime geometry and quantum gravity to quantum information theoretic tools, such as entanglement of information, algorithmic complexity, random quantum networks, quantum holography, error correcting codes.

A discrete and finite spacetime for quantum gravity is a possible way for describing the remarkable fact that the Hilbert space of states of the BH microscopic degrees of freedom is finite-dimensional. Its dimensionality equals to the exponential of the Bekenstein-Christodoulou-Hawking black hole entropy, which is of quantum origin. The generalization of the Bekenstein entropy bounds implies that, for any pair of local observers in a general gravitational background, the physics inside their causal diamond is also described by a finite dimensional Hilbert space of states [16]. This result has been exploited further and consistently under the name of holographic spacetime, in the works of refs. [17–19].

Our idea about the nature of spacetime at the Planck scale, takes the notion of a holographic spacetime one step further: Namely, that the finite dimensionality of the Hilbert space of local spacetime regions originates from a discrete and finite spacetime, which underlies the emergent continuous geometric description [1, 20].

Our starting point, therefore, is the hypothesis that space-time, at the Planck scale, is fundamentally discrete and finite and, moreover, does not emerge from any other continuous description (conformal field theory, string theory, or anything else). We claim that, at “large” distances (in units of the Planck length), the continuous spacetime geometry can be described as an infrared limit thereof. This hypothesis, indeed, is similar to the proposal by ’t Hooft [5].

This assumption implies developing and using the appropriate mathematical tools, that can describe the properties and dynamics of discrete and finite geometries as well as the emergence, in their infrared limits, of continuous geometries. So what we shall show in this contribution is an explicit example of how the continuous geometry of AdS_2 can emerge as a scaling limit of a specific discretization procedure.

We do not wish to imply that it is not possible to define quantum gravity, with a finite dimensional Hilbert space, in any other way; just that this is one possible way to describe quantum physics with finite dimensional Hilbert space.

2. Discretization and toroidal compactification of the AdS_2 geometry

2.1 The UV cutoff, the lattice of integral points and the $SO(2, 1, \mathbb{Z})$ isometry of $AdS_2^M[\mathbb{Z}]$

We shall now present and study in detail the lattice of integral points of AdS_2 , along with its isometries.

The physical lengthscale in our problem is the radius of the AdS_2 spacetime, R_{AdS_2} . We set $R_{AdS_2} = 1$ and we divide it into M segments, of length $a = R_{AdS_2}/M$. This defines a as the UV cutoff (lattice spacing) and $M \in \mathbb{N}$ and, hence, a lattice in $\mathcal{M}^{2,1}$.

The continuum limit is defined by taking $M \rightarrow \infty$ and $a \rightarrow 0$ with $R_{AdS_2} = 1$ fixed.

The global embedding coordinates (x_0, x_1, x_2) of this lattice are $(ka, la, ma) = a(k, l, m)$, where $k, l, m \in \mathbb{Z}$. They are measured in units of the lattice spacing a . Therefore the lattice points, that lie on AdS_2 satisfy the equation

$$k^2 + l^2 - m^2 = M^2 \quad (1)$$

whose solutions define $AdS_2^M[\mathbb{Z}]$, the set of all integral points of AdS_2 with integer radius M .

In the literature there has been considerable effort in counting the number of solutions to the above equation, in particular the asymptotics of the density of such points [21–26]. This problem

can be mapped to a problem whose solution is known, namely the Gauss circle problem. This pertains to finding the number $r_2(m, M)$ of solutions to the equation $k^2 + l^2 = M^2 + m^2$. This number is determined by factoring $M^2 + m^2$ into its prime factors [21] and counting the number of primes, p_i , of the form $p_i \equiv 1 \pmod{4}$ (this is described in detail in [27]; the dependence on M is a topic of current research [25, 26]).

This factorization procedure generates a sequence of primes that contains an element of inherent randomness. It is this property that captures the random distribution of the integral points on AdS₂—this is illustrated in figs. 1.

