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Tachyonic Dirac Equation in Schwarzschild Metric

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Heisenberg's uncertainty principle at the Planck scale extends to extensions of Dirac equations. In this paper, the generalized uncertainty problem (GUP) theory is used as an extension of the Dirac equation with the mass term $m_1 + i\gamma^5 m_2$ (tachyonic) in the Schwarzschild metric. The eigenvalue problem for a particle in a gravitational field a central mass creates is also solved. The fundamental spinor solution for the tachyonic Dirac equation is found on a helicity basis. This study shows that it is impossible with current theories to formulate a covariant equation that could be repulsed by gravity in the framework of space-like particles.

Keywords: Generalized Uncertainty Principle, Relativistic Quantum Mechanics, Tachyonic Dirac Equation, Schwarzschild Metric

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1. Introduction

Theoretically, theoretical physics aims to merge Quantum Mechanics and General relativity. All these theories indicate the minimal observable distance of Planck length order $\ell_p = \sqrt{G_N \frac{\hbar}{c^3}}$ exists, where G_N is the Newton Constant because, at the Planck scale, the gravitational fluctuations must be considered. Several justifications extend to a generalization of the uncertainty principle (GUP) by the form like

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left[1+\beta\left(\frac{\Delta p}{m_p c}\right)^2\right] \tag{1}$$

$m_p = \sqrt{\frac{\hbar c}{G_N}}$ is the Planck mass, and β is the deformation parameter. The GUP and its associated definitions are used to find a generalized Dirac equation and solve its eigenvalue problem for a free particle [1, 2]. The Dirac equation under the theory of GUP corresponds to a Schrödinger-like equation with effective potential. The consequences of GUP have been studied extensively [3–12]. The primary purpose of our work is to modify the original Dirac equation for tachyons [17] in a curved space-time using the Schwarzschild metric and study the effect of GUP on the definition of momentum operator and solve the eigenvalue problem. We choose $\hbar = c = 1$.

The Schwarzschild solution describes space-time under the influence of a massive, non-rotating, spherically symmetric object. It is considered by some to be one of the most straightforward and valuable solutions to Einstein's field equations. The Schwarzschild metric is the solution to the Einstein field equations describing the gravitational field outside a spherical mass by supposing the mass's electric charge, the angular momentum of the mass, and the universal cosmological constant are all zero.

The Dirac equation has been generalized in curved space-time to investigate the behavior of spin-half particles in gravitational fields [18–20]. But the Dirac equation has never been considered in the framework of relativistic quantum physics of tachyons because the classical Dirac equation is not suitable for describing the dynamics of a superluminal particle that violates the Lorentz invariance. However, some experiments show that neutrinos fulfill the energy-momentum relationship of tachyons [21–23]. In this study, another approach is based on introducing superluminal Lorentz transformation. The results from this approach show that the tachyonic Dirac equation is more suitable for describing the dynamics of a particle closest to the classical concept. Based on our knowledge, even with experimental data, current theoretical models fail to reconcile the physics of tachyons with relativistic quantum mechanics.

2. Gravitational Dirac Equation

The original Dirac equation is written as

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0 \tag{2}$$

And

$$\{\gamma^\mu \gamma^\nu\} = 2\eta^{\mu\nu} I_4 \tag{3}$$

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$\eta^{\mu\nu}$ is a special relativity metric. In the gravitational field satisfied by Einstein's equations,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{C^4} T_{\mu\nu} \tag{4}$$

Which $R_{\mu\nu}$ is the Ricci curvature tensor, R is the scalar curvature, and $T_{\mu\nu}$ is the stress-energy tensor. The generalization of the Dirac equation of a particle inside a gravitational field has been considered [24]. In this study, the tachyonic Dirac equation $(i\gamma^\mu \partial_\mu$- m_1- $i\gamma^5 m_2)\psi(x)$=0 with metric tensor $g_{\mu\nu}$ considered which is generated by the gravitational field. Following the spinor representation of the Dirac wave function and spinor connection, the tachyonic Dirac equation with a metric tensor $g_{\mu\nu}$ is $\left(i\gamma^\mu D_\mu - m_1 - i\gamma^5 m_2\right)\psi\,(x) = 0$. ∂_μ is replaced by the covariant derivative D_μ, which satisfies

$$(D_\mu \widetilde{\psi})^\rho = \widetilde{\psi}^\rho_{;\mu} = \partial_\mu \widetilde{\psi}^\rho + \Gamma^\rho_{\sigma\mu} \widetilde{\psi}^\rho \tag{5}$$

$\Gamma^\rho_{\sigma\mu}$ are the Christoffel symbols associated with the metric $g_{\mu\nu}$.

