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Celestial soft dressings from generalised Wilson lines

Kevin Nguyen,[a,*] **Alan Rios Fukelman**[a] **and Chris White**[b]

[a] *Department of Mathematics, King's College London, The Strand, London WC2R 2LS, United Kingdom*

[b] *Department of Physics and Astronomy, Queen Mary University of London, 327 Mile End Road, London E1 4NS, United Kingdom*

E-mail: kevin.nguyen@kcl.ac.uk, alan.rios_fukelman@kcl.ac.uk, christopher.white@qmul.ac.uk

In this review article, we revisit the connection between dressing of scattering states in quantum electrodynamics by clouds of soft photons, and their dressing by (generalised) Wilson line operators. In particular, we show that the leading and subleading soft conformal dressings considered in the context of celestial holography can be straightforwardly obtained from generalised Wilson lines, and that this only requires knowledge of the asymptotic behaviour of the photon field near null and timelike infinity.

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1. Introduction

In quantum electrodynamics (QED) and other gauge theories, the traditional scattering S-matrix between Fock states infamously suffers from infrared divergences [1–6]. In order to overcome this issue, dressing of Fock states by coherent states of soft photons, commonly referred to as Faddeev–Kulish (FK) dressing, was designed such that S-matrix elements be free of these infrared divergences [7–12]. A detailed account of this and other related topics is provided in [13]. Importantly, a close relation between these dressed states and the gauge-invariant quantum fields constructed through the introduction of Wilson lines [14, 15] was also later uncovered [16, 17].

More recently it has been observed that many infrared features of the S-matrix are actually controlled by large gauge transformations (LGT), i.e., gauge transformations with support at infinity [18–25]. In particular it has been understood that FK states are eigenstates of the LGT charge, and that the latter characterises scattering superselection sectors [24–26]. In fact conservation of the LGT charge is equivalent to the leading soft photon theorem [19–22], which trivialises when considering the scattering of dressed states rather than that of Fock states. Given that there also exists a subleading soft photon theorem [27–30], a natural extension was the construction of dressed states that trivialise this subleading soft theorem as well [31]. However the subleading dressings do not relate to infrared divergences and are therefore not strictly needed. In this article we observe that the subleading soft dressing can be given a different interpretation in terms of a generalised Wilson line operator, introduced in the literature to efficiently compute scattering amplitudes at subleading order in a soft expansion [32–38].

Largely motivated by the connection between soft theorems and LGT charge conservation, a new approach to scattering amplitudes emerged under the name of *celestial holography*. For recent reviews of this rapidly growing field we refer the reader to [39, 40]. In that approach a different basis of one-particle states is used, given by the set of conformal primary states that are boost rather than momentum eigenstates [41–50]. These behave precisely as (quasi)-conformal primaries for the Lorentz group $SO(1,3) \simeq SL(2,\mathbb{C})$. In this conformal basis the soft theorems take the form of Ward identities associated with conserved currents of a two-dimensional conformal field theory [51, 52]. On the other hand infrared divergences are then accounted for by a particular conformal field, namely the Goldstone mode of spontaneously broken LGT [53]. In order to define an S-matrix free of these infrared divergences, a notion of dressing appropriate to the conformal primary states has been similarly proposed [25, 54, 55], which is different but closely related to the FK dressing.

In this article we revisit the connection between (generalised) Wilson line operators and the various notions of dressings discussed above. Our aims are not only to collect useful results in one place, but also to explain the connection between them. In particular we show how the leading and subleading soft dressings are easily reproduced from the Wilson line operators through use of the asymptotic expansions of the gauge potential near null infinity $\mathscr{I}$ and timelike infinity i^+. This approach therefore ties together the asymptotic structure of QED with the soft dressings more directly, and gives a nice geometrical picture for the latter in terms of Wilson lines tracking the classical trajectories of the scattered particles.

The article is organised as follows. We start in section 2 by reviewing the leading and subleading FK soft dressings, the conformally soft dressings and the Wilson line dressings, together with the known relations between them. In section 3 we describe the asymptotic regions near $\mathscr{I}$ and i^+, as well as the corresponding parametrisation of classical particle trajectories. In appendix A and B we study the asymptotic structure of the electromagnetic potential near null infinity $\mathscr{I}$ and timelike infinity i^+, respectively. We use it in sections 4 and 5 to derive the soft dressings from the (generalised) Wilson line operators associated with massless and massive scattering states, respectively.

2. Preliminaries

We start by recalling three different versions of dressings and the known relations between them. These are Faddeev–Kulish (FK), conformally soft and Wilson line dressings. We will restrict the discussion to QED for simplicity.