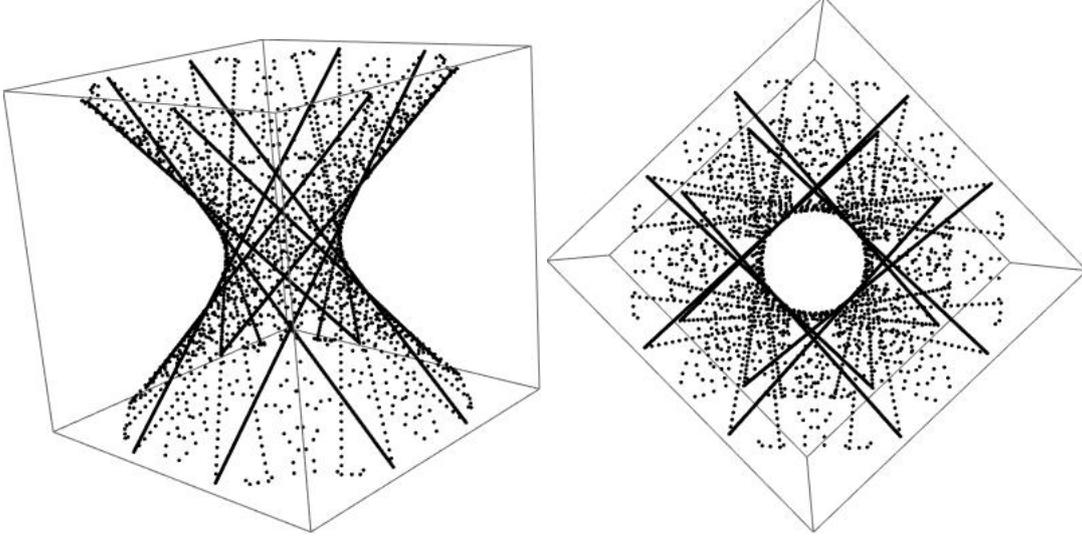


Figure 1: Integral points on AdS₂.

Therefore, from these facts, the number of integral points of the hyperboloid, up to height m , is given by the expression

$$\text{Sol}(m) = 4 + 2 \sum_{j=1}^m r_2(j, M) \tag{2}$$

We plot this function—in fig. 2, for $M = 1$, when m runs from -200 to 200 (due to the symmetry, $m \leftrightarrow -m$, we plot only the positive values of m .)

It is, indeed, striking that the result is an almost straight line [25, 26].

We shall now discuss how to actually construct these points, using the property that they belong to light-cone lines, which emerge from the rational points of the circle on the throat of AdS₂.

Using the ruling property of AdS₂,

$$\begin{aligned} k &= \cos \phi - \mu \sin \phi \\ l &= \sin \phi + \mu \cos \phi \\ m &= \mu \end{aligned} \tag{3}$$

we may repackage these as follows

$$x_0 + ix_1 = k + il = e^{i\phi}(1 + i\mu) = e^{i\phi}(1 + im) \Leftrightarrow e^{i\phi} = \frac{k + il}{1 + im} \tag{4}$$

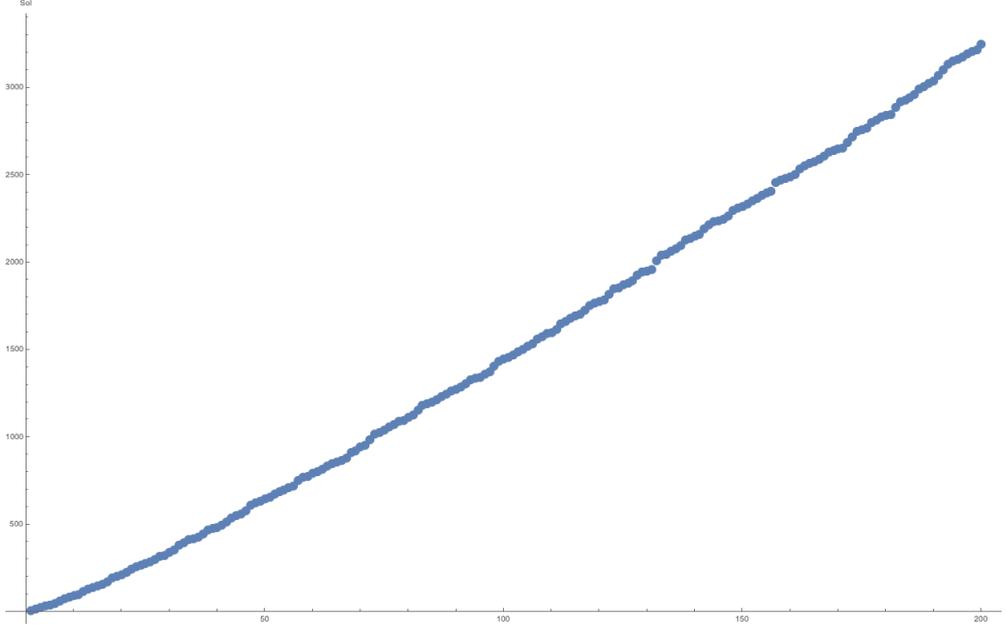


Figure 2: The number of integral points, on AdS₂, as a function of the height, m , for $M = 1$. Due to symmetry, $m \leftrightarrow -m$, we plot only the positive values of m .

hence

$$\cos \phi = \frac{k + lm}{1 + m^2} \quad \text{and} \quad \sin \phi = \frac{l - mk}{1 + m^2} \quad (5)$$

We remark that these are rational numbers—therefore they label rational points on the circle [28].