Now we can consider the Schwarzschild metric and study the case of a central mass M that affects the tachyonic Dirac equation. This metric is the solution to Einstein's field equations in empty space only outside a spherical gravitational body with Schwarzschild coordinates (t, r,θ, φ). The line element for a proper time is:

$$ds^2 = (1 - \frac{r_s}{r})\, dt^2 - \, (1 - \frac{r_s}{r})^{-1} dr^2 - \, r^2 d\theta^2 - \, r^2 sin^2\theta d\varphi^2 \tag{6}$$

Where r_s is the Schwarzschild radius of the massive body related to its mass by r_s= $\frac{2GM}{C^2}$.

Now we should simplify the problem. The Schwarzschild metric is symmetrical about θ= $\frac{\pi}{2}$; any geodesic that begins moving in that plane will remain in that plane indefinitely. We fix the θ coordinate to be $\frac{\pi}{2}$ so that the metric simplifies to:

$$c^2 d\tau^2 = (1 - \frac{r_s}{r})\, c^2 dt^2 - \frac{dr^2}{1 - \frac{r_s}{r}} - r^2 d\varphi^2 \tag{7}$$

In this equation, we have two constants of motion:

- **Specific Angular Momentum**

$$h = \frac{L}{\mu} = r^2 \frac{d\varphi}{d\tau}$$

$$\overrightarrow{h} = \overrightarrow{r} \times \overrightarrow{v} = \frac{\overrightarrow{L}}{M}, \overrightarrow{L} = \overrightarrow{r} \times mv$$

$$\frac{dh}{dt} = r \times \frac{F}{m} = r \times (-\frac{GM}{r^2})\hat{r} = 0$$

  L is the total angular momentum of two bodies, μ is reduced mass.

- **Total Energy**
  If the metric of our space-time is $ds^2 = g_{\mu\nu} dx^\mu\, dx^\nu$, the Schwarzschild expression for the metric around a mass M is [25]:

$$c^2 d\tau^2 = \, c^2 g_{00} dt^2 - \, g_{rr} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\varphi^2 \tag{8}$$

With $g_{00} = \frac{1}{g_{rr}} = 1 - \frac{2GM}{C^2 r}$
The square of velocities will be

$$c^2 \rightarrow c^2 g_{00}$$
$$v^2 \rightarrow g_{rr}\ (dr/dt)^2 + r^2 (d\theta/dt)^2$$

Thus the Schwarzschild metric can be written

$$g_{00} = \frac{dt}{d\tau} = \text{constant}$$

The relativistic equations of motion must go over into the standard forms given by Newton's theory in the non-relativistic limit. So we should identify the constant in the non-relativistic limit. In the non-relativistic limit, we have

$$g_{00}\frac{dt}{d\tau} \cong \frac{1}{mc^2}\left(mc^2 + \frac{1}{2}mv^2 - \frac{GMm}{r}\right) \tag{9}$$

m is the mass of a particle. By comparison with Newton's theory, we can identify the constant with

$$g_{00}\frac{dt}{d\tau} = \frac{1}{mc^2}\left(mc^2 + E\right) \tag{10}$$

Consequently, it yields the expression

$$E = mc^2 g_{00}\frac{dt}{d\tau} - mc^2$$

Therefore, the total energy is

$$\left(1 - \frac{r_s}{r}\right)\frac{dt}{d\tau} = \frac{E}{mc^2} \tag{11}$$

By substituting these constants into the definition of the Schwarzschild metric, we have

$$c^2 d\tau^2 = (1 - \frac{r_s}{r})\ c^2 dt^2 - \frac{dr^2}{1 - \frac{r_s}{r}} - r^2 d\theta^2 - r^2 sin^2\theta d\varphi^2 \tag{12}$$
$$c^2 = (1 - \frac{r_s}{r})\ c^2 (\frac{dt}{d\tau})^2 - \frac{1}{1 - \frac{r_s}{r}}\ (\frac{dr}{d\tau})^2 - r^2 (\frac{d\varphi}{d\tau})^2$$