Faddeev–Kulish dressings. As reviewed in the introduction, the FK dressing of charged particle states by clouds of soft photons were introduced in order to define an S-matrix free of infrared divergences. For a one-particle state $|p,J\rangle$ with momentum p, helicity $J=\pm s$ and electric charge eQ, the corresponding FK dressed state can be written

$$||p,J\rangle\rangle = \tilde{W}_0\,\tilde{W}_1\,|p,J\rangle\,, \tag{1}$$

with the *leading* FK dressing given by [12]

$$\tilde{W}_0 = \exp\left[eQ\int\frac{d^3\vec{q}}{(2\pi)^3}\frac{f(\omega)}{2\omega}\frac{p^\mu}{p\cdot q}\left(a^\dagger_\mu(q)-a_\mu(q)\right)\right], \tag{2}$$

and the *subleading* FK dressing given by [31]

$$\tilde{W}_1 = \exp\left[-ieQ\int\frac{d^3\vec{q}}{(2\pi)^3}\frac{f(\omega)}{2\omega}\frac{q_\nu J^{\nu\mu}}{p\cdot q}\left(a^\dagger_\mu(q)+a_\mu(q)\right)\right]. \tag{3}$$

In the above expressions $a^\dagger_\mu(q)$ is the creation operator for a photon of null momentum q^μ with frequency $\omega = |\vec{q}\,|$, $J_{\mu\nu}$ is the Lorentz generator (or total angular momentum), and $f(\omega)$ a distribution with vanishingly small support around $\omega = 0$ satisfying $f(0) = 1$. The latter conditions simply mean that the dressing only retains the leading nontrivial contribution in the limit $\omega \to 0$.

Conformally soft dressings. This second type of dressing naturally arises in the description of scattering amplitudes in a basis of conformal primary states [25, 54, 55]. Conformal primary states are typically denoted $|\Delta, J, w, \bar{w}\rangle$, and depend on the conformal dimension Δ and helicity $J = \pm s$ as well as some insertion point $(w, \bar{w})$ on the celestial sphere at null infinity. Conformally soft dressings are then the analogue of the FK dressings in this context and are still formally given by the expressions (2)-(3), however with the important difference that $f(\omega) = 1$ across the whole spectrum. This means in particular that both hard and soft photons contribute to the dressing, and that conformally dressed states have infinite energy. Denoting by W_0 and W_1 the leading and subleading conformally soft dressing operators, we have the relation

$$W_0\big|_{\text{soft}} = \tilde{W}_0\,, \qquad W_1\big|_{\text{soft}} = \tilde{W}_1\,, \tag{4}$$

where, given a generic quantity $Q = \int_0^\infty d\omega\, Q(\omega)$ involving photons of all frequencies ω, we define its *soft component* through insertion of a function with vanishingly small support around $\omega = 0$ and satisfying $f(0) = 1$,

$$Q\big|_{\text{soft}} \equiv \int_0^\infty d\omega f(\omega) Q(\omega)\,. \tag{5}$$

The explicit expressions given in [25, 54, 55] for the conformally soft dressings of a massless state $|\Delta, J, w, \bar{w}\rangle$ are

$$W_0 = e^{-iQ\,\mathcal{S}(w,\bar{w})}\,, \tag{6}$$

and

$$W_1 = \exp\left[\frac{Q}{\omega}\left(2h\,\partial_w \mathcal{Y}^w + \mathcal{Y}^w \partial_w + 2\bar{h}\,\partial_{\bar{w}} \mathcal{Y}^{\bar{w}} + \mathcal{Y}^{\bar{w}} \partial_{\bar{w}}\right)\right], \tag{7}$$

where $\Delta = h + \bar{h}$ and $J = h - \bar{h}$. Here $\mathcal{S}(w, \bar{w})$ and $\mathcal{Y}^w(w, \bar{w})$ are conformal fields of weights $h = \bar{h} = 0$, given in terms of photon creation and annihilation operators by [55]

$$\begin{aligned}
\mathcal{S}(w,\bar{w}) &= \frac{ie}{\sqrt{2}(2\pi)^2}\int_0^\infty d\omega \left[\partial_{\bar{w}}^{-1}(a_-(q) - a_+^\dagger(q)) + \partial_w^{-1}(a_+(q) - a_-^\dagger(q))\right], \\
\mathcal{Y}^w(w,\bar{w}) &= \frac{e}{\sqrt{2}(2\pi)^2}\int_0^\infty d\omega\,\omega\,\partial_w^{-2}(a_+(q) + a_-^\dagger(q)), \\
\mathcal{Y}^{\bar{w}}(w,\bar{w}) &= \frac{e}{\sqrt{2}(2\pi)^2}\int_0^\infty d\omega\,\omega\,\partial_{\bar{w}}^{-2}(a_-(q) + a_+^\dagger(q)),
\end{aligned} \tag{8}$$

where there is an implicit parametrisation $q(\omega, w, \bar{w})$ of the photon momentum, given below in (20). For massive states the leading conformally soft dressing can also be found in [25, 54], however the subleading dressing expressed in terms of conformal fields has yet to be worked out.