The light cone lines at (k, l, m) are, therefore, parametrized by $\mu \in (-\infty, \infty)$, as

$$\begin{aligned} x_0 &= \frac{k+lm}{1+m^2} - \mu \frac{l-mk}{1+m^2} \\ x_1 &= \frac{l-mk}{1+m^2} + \mu \frac{k+lm}{1+m^2} \\ x_2 &= \mu \end{aligned} \quad (6)$$

(When $\mu = x_2 = m$, $x_0 = k$ and $x_1 = l$.)

Proposition 1. *On these specific light-cone lines there exist infinitely many integral points, when μ , that labels the space-like direction x_2 , takes appropriate integer values.*

Proof. We write

$$x_0(\mu) + ix_1(\mu) = e^{i\phi} (1 + i\mu) \quad (7)$$

where ϕ is defined by eq. (5).

We look for integer values of $\mu = n \in \mathbb{Z}$, such that $x_0(n)$ and $x_1(n)$ are, also, integers.

That is

$$x_0(n) + ix_1(n) = \frac{k + il}{1 + im} (1 + in) \quad (8)$$

should be a Gaussian integer and this can happen iff $(1 + in)/(1 + im) = a + ib$ with $a, b \in \mathbb{Z}$.

Therefore

$$1 + in = (a - mb) + i(am + b) \Leftrightarrow \begin{cases} 1 = a - mb \\ n = am + b \end{cases} \quad (9)$$

Thus on the light cone line passing through the point (k, l, m) there are infinite integer points parametrized as:

$$\begin{aligned}x_0 &= k + b(km - l) \\x_1 &= l + b(k + lm) \\x_2 &= n = m + b(1 + m^2)\end{aligned}\tag{10}$$

□

Proposition 2. *Conversely, on any light cone line emanating from any rational point of the circle on the throat of the hyperboloid there is an infinite number of integer points.*

Proof. Indeed, we have

$$e^{i\phi} \equiv \frac{a + ib}{a - ib} \Leftrightarrow x_0 + ix_1 = \frac{a + ib}{a - ib}(1 + in)\tag{11}$$

with $a, b \in \mathbb{Z}$. In order to obtain an integral point, for $\mu = n$, we must have

$$\frac{1 + in}{a - ib} = d + ic\tag{12}$$

with $c, d \in \mathbb{Z}$

We immediately deduce that

$$\begin{aligned}1 &= ad - bc \\n &= ac + bd\end{aligned}\tag{13}$$

These expressions imply that, given the integers a and b , it's possible to find the integers c and d and to express the coordinates x_0, x_1 and x_2 as

$$\begin{aligned}x_0 &= ad + bc \\x_1 &= ac - bd \\x_2 &= ac + bd\end{aligned}\tag{14}$$

The Diophantine equation $1 = ad - bc$ is solved for c and d , given two coprime integers a and b , by the Euclidian algorithm—which seems to lead to a unique solution, implying that the point (x_0, x_1, x_2) is unique.

However there's a subtlety! There are *infinitely many* solutions (c, d) , to the equation $ad - bc = 1$! The reason is that, given any one solution (c, d) , the pair $(c + \kappa a, d + \kappa b)$, with $\kappa \in \mathbb{Z}$, is, also, a solution, as it can be checked by substitution.

Therefore there is a one-parameter family of points, labeled by the integer κ :

$$\begin{aligned}x_0 &= ad + bc + 2\kappa ab \\x_1 &= ac - bd + \kappa(a^2 - b^2) \\x_2 &= ac + bd + \kappa(a^2 + b^2)\end{aligned}\tag{15}$$

We remark, however, that the vector $(2ab, a^2 - b^2, a^2 + b^2)$ is light-like, with respect to the $(+ + -)$ metric: $(2ab)^2 + (a^2 - b^2)^2 - (a^2 + b^2)^2 = 0$. So eq. (15) describes a shift of the point $(ad + bc, ac - bd, ac + bd)$, along a light-like direction. Since the shift is linear in the “affine parameter”, κ , it generates a light-like line, passing through the original point.

In this way we have established the dictionary between the rational points of the circle and the integral points of the hyperboloid.

□

Now we proceed with the study of the discrete symmetries of the integral Lorentzian lattice of $\mathcal{M}^{2,1}$, where the lattice of integral points on AdS₂ is embedded. The lattice of integral points of $\mathcal{M}^{2,1}$, with one space-like and two time-like dimensions, carries as isometry group the group of integral Lorentz boosts $\text{SO}(2, 1, \mathbb{Z})$, as well as integral Poincaré translations. The double cover of this infinite and discrete group is $\text{SL}(2, \mathbb{Z})$, the modular group. This has been shown by Schild [29, 30] in the 1940s. The group $\text{SO}(2, 1, \mathbb{Z})$ can be generated by reflections, as has been shown by Coxeter [31], Vinberg [32]. This work culminates in the famous book by Kac [33], where he introduced the notion of hyperbolic, infinite dimensional, Lie algebras. The characteristic property of such algebras is that the discrete Weyl group of their root space is an integral Lorentz group. Generalization from $\text{SL}(2, \mathbb{Z})$ to other normed algebras has been studied in [34].