Finally, the equation of motion for the radius as a function of the proper time τ

$$(\frac{dr}{d\tau})^2 = \frac{E^2}{m^2 c^2} - \left(1 - \frac{r_s}{r}\right)\left(c^2 + \frac{h^2}{r^2}\right) \tag{13}$$

3. Solving The Geodesic Equation

The question is how to characterize motion in general relativity. The world line of a free test particle between two time-like separated points extremes the proper time between them. A test particle is not a significant source of space-time curvature, and a free particle is only under curved space-time. Now we use Lagrangian mechanics to deduce the equations of motion for a metric.

The proper time along the time-like world line between point A and point B for the metric $g_{\mu\nu}$ is given by

$$\tau_{AB} = \int_A^B d\tau = \int_A^B \left(-g_{\mu\nu}(x)\, dx^\mu dx^\nu\right)^{\frac{1}{2}} \tag{14}$$

$\mu, \nu = 0, 1, 2, 3 \longrightarrow$ Einstein summation notation
Parametrize the four coordinates with the σ parameter.

$$\begin{cases} \sigma = 0 \ \ at\ A \\ \sigma = 1 \ \ at\ B \end{cases}$$

$$\tau_{AB} = \int_0^1 d\sigma \left(-g_{\mu\nu}(x) \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}\right)^{\frac{1}{2}} \tag{15}$$

We can treat that integrated as Lagrangian

$$\mathcal{L} = \left(-g_{\mu\nu}(x) \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}\right)^{\frac{1}{2}} \tag{16}$$

World lines extremizing proper time are those that satisfy the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{d}{d\sigma} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{dx^\mu}{d\sigma}\right)} \right) = 0 \tag{17}$$

The resulting equations will all have second derivatives concerning σ that can be changed. These four equations together give the equation for the world line extremizing the proper time. This world line is called the geodesic. Each of the equations will have the form:

$$\frac{d^2 x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \tag{18}$$

Christoffel symbols depend on the metric and are taken to be symmetric in the lower indices. These equations together are the geodesic equation. We use the Schwarzschild metric to compute the Christoffel symbols $\Gamma^\rho_{\sigma\mu}$ and to write the tachyonic Dirac equation. We attempt to solve the eigenvalue equation for the tachyonic Dirac equation. We should calculate the generalized potential from a set of geodesic equations for the Schwarzschild metric to solve the problem. To derive the effective potential for a test particle around a massive object from the Schwarzschild metric, we should assume the particle is on the equatorial plane where $\phi = \frac{\pi}{2}$

$$-c^2 d\tau = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\theta \tag{19}$$

The radial equation is

$$\ddot{r} = -\Gamma^1_{\mu\nu}\, \dot{x}^\mu\, \dot{x}^\nu \tag{20}$$

The appropriate Christoffel symbols are

$$\Gamma^1_{00} = \frac{c^2 r_s}{2r^2}$$

$$\Gamma^1_{11} = -\frac{r_s}{2r^2}(1-\frac{r_s}{r})^{-1}$$
$$\Gamma^1_{22} = -r\,(1-\frac{r_s}{r})$$

Making the radial equation

$$\ddot{r} = -\frac{c^2 r_s}{2r^2}\left(1-\frac{r_s}{r}\right)\dot{t}^2 + \frac{r_s}{2r^2}\left(1-\frac{r_s}{r}\right)^{-1}\dot{r}^2 + r(1-\frac{r_s}{r})\dot{\theta}^2 \tag{21}$$

From the metric, we know

$$\dot{t}^2 = \left(1-\frac{r_s}{r}\right)^{-1}\left(1+\frac{1}{c^2}\left(1-\frac{r_s}{r}\right)^{-1}\dot{r}^2 + \frac{r^2}{c^2}\dot{\theta}^2\right) \tag{22}$$