The fundamental domain of $\text{SO}(2, 1, \mathbb{Z})$ is the minimum set of points of the integral lattice of $\mathcal{M}^{2,1}$, which are not related by any element of the group and from which, all the other points of the lattice can be generated by repeated action of the elements of the group. It turns out that the fundamental region is an infinite set of points which can be generated by repeated action of reflections in the following way:

Using the metric $h \equiv \text{diag}(1, 1, -1)$ on $\mathcal{M}^{2,1}$ the generating reflections, elements of $\text{SO}(2, 1, \mathbb{Z})$, are given by the matrices

$$R_1 = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & & \\ & & & 1 \end{pmatrix} \quad (16)$$

$$R_4 = \begin{pmatrix} 1 & -2 & -2 \\ 2 & -1 & -2 \\ -2 & 2 & 3 \end{pmatrix}$$

If (k, l, m) are the coordinates of the integral lattice, the fundamental domain of $\text{SO}(2, 1, \mathbb{Z})$ can be defined by the conditions $m \geq k + l \geq 0$ and $k \geq l \geq 0$. This fundamental domain, restricted on $\text{AdS}_2^M[\mathbb{Z}]$, defines the corresponding fundamental domain of $\text{SO}(2, 1, \mathbb{Z})$, acting on $\text{AdS}_2^M[\mathbb{Z}]$. This region of $\text{AdS}_2[\mathbb{Z}]$ lies in the positive octant of $\mathcal{M}^{2,1}$ and between the two planes, that define the conditions—cf. fig. 3. It is of *infinite* extent.

2.2 The IR cutoff and the toroidal compactification of AdS₂

Having introduced the lattice of integral points on AdS₂, which we consider as defining an UV cutoff, we proceed, now, to impose an infrared (IR) cutoff. The crucial reason for such a cutoff is that in order to study chaotic Hamiltonian dynamics on this spacetime. [35], we have made use of the interpretation of AdS₂ as a phase space of single particles, due to the symplectic nature of the isometry $\text{SL}(2, \mathbb{R}) = \text{Sp}(2, \mathbb{R})$. The additional requirement of mixing (scrambling) imposes the condition of the compactness of the phase space and therefore the necessity of imposing of an infrared cutoff (for a detailed discussion of this point cf. [36]).

Having embedded the AdS₂ hyperboloid,

$$x_0^2 + x_1^2 - x_2^2 = R_{\text{AdS}_2}^2 \quad (17)$$

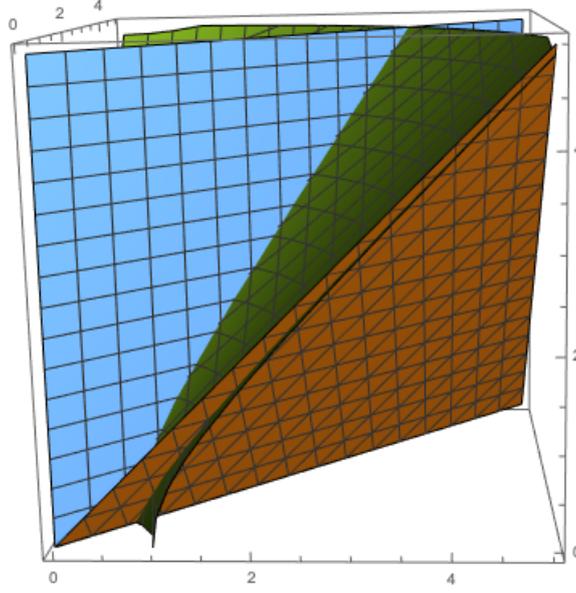


Figure 3: The fundamental domain of $SO(2, 1, \mathbb{Z})$ on $AdS_2^M[\mathbb{Z}]$ is the dark green part of the hyperboloid, in the positive octant, that lies between the two planes, $m \geq k + l \geq 0$ and $k \geq l \geq 0$.

in $\mathcal{M}^{2,1}$, the IR cutoff, L is defined by periodically identifying all the spacetime points of $\mathcal{M}^{2,1}$, if the difference of their coordinates is an integral vector $\times L$:

$$x \sim y \Leftrightarrow x - y = (k, l, m)L \quad (18)$$

where $k, l, m \in \mathbb{Z}$. In this way we have compactified $\mathcal{M}^{2,1}$ to the three-dimensional torus, of size L , $\mathbb{T}^3(L)$.

More concretely, $\mathbb{T}^3(L)$ is the fundamental domain of the group of integral translations, $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, acting on $\mathcal{M}^{2,1}$. To describe this geometric property by the algebraic operation, $\text{mod } L$, that acts on the coordinates of $\mathcal{M}^{2,1}$, we are led to identify the fundamental domain with the positive octant of $\mathcal{M}^{2,1}$, i.e. $x_0, x_1, x_2 \geq 0$

After this compactification, the spacetime geometry of AdS₂ becomes a foliation of the 3-torus, with leaves the images of AdS₂ under the operation $\text{mod } L$. So the equation, whose solutions define the points of the compactified AdS₂, is

$$x_0^2 + x_1^2 - x_2^2 \equiv R_{AdS_2}^2 \text{ mod } L \quad (19)$$

where $(x_0, x_1, x_2) \in \mathbb{T}^3(L)$.

It is obvious, that inside the 3-torus, there is a part of the AdS₂ surface, which corresponds to solutions of eq. (19), without the $\text{mod } L$ operation. On the other hand, the infinite part of AdS₂, that lies outside the torus, is partitioned in infinitely many pieces, which belong to images of $\mathbb{T}^3(L)$ in $\mathcal{M}^{2,1}$. These pieces are brought inside the torus by the $\text{mod } L$ operation.

Now we choose the IR cutoff L in units of a , so that $L = aN$, where N is an integer, independent of M . It is constrained by $N > M$, since the cube should contain, at least, the throat of AdS₂.

So the scaling limit entails taking $M \rightarrow \infty$, $N \rightarrow \infty$, but keeping L fixed.

The periodic nature of the IR cutoff implies that we must take the images of all integral points of $AdS_2[\mathbb{Z}]$ under the mod N operation, inside the cubic lattice of N^3 points.

The set of these images satisfy the equations

$$k^2 + l^2 - m^2 \equiv M^2 \pmod{N} \quad (20)$$

The set of points satisfying this condition will be called $AdS_2^M[\mathbb{Z}_N]$.

Our definition for $AdS_2[\mathbb{Z}_N]$ in our previous work was similar to the one given here. The only difference being that the RHS of eq. (20) was $1 \pmod{N}$, which was chosen for convenience, rather than for any intrinsic reason. We remark that the two definitions are consistent iff $M^2 \equiv 1 \pmod{N}$.

The solutions of eq. (20), when $M^2 \equiv 1 \pmod{N}$, produce the $AdS_2[\mathbb{Z}_N]$ geometry introduced in our previous work.

3. Continuum limit for large N

3.1 Constraints on the double sequences of the UV/IR cutoffs

Having constructed the finite geometry, $AdS_2^M[\mathbb{Z}_N]$ and established its relation with $AdS_2[\mathbb{Z}_N]$, we shall discuss the meaning of the limit, $M, N \rightarrow \infty$. It is in this limit that we hope to recover the continuum AdS_2 geometry.

Such a limit can be defined using the topology of the ambient Minkowski spacetime $\mathcal{M}^{2,1}$.

Specifically, we use a reverse, two-step, process: Firstly, by removing the UV cutoff; next, by removing the IR cutoff. This is realized by choosing any sequence of pairs of integers, (M_n, N_n) , $n = 1, 2, 3, \dots$, such that, for any $n = 1, 2, 3, \dots$

- $N_n > M_n$,
- $M_n^2 \equiv 1 \pmod{N_n}$,
- The limit of the ratio N_n/M_n takes a finite value, > 1 (as $n \rightarrow \infty$), which we can identify with L/R_{AdS_2} .

Below we shall present the general solution to the equation $M^2 \equiv 1 \pmod{N}$. Subsequently, we shall select those solutions that satisfy the other requirements.

The first step is to factor N into (powers of) primes, $N = N_1 \times N_2 \times \dots \times N_l = q_1^{k_1} q_2^{k_2} \dots q_l^{k_l}$. Then the equation $M^2 \equiv 1 \pmod{N}$, is equivalent to the system

$$M^2 \equiv 1 \pmod{q_I^{k_I}} \quad (21)$$

where $I = 1, 2, \dots, l$. The Chinese Remainder Theorem [27] then implies that all the solutions of eq. (21) can be used to construct M , with $M = M_1 m_1 n_1 + \dots + M_l m_l n_l$, where $M_I \equiv M \pmod{N_I}$, $m_I = N/N_I$, $n_I \equiv m_I^{-1} \pmod{N_I}$.

When $q_I \neq 2$, the solutions are $M_I = 1$ and $q_I^{n_I} - 1$. When $q_I = 2$, there exist four solutions, $M_I = 1, 2^{n_I} - 1, 2^{n_I - 1} \pm 1$.

Now we must choose sequences, N_n and determine the corresponding M_n , satisfying the constraints listed above.

In the next two subsections we shall present nontrivial examples of sequences of pairs, (M_n, N_n) satisfying the above constraints, whose limiting ratio, $\lim_{n \rightarrow \infty} N_n/M_n$, is the “golden” or “silver” ratios. The general question of determining sequences which have an arbitrary, but given, limiting ratio, is an interesting question, which is deferred to a future work.