By substituting, we have

$$\ddot{r} = -\frac{c^2 r_s}{2r^2} + r\dot{\theta}^2 - \frac{3r_s}{2}\dot{\theta}^2$$
$$\dot{\theta} = \frac{L}{\mu r^2}$$
$$r_s = \frac{2GM}{c^2}$$

$$m\ddot{r} = -\frac{GMm}{r^2} + \frac{mL^2}{\mu^2 r^3} - \frac{3L^2 G(M+m)}{\mu c^2 r^4} \tag{23}$$

The potential is found to be

$$U\,(r) = -\frac{GMm}{r} + \frac{mL^2}{2\mu r^2} - \frac{G\,(M+m)\,L^2}{c^2\mu r^3} \tag{24}$$

By rewriting the equation of motion (13) that explains the orbit of the infinitesimal mass around the central mass M, we have

$$\frac{1}{2}m_1(\frac{dr}{d\tau})^2 - \left[\frac{E^2}{2m_1c^2} - \frac{1}{2}m_1c^2\right] = \frac{GMm_1}{r} - \frac{m_1L^2}{2\mu r^2} + \frac{G\,(M+m_1)\,L^2}{c^2\mu r^3} \tag{25}$$

The left term is relativistic energy, and the right term is effective potential. If $M = 0$, $V_1 = 0$, the relativistic energy becomes

$$\frac{1}{2}m_1(\frac{dr}{d\tau})^2 - \left[\frac{E^2}{2m_1c^2} - \frac{1}{2}m_1c^2\right] = 0 \tag{26}$$

The mass shell condition is ${E_0}^2 = p^2c^2 + m^2c^4$, $E = E_0$. If $V_1 \neq 0$, we have

$$E^2 = p^2c^2 + {m_1}^2c^4 + 2m_1c^2V_1 = {E_0}^2 + 2m_1c^2V_1 = {E_0}^2\big(1+\frac{2m_1c^2V_1}{E_0}\big) \tag{27}$$

By using perturbation of energy

$$E \cong E_0\big(1+\frac{m_1c^2V_1}{{E_0}^2}\big) \cong E_0\big(1+\frac{m_1V_1}{m_1c^2}\big) \cong E_0\big(1+\frac{V_1}{m_1c^2}\big) \tag{28}$$

We must consider the tachyonic Dirac equation $\left(i\gamma^{\mu}\partial_{\mu}-m_1 - i\gamma^5 m_2\right)\psi\,(x) = 0$, of the form like

$$i\partial_0\,\widetilde{\psi} = -\gamma_0\left(i\gamma^i\partial_i + m_1 + i\gamma^5 m_2\right)\left(1+\frac{V_1}{m_1}\right)\widetilde{\psi} \tag{29}$$

$$i\gamma^i\partial_i + m_1 = \alpha.p$$

This expression is equivalent to

$$(E-\beta V_1)\,\widetilde{\psi} = \left[(\alpha.p) + i\gamma^5 m_2\left(1+\frac{V_1}{m_1}\right) + \beta m_1\right]\widetilde{\psi} \tag{30}$$

The momentum is

$$\tilde{p} = p\left(1+\frac{V_1}{m_1}\right) \tag{31}$$

So the effective potential is

$$(E-\beta V_1)\,\widetilde{\psi} = \overline{H} = \overrightarrow{\alpha}.\overrightarrow{p} + \beta m_1 + i\beta\gamma^5 m_2 \tag{32}$$

Where $\overline{H}$ is the Hamiltonian. For solving this tachyonic Dirac equation, we use the following solution

$$\widetilde{\psi}\,(z,t) = \widetilde{\psi}(z)e^{-i\lambda(E+V_1)t} \tag{33}$$

We assume the particle is moving in the z direction. $\lambda = \pm 1$ corresponds to the particle and antiparticle spectrum. We only study the case $\lambda = 1$. Regarding the 4-dimensional nature of spinors, we have

$$\widetilde{\psi}\,(z) = \begin{pmatrix}\varphi\\ \chi\end{pmatrix} = \begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\end{pmatrix} \tag{34}$$