3.2 Removing the UV cutoff by the Fibonacci sequence

Although it is easy to demonstrate the existence of such sequences—for example, $N_n = 2^n$ and $M_n = 2^{n-1} \pm 1$, where $M_n^2 \equiv 1 \pmod{N_n}$ and $N_n/M_n \rightarrow 2$, which implies that $L/R_{\text{AdS}_2} = 2$, in this section we focus on another particular class of sequences, based on the Fibonacci integers, f_n [27]. This case is of particular interest, since, in our previous paper [35], where we studied fast scrambling, we found that, for geodesic observers, moving in AdS₂[N], with evolution operator the Arnol’d cat map, the fast scrambling bound is saturated, when N is a Fibonacci integer.

The Fibonacci sequence, defined by

$$\begin{aligned} f_0 &= 0; f_1 = 1 \\ f_{n+1} &= f_n + f_{n-1} \end{aligned} \quad (22)$$

can be written in matrix form

$$\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}_A \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix} \quad (23)$$

We remark that the famous Arnol’d cat map can be written as

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = A^2 \quad (24)$$

Since the matrix A doesn’t depend on n , we can solve the recursion relation in closed form, by setting $f_n \equiv C\rho^n$ and find the equation, satisfied by ρ

$$\rho^{n+1} = \rho^n + \rho^{n-1} \Leftrightarrow \rho^2 - \rho - 1 = 0 \Leftrightarrow \rho \equiv \rho_{\pm} = \frac{1 \pm \sqrt{5}}{2}$$

Therefore, we may express f_n as a linear combination of ρ_+^n and $\rho_-^n = (-)^n \rho_+^{-n}$:

$$f_n = A_+ \rho_+^n + A_- \rho_-^n \Leftrightarrow \begin{cases} f_0 = A_+ + A_- = 0 \\ f_1 = A_+ \rho_+ + A_- \rho_- = 1 \end{cases} \quad (25)$$

whence we find that

$$A_+ = -A_- = \frac{1}{\rho_+ - \rho_-} = \frac{1}{\sqrt{5}}$$

therefore,

$$f_n = \frac{\rho_+^n - (-)^n \rho_+^{-n}}{\sqrt{5}} \quad (26)$$

It’s quite fascinating that the LHS of this expression is an integer!

The eigenvalue $\rho_+ > 1$ is known as the ‘‘golden ratio’’ (often denoted by ϕ in the literature) and it’s straightforward to show that $f_{n+1}/f_n \rightarrow \rho_+$, as $n \rightarrow \infty$.

Furthermore, it can be shown, by induction, that the elements of A^n are, in fact, the Fibonacci numbers themselves, arranged as follows:

$$A^n = \begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix} \quad (27)$$

One reason this expression is useful is that it implies that $\det A^n = (-)^n = f_{n-1}f_{n+1} - f_n^2$.

For $n = 2l + 1$, we remark that this relation takes the form $f_{2l+1}^2 = 1 + f_{2l}f_{2l+2}$.

Now, since f_{2l+1} and f_{2l+2} are successive iterates, they’re coprime, which implies, that $f_{2l+1}^2 \equiv 1 \pmod{f_{2l+2}}$.

Therefore, the sequence of pairs, $(M_l = f_{2l+1}, N_l = f_{2l+2})$, where $l = 1, 2, 3, \dots$, satisfy all of the requirements and the corresponding limiting ratio, L/R_{AdS_2} , can be found analytically. It is, indeed, equal to $\rho_+ = (1 + \sqrt{5})/2$, the golden ratio.

In the next subsection we shall consider the so-called k -Fibonacci sequences, which will be important for obtaining other values for the ratio L/R_{AdS_2} , as well as for removing the IR cutoff.

3.3 Removing the IR cutoff using the generalized k -Fibonacci sequences

It’s possible to generalize the Fibonacci sequence in the following way:

$$g_{n+1} = kg_n + g_{n-1} \quad (28)$$

with $g_0 = 0$ and $g_1 = 1$ and k an integer. This is known as the ‘‘ k -Fibonacci’’ sequence [37].

We may solve for $g_n \equiv C\rho^n$; the characteristic equation for ρ , now, reads

$$\rho^2 - k\rho - 1 = 0 \Leftrightarrow \rho_{\pm}(k) = \frac{k \pm \sqrt{k^2 + 4}}{2} \quad (29)$$

and express g_n as a linear combination of the ρ_{\pm} :

$$g_n = A_+\rho_+(k)^n + A_-\rho_-(k)^n = \frac{\rho_+(k)^n - (-)^n\rho_+(k)^{-n}}{\sqrt{k^2 + 4}} \quad (30)$$

that generalizes eq. (26).

In matrix form

$$\begin{pmatrix} g_n \\ g_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}}_{A(k)} \begin{pmatrix} g_{n-1} \\ g_n \end{pmatrix} \quad (31)$$

Similarly as for the usual Fibonacci sequence, we may show, by induction, that

$$A(k)^n = \begin{pmatrix} g_{n-1} & g_n \\ g_n & g_{n+1} \end{pmatrix} \quad (32)$$

We find that $\det A(k)^n = (-)^n$, therefore that $g_{2l+1}^2 \equiv 1 \pmod{g_{2l+2}}$; thus, $g_{2l+2}/g_{2l+1} \rightarrow L/R_{\text{AdS}_2} = \rho_+(k)$, where the eigenvalue of $A(k)$, $\rho_+(k)$, that’s greater than 1, of course, depends on k . In this

way it is possible to obtain infinitely many values of the ratio L/R_{AdS_2} . Furthermore, we have determined L , the IR cutoff, in terms of R_{AdS_2} .

What is remarkable is that, using the additional parameter, k , of the k -Fibonacci sequence, it is, now, possible to remove the IR cutoff, as well, since it is possible to send $L \rightarrow \infty$, as $k \rightarrow \infty$, keeping R_{AdS_2} fixed.

While k remains finite, the periodic box cannot be removed and, in the continuum limit, $a \rightarrow 0$, we obtain infinitely many foldings of the AdS₂ surface inside the box due to the mod L operation.

The Fibonacci sequence, taken mod N , is periodic, with period $T(N)$; this turns out to be a “random” function of N . The “shortest” periods, as has been shown by Falk and Dyson [38], occur when $N = F_l$, for any l . In that case, $T(F_l) = 2l$.

We may, thus, ask the same question for the k -Fibonacci sequence, where the ratio of its successive elements, g_{n+1}/g_n tend to the so-called “ k -silver ratio”,

$$\rho_+(k) = \frac{k + \sqrt{k^2 + 4}}{2} \quad (33)$$

(the “silver ratio” is $\rho_+(k = 2)$)

From eq. (32), taking mod g_l on both sides, we find that, when $n = l$, the matrix becomes \pm (the identity matrix), so $T(g_l) = l$ or $2l$, respectively; thereby generalizing the Falk–Dyson result for the k -Fibonacci sequences.

4. Conclusions

The logical approach for discussing the relation between classical and quantum physics involves showing how the former can be obtained as a limit of the latter, since it is the quantum description that is more “fundamental” and it isn’t possible to describe quantum effects in terms of classical physics. In this contribution we have, therefore, shown how the smooth AdS₂ geometry, that is a hallmark of the near horizon geometry of extremal black holes, can, indeed, be obtained by a limiting process from a finite and discrete geometry, that has been shown to capture the consistent description of the single-particle probes of the near horizon geometry, that can resolve the individual black hole microstates.

This approach can be readily generalized to higher dimensional AdS _{k} spacetimes, $k > 2$.

The next step involves describing the near horizon degrees themselves, as a many-body system.

References

- [1] M. Axenides, E. G. Floratos and S. Nicolis, *Modular discretization of the AdS_2/CFT_1 holography*, *JHEP* **02** (2014) 109, [[1306.5670](#)].
- [2] S. Hawking, *Space-Time Foam*, *Nucl. Phys. B* **144** (1978) 349–362.
- [3] S. Carlip, *The Small Scale Structure of Spacetime*, in *Proceedings, Foundations of Space and Time: Reflections on Quantum Gravity: Cape Town, South Africa*, pp. 69–84, 2009. [1009.1136](#).
- [4] S. Carlip, R. A. Mosna and J. P. M. Pitelli, *Vacuum Fluctuations and the Small Scale Structure of Spacetime*, *Phys. Rev. Lett.* **107** (2011) 021303, [[1103.5993](#)].
- [5] G. 't Hooft, *How quantization of gravity leads to a discrete space-time*, *J. Phys. Conf. Ser.* **701** (2016) 012014.
- [6] E. P. Verlinde, *On the Origin of Gravity and the Laws of Newton*, *JHEP* **04** (2011) 029, [[1001.0785](#)].
- [7] A. Sen, *Quantum Entropy Function from $AdS(2)/CFT(1)$ Correspondence*, *Int. J. Mod. Phys. A* **24** (2009) 4225–4244, [[0809.3304](#)].
- [8] A. Sen, *Arithmetic of Quantum Entropy Function*, *JHEP* **08** (2009) 068, [[0903.1477](#)].
- [9] S. Hawking, *Information loss in black holes*, *Phys. Rev. D* **72** (2005) 084013, [[hep-th/0507171](#)].
- [10] J. Maldacena, *Black holes and quantum information*, *Nature Rev. Phys.* **2** (2020) 123–125.
- [11] A. Almheiri, X. Dong and D. Harlow, *Bulk Locality and Quantum Error Correction in AdS/CFT* , *JHEP* **04** (2015) 163, [[1411.7041](#)].
- [12] A. Almheiri, D. Marolf, J. Polchinski and J. Sully, *Black Holes: Complementarity or Firewalls?*, *JHEP* **02** (2013) 062, [[1207.3123](#)].
- [13] K. Papadodimas and S. Raju, *An Infalling Observer in AdS/CFT* , *JHEP* **10** (2013) 212, [[1211.6767](#)].
- [14] J. Maldacena and L. Susskind, *Cool horizons for entangled black holes*, *Fortsch. Phys.* **61** (2013) 781–811, [[1306.0533](#)].
- [15] L. Susskind, *Dear Qubitizers*, *GR=QM*, [1708.03040](#).
- [16] R. Bousso, *Black hole entropy and the Bekenstein bound*, pp. 139–158. 2020. [1810.01880](#). [10.1142/9789811203961_012](#).
- [17] S. B. Giddings, *Black holes, quantum information, and unitary evolution*, *Phys. Rev. D* **85** (2012) 124063, [[1201.1037](#)].