By using an operator like the form of $\overline{H}$, we have

$$\begin{aligned}&\left[E\begin{pmatrix}1&0\\0&1\end{pmatrix} - V_1\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right]\begin{pmatrix}\varphi\\ \chi\end{pmatrix} =\\ &= \begin{pmatrix}0&\sigma\\ \sigma&0\end{pmatrix}.p\left(1+\frac{V_1}{m_1}\right)\begin{pmatrix}\varphi\\ \chi\end{pmatrix} + m_1\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} + im_2\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}\gamma^5\end{aligned} \tag{35}$$

We should find the eigenvalues and eigenvectors of this equation. If suppose that

$$\varphi\,(z) = \varphi_0 e^{ipz}$$

$$\chi\,(z) = \chi_0 e^{ipz}$$

After substituting these solutions into the previous equation, we have

$$(E-V_1)\,\varphi_0 = (\sigma_z\cdot p)\left(1+\frac{V_1}{m_1}\right)\chi_0 + m\varphi_0 + i\gamma^5 m_2\varphi_0$$

$$(E+V_1)\,\chi_0 = (\sigma_z \cdot p)\left(1+\frac{V_1}{m_1}\right)\varphi_0 - m\chi_0 - i\gamma^5 m_2\chi_0 \tag{36}$$

That implies

$$\chi_0 = \frac{(\sigma_z \cdot p)\,(1+\frac{V_1}{m_1})}{E - V_1 - m_1 - i\gamma^5 m_2}\varphi_0 \tag{37}$$

Supposing $\varphi_0 = \binom{1}{0}$, then we have

$$\chi_0 = \frac{(\sigma_z \cdot p)\,(1+\frac{V_1}{m_1})}{E - V_1 - m_1 - i\gamma^5 m_2}\begin{pmatrix}1\\0\end{pmatrix} \tag{38}$$

The following determinant gives the energy spectrum

$$\begin{vmatrix} E - V_1 - m_1 - i\gamma^5 m_2 & -(\sigma_z.p)(1+\frac{V_1}{m_1}) \\ -(\sigma_z \cdot p)(1+\frac{V_1}{m_1}) & E + V_1 + m_1 + i\gamma^5 m_2 \end{vmatrix} = 0 \tag{39}$$

The time-independent wave function is

$$\widetilde{\psi} = Ne^{i[pz-\lambda(E+V_1)t]}\begin{pmatrix} \varphi \\ \frac{(\sigma_z \cdot p)(1+\frac{V_1}{m_1})}{E-V_1-m_1-i\gamma^5 m_2}\varphi \end{pmatrix} \tag{40}$$

By using the orthogonality condition, the normalization constant could be computed

$$\int \widetilde{\varphi}^t_{\lambda,p}\,(z,t)\,\widetilde{\varphi}_{\lambda,p}\,(z,t)\,d^3z = \delta_{\lambda,\lambda'}\delta(p-p') \tag{41}$$

Which leads to

$$N^2\left(\varphi^\dagger\varphi + \varphi^\dagger\left(\frac{(\sigma_z \cdot p)\left(1+\frac{V_1}{m_1}\right)}{E - V_1 - m_1 - i\gamma^5 m_2}\right)^2\varphi\right) = 1 \tag{42}$$

Or

$$N = \sqrt{\frac{(E - V_1 - m_1 - i\gamma^5 m_2)^2}{\left(E - V_1 - m_1 - i\gamma^5 m_2\right) + \left((\sigma_z \cdot p)(1+\frac{V_1}{m_1})\right)^2}} \tag{43}$$

Finally, the normalization factor becomes

$$N = \frac{(E - V_1 - m_1 - i\gamma^5 m_2)}{\sqrt{(E - V_1 - m_1 - i\gamma^5 m_2)^2 + p^2(\frac{V_1}{m_1})}} \tag{44}$$

The wave function for a particle in the gravitational field created by a central mass M is

$$\widetilde{\varphi}_{p,\lambda}(z,t) = \frac{(E - V_1 - m_1 - i\gamma^5 m_2)}{\sqrt{(E - V_1 - m_1 - i\gamma^5 m_2)^2 + p^2(1+\frac{V_1}{m_1})}}\begin{pmatrix} \binom{1}{0} \\ \left(\frac{(\sigma_z \cdot p)\left(1+\frac{V_1}{m_1}\right)}{E-V_1-m_1-i\gamma^5 m_2}\right)\binom{0}{1} \end{pmatrix} e^{i[pz-\lambda(E+V_1)t} \tag{45}$$

For $\lambda = \pm 1$
Now we should consider the helicity. In relativistic quantum mechanics, helicity is defined as

$$\Lambda_s = \Sigma \cdot \hat{p}$$

$$\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, \hat{p} = \frac{p}{|p|} \tag{46}$$

The helicity operator does not change in this case. Two possible eigenvalues for the helicity operator: $\pm\frac{1}{2}$ are available because we only considered the z-direction

$$\widetilde{\varphi}_{p,\lambda,+\frac{1}{2}} = \frac{(E - V_1 - m_1 - i\gamma^5 m_2)}{\sqrt{(E - V_1 - m_1 - i\gamma^5 m_2)^2 + p^2(1 + \frac{V_1}{m_1})}} \begin{pmatrix} \binom{1}{0} \\ \left(\frac{(\sigma_1 \cdot p)\left(1+\frac{V_1}{m_1}\right)}{E - V_1 - m_1 - i\gamma^5 m_2}\right)\binom{1}{0} \end{pmatrix} e^{i[pz - \lambda(E+V_1)t} \tag{47}$$

For positive helicity

$$\widetilde{\varphi}_{p,\lambda,-\frac{1}{2}} = \frac{(E - V_1 - m_1 - i\gamma^5 m_2)}{\sqrt{(E - V_1 - m_1 - i\gamma^5 m_2)^2 + p^2(1 + \frac{V_1}{m_1})}} \begin{pmatrix} \binom{0}{1} \\ \left(\frac{(\sigma_1 \cdot p)\left(1+\frac{V_1}{m_1}\right)}{E - V_1 - m_1 - i\gamma^5 m_2}\right)\binom{0}{1} \end{pmatrix} e^{i[pz - \lambda(E+V_1)t} \tag{48}$$

For negative helicity
Another thing that should be considered is whether the gravitational tachyonic Dirac equation can derive the scale length quantization of a region where a particle is confined when it can derive from the curved space-time but does not derive from the generalized uncertainty principle.

4. Length Quantization in the Gravitational Tachyonic Dirac Equation

Here, we investigate whether finding a length quantization from the gravitational tachyonic Dirac equation is possible. We postulate a particle confined by a spherical cavity of radius R, which can be explained by potential

$$U(r) = 0, \quad \text{for } r \leq R$$

$$U(r) = U_0 \to \infty, \quad \text{for } r > R$$

Furthermore, we should determine a parameter that checks whether a system is in curved space-time. The parameter is a. Since the generalized potential V_1 is expressed in Schwarzschild coordinates (t, r, θ, φ), we cannot use wave functions $\widetilde{\varphi}_{p,\lambda,\pm\frac{1}{2}}$. First, we should write the tachyonic Dirac equation in spherical coordinates

$$-\gamma_0\left[i(\gamma^r \partial_r + \gamma^\theta \cdot \frac{1}{r}\partial_\theta + \gamma^\phi \frac{1}{r\sin\theta}\ \partial\phi) + (m_1 + i\gamma^5 m_2 + U\right](1 + \frac{aV_1}{m_1}) \tag{49}$$

This could be written as

$$i\partial_0\widetilde{\psi} + \gamma_0\left(m_1 + i\gamma^5 m_2 + U\right)\frac{aV_1}{m_1}$$

$$= \left[-\gamma_0 i(\gamma^r \partial_r + \gamma^\theta \cdot \frac{1}{r}\partial_\theta + \gamma^\phi \frac{1}{r\sin\theta}\partial_\phi)(1 + \frac{aV_1}{m_1}) - \gamma_0(m_1 + i\gamma^5 m_2 + U)\widetilde{\psi}\right] \quad (50)$$