- [18] T. Banks, *Holographic Space-time and Quantum Information*, *Front. in Phys.* **8** (2020) 111, [2001.08205].
- [19] N. Bao, S. M. Carroll and A. Singh, *The Hilbert Space of Quantum Gravity Is Locally Finite-Dimensional*, *Int. J. Mod. Phys. D* **26** (2017) 1743013, [1704.00066].
- [20] E. G. Floratos, *The Heisenberg-weyl Group on the $Z(N) \times Z(N)$ Discretized Torus Membrane*, *Phys. Lett.* **B228** (1989) 335–340.
- [21] D. Shanks, *A sieve method for factoring numbers of the form $n^2 + 1$* , *Mathematical Tables and Other Aids to Computation* **13** (1959) 78–86.
- [22] A. Baragar, *Lattice points on hyperboloids of one sheet*, *New York Journal of Mathematics* **20** (2014) 1253–1268.
- [23] A. V. Kontorovich, *The Hyperbolic Lattice Point Count in Infinite Volume with Applications to Sieves*, *arXiv e-prints* (Dec, 2007) arXiv:0712.1391, [0712.1391].
- [24] W. Duke, Z. Rudnick and P. Sarnak, *Density of integer points on affine homogeneous varieties*, *Duke Math. J.* **71** (07, 1993) 143–179.
- [25] D. Lowry-Duda, *On Some Variants of the Gauss Circle Problem*, *arXiv e-prints* (Apr., 2017) arXiv:1704.02376, [1704.02376].
- [26] H. Oh and N. A. Shah, *Limits of translates of divergent geodesics and integral points on one-sheeted hyperboloids*, *Israel Journal of Mathematics* **199** (2014) 915–931.
- [27] D. Bressoud and S. Wagon, *A Course in Computational Number Theory*. No. vol. 1 in A Course in Computational Number Theory. Key College Pub., 2000.
- [28] L. Tan, *Rational points on the circle*, *Mathematics Magazine* **69** (1996) 163–171.
- [29] A. Schild, *Discrete Space-Time and Integral Lorentz Transformations*, *Canadian J. Math.* **1** (1949) 29–47.
- [30] A. Schild, *Discrete Space-Time and Integral Lorentz Transformations*, *Phys. Rev.* **73** (Feb, 1948) 414–415.
- [31] H. S. M. Coxeter, *Discrete groups generated by reflections*, *Annals of Mathematics* **35** (1934) 588–621.
- [32] È. B. Vinberg, *Discrete Groups generated by reflections in Lobačevskiĭ spaces*, *Mathematics of the USSR-Sbornik* **1** (apr, 1967) 429–444.
- [33] V. G. Kac, *Infinite dimensional Lie algebras*. Cambridge, UK: Univ. Pr. (1990) 400 p, 1990.
- [34] A. J. Feingold, A. Kleinschmidt and H. Nicolai, *Hyperbolic Weyl groups and the four normed division algebras*, *J. Algebra* **322** (2009) 1295–1339, [0805.3018].

- [35] M. Axenides, E. Floratos and S. Nicolis, *The quantum cat map on the modular discretization of extremal black hole horizons*, *Eur. Phys. J.* **C78** (2018) 412, [[1608.07845](#)].
- [36] V. I. Arnol'd and A. Avez, *Ergodic problems of classical mechanics*. The mathematical physics monograph series. W. A. Benjamin, New York, NY, 1968.
- [37] A. F. Horadam, *A Generalized Fibonacci Sequence*, *The American Mathematical Monthly* **68** (1961) 455–459.
- [38] F. J. Dyson and H. Falk, *Period of a discrete cat mapping*, *The American Mathematical Monthly* **99** (1992) 603–614.