If we suppose $\beta = -\gamma_0$, $\overrightarrow{\alpha} = -i\gamma_0(\gamma^r, \gamma^\theta, \gamma^\phi)$ and $\overline{p} = \left(\partial_r, \frac{1}{r}\partial_\theta, \frac{1}{r\sin\theta}\partial_\phi\right)$ the equation is equivalent to

$$\left[E - \beta(m_1 + i\gamma^5 m_2 + U)\frac{aV_1}{m_1}\right]\widetilde{\psi} = \left[\left(\overrightarrow{\alpha} \cdot \overrightarrow{p}\right)\left(1 + \frac{aV_1}{m_1}\right) + \beta(m_1 + i\gamma^5 m_2 + U)\right]\widetilde{\psi} \quad (51)$$

Now we see an eigenvalue problem in the form

$$E\widetilde{\psi} = \left(\tilde{H} + \beta(m_1 + i\gamma^5 m_2 + U)\,\frac{aV_1}{m_1}\right)\widetilde{\psi} \quad (52)$$

$$\tilde{H} = \overrightarrow{\alpha} \cdot \overrightarrow{p}\left(1 + \frac{aV_1}{m_1}\right) + \beta(m_1 + i\gamma^5 m_2 + U) \quad (53)$$

For solving this tachyonic Dirac equation, Eq. (52), the following solution is used

$$\widetilde{\psi}\,(r,t) = \widetilde{\psi}\,(r)\, e^{-i\lambda(E+aV_1)} \quad (54)$$

In this case, we postulate that the particle moves in a radial direction and $\lambda = \pm 1$. The case $\lambda = 1$ only considered. The 4-dimensional Dirac spinor is

$$\widetilde{\psi}(r) = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \quad (55)$$

So, the tachyonic Dirac equation (38) has the form

$$\left(\overrightarrow{\alpha} \cdot \overrightarrow{p}\right)\left(1 + \frac{aV_1}{m_1}\right)\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} + \left(m_1 + i\gamma^5 m_2 + U\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$$
$$+ \left(m_1 + i\gamma^5 m_2 + U\right)\frac{aV_1}{m_1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = E\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \quad (56)$$

By using similar reasoning, a Dirac spinor is obtainable in the form

$$\widetilde{\psi}\,(r) = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} g(r)\widetilde{\psi}_1(\hat{r}) \\ f(r)\widetilde{\psi}_1(\hat{r}) \end{pmatrix} \quad (57)$$

This solution is separable into radial and angular parts. These solutions extend to the perturbation of energy quantification because of the potentials V_1 and U. Moreover, $f(r)$ and $g(r)$ satisfy Eq.(56) for $a \neq 0$ if

$$(\overline{\alpha} \cdot \overline{p}) = \left(\frac{aV_1}{m}\right)\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} + (m + U)\,\frac{aV_1}{m}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0 \quad (58)$$

By studying the reasoning presented in [26], it is worth noting that exploring possible phenomenological implications of space quantization if it has any measurable effects at distance scales far greater than the Planck length, such as at about $10^{-4} fm$, the length scale to be probed by the Large Hadron Collider.

5. Tachyonic Dirac Equation through Superluminal Lorentz Transformation of Dirac Wave functions

We need to have a suitable covariant equation to describe the dynamics of spin-half particles, which could help construct a theory capable of interpreting the experimental results [21–23], especially for designing new experiments of precise measurements of the neutrino mass square.

By tachyonic Dirac equation, we mean a covariant equation that describes the dynamics of a spin-half particle moving with superluminal velocity in the quantum mechanics framework. In this section, the tachyonic Dirac equation is formulated by performing a superluminal Lorentz transformation on the $\widetilde{\psi}$ matrix. This approach implies that the tachyonic wave functions cannot be used to understand an irreducible representation of the Lorentz group [27]. The superluminal Lorentz transformation denoted by $\Lambda_{\text{superluminal}}$ transforms a subliminal reference frame $\tilde{S}$ in a superluminal frame S. This operator must satisfy the following terms:

$$(\Lambda_{\text{superluminal}})^{\dagger}{=}(\Lambda_{\text{superluminal}})^{-1} \tag{59}$$

$$\gamma^0\Lambda_{\text{superluminal}}\ \gamma^0 = \Lambda_{\text{superluminal}} \tag{60}$$

So the tachyonic spinor matrix can be written as $\psi = \Lambda_{\text{superluminal}}\ \widetilde{\psi}$. We use the explicit form of $\Lambda_{\text{superluminal}}$ has been proposed. It reads [28]:

$$\Lambda_{\text{superluminal}}\ = \frac{(1 + i\boldsymbol{\alpha}\cdot\boldsymbol{n})}{\sqrt{2}} \tag{61}$$

$\boldsymbol{\alpha}$ Is the vector whose components are the three Dirac matrices α^{μ}, and $\boldsymbol{n}$ is the vector whose components are the direction cosines that the relative direction of motion of the reference frame $\tilde{S}$ with respect to S. This matrix is Unitary and Hermitian since 1 is symmetric and $\alpha.n$ is antisymmetric. For simplicity, suppose that the relative motion between the two reference frames occurs along the x axis. In this case, the $\Lambda_{\text{superluminal}}$ matrix is $\Lambda_{\text{superluminal}}\ = \frac{(1+i\gamma^5)}{\sqrt{2}}$.
To obtain the tachyonic Dirac equation, we perform the following transformation on the Dirac spinor:

$$\psi = \Lambda_{\text{superluminal}}\ \widetilde{\psi} \longrightarrow \psi = (\Lambda_{\text{superluminal}}\ )^{-1}\widetilde{\psi} = \frac{1}{\sqrt{2}}(1 - i\alpha\cdot n)\psi \tag{62}$$

Since $\widetilde{\psi}$ is a solution of the ordinary Dirac equation, we can write:

$$\left(i\hbar\gamma^0\partial_t - i\hbar c\gamma^{\mu}n_{\mu}\partial_{\mu} - 1mc^2\right)\frac{1}{\sqrt{2}}(1 - i\alpha\cdot n)\psi = 0 \tag{63}$$

We used the scalar product between α and n vectors for consistency of the formalism by multiplying the left-handed with $(\Lambda_{\text{superluminal}}\ )$. We have

$$\begin{aligned}\Lambda_{\text{superluminal}}\ \gamma^0(\Lambda_{\text{superluminal}}\ )^{-\mathbf{1}} &= i\gamma^{\mu}n_{\mu}\\ \Lambda_{\text{superluminal}}\ \gamma^{\mu}n_{\mu}(\Lambda_{\text{superluminal}}\ )^{-\mathbf{1}} &= i\gamma^0\end{aligned} \tag{64}$$

The superluminal transformation interchanges the temporal variable with the spatial one and vice versa. This means operator ∂_t is interchanged with ∂_{μ}. By substituting relations (64) in equation (63), we have the tachyonic Dirac equation:

$$(\hbar\gamma^0\partial_t - \hbar c\gamma^{\mu}n_{\mu}\partial_{\mu} + \mathbb{1}mc^2)\psi = 0 \tag{65}$$

Equation (65) is the superluminal transformation of the Dirac equation. Its covariance is trivially inherent from the fact that superluminal transformations are the components of the Lorentz group of the theory of relativity extended to superluminal motions [29], which, in turn, is consistent with the postulates of the theory of relativity [28]. So we can say that superluminal transformation transforms in a covariant way any other equation that satisfies the formalism of the ordinary theory of relativity.

6. Conclusions

On the Planck scale, the quantum fluctuations of space-time gravity modify Heisenberg's uncertainty principle. To reconcile quantum mechanics and general relativity, these effects must be considered leading to Heisenberg's uncertainty principle. As a result, we find the generalization of momentum and energy that causes the modified Dirac equation like the tachyonic one.

In this study, the eigenvalue problem of the tachyonic Dirac equation for a particle under the gravitational field of the central mass is solved. We assumed a Schwarzschild metric to solve the problem and considered the particle's motion equation inside the gravitational field created by central mass.

Finally, we conclude that it is not currently possible to formulate a covariant tachyonic Dirac equation capable of describing the neutrino's superluminal dynamics. If we choose the theory of describing the pseudo-tachyon neutrino, the experimental research should be directed toward the measurements of the square mass of the neutrino. On the contrary, if we choose the theory of relativity extended to superluminal motions, we have an equation that does not adapt correctly to the neutrino's symmetries. Choosing between these theories depends on the considered experiment. The enigmatic nature of neutrino affects the theoretical physics to find a perfect theory in the tachyonic framework.

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