Wilson line dressings. The leading conformally soft factor W_0 can be alternatively understood as a dressing by a Wilson line along the particle classical trajectory [16, 17, 56]. One justification for the presence of a Wilson line is simply the construction of gauge-invariant field variables at the cost of introducing some controlled degree of nonlocality [14, 15]. Indeed, consider a local quantum field $\phi(x)$ with electric charge eQ, which acquires a phase under local $U(1)$ transformations,

$$\phi(x) \mapsto e^{-ieQ\Lambda(x)}\phi(x)\,, \qquad A_\mu(x) \mapsto A_\mu(x) + e\,\partial_\mu \Lambda(x)\,. \tag{9}$$

The phase factor can be compensated for by considering a line segment $\Gamma[x_0, x]$ with endpoints x_0 and x, and dressing the local field $\phi(x)$ with the corresponding Wilson line operator,

$$\phi(x|\Gamma) \equiv e^{iQ\int_\Gamma A}\,\phi(x)\,. \tag{10}$$

The transformation of the dressed field is now instead

$$\phi(x|\Gamma) \mapsto e^{-ieQ\Lambda(x_0)}\phi(x|\Gamma)\,. \tag{11}$$

Fixing the reference point x_0 once and for all (such as the origin of Minkowski space), the above is a global phase shift common to all dressed field variables $\phi(x|\Gamma)$ independently of x, and local gauge invariance is therefore achieved.

The connection with the conformally soft dressing W_0 of a single-particle state $|p, J\rangle$ arises when choosing the path Γ to be the classical trajectory of the corresponding particle. As is well-known a Wilson line carries infinite energy, and thus corresponds to the dressings W_0 for which $f(\omega) = 1$ across the whole spectrum. Specifically, considering A_μ to be the *radiation* field[1], we have the equality [17]

$$W_0 = e^{-iQ\int_\Gamma A}\,. \tag{12}$$

This equivalence extends to the subleading dressing as well. Still considering the path Γ to be the classical trajectory of the particle, the subleading FK dressing (3) can be written[2]

$$W_1 = \exp\left[-\frac{iQ}{8}\int_\Gamma F_{\mu\nu}J^{\mu\nu}\right]\,. \tag{13}$$

In the leading soft case, we saw that FK dressings are in turn related to Wilson lines. The question then naturally arises as to whether the subleading FK dressing of eq. (13) can itself be associated with a generalisation of the conventional Wilson line. Indeed this is the case, where the relevant generalised Wilson line was first discussed in refs. [32, 33], in the context of collider physics. That reference discussed gluon radiation from scalar particles, and argued that one may indeed write a generalised Wilson line describing radiation up to next-to-soft level, where the additional contribution to the Wilson line involves a contraction of the field strength tensor with the generator of spin transformations, consistent with eq. (13). Further work has established that the full angular momentum generator is indeed obtained from conventional QCD approaches [34], which must anyway be the case on general grounds (e.g. Lorentz and gauge invariance). Related results in gravity were obtained in ref. [36], and applications to collider physics may be found in refs. [35, 37, 38].

[1]That part of the gauge potential which is sourced by onshell particles is known to account for Coulomb phases [17]. This will not be further discussed here.

[2]We fixed the normalisation of the exponent in order to agree with the expression (7) in section 4.

3. Asymptotic regions and classical trajectories

We aim to describe the FK dressings in terms of asymptotic components of the photon field. To achieve this we will exploit the relation between soft dressings and (generalised) Wilson lines reviewed in the preceding section.

A distinction between massless and massive fields will be made, as the asymptotic regions where the corresponding wavefunctions are supported are of a different nature. Massless fields and the corresponding classical null rays extend to future (past) null infinity $\mathscr{I}^+$ ($\mathscr{I}^-$), while massive fields and the corresponding timelike geodesics extend to future (past) timelike infinity i^+ (i^-). We present the coordinate systems adapted to $\mathscr{I}^+$ and i^+ and describe some relevant aspects of the corresponding classical trajectories.

Null infinity. Retarded coordinates $x = (r, u, z, \bar{z})$ are best adapted to the description of $\mathscr{I}^+$. The relation to cartesian coordinates X^μ can be conveniently written [57]

$$X^\mu = u\, n^\mu + r\, \hat{q}^\mu(z, \bar{z})\,, \tag{14}$$

where n^μ and $\hat{q}^\mu(z,\bar{z})$ are null vectors with cartesian components given by

$$\begin{aligned} \hat{q}^\mu(z,\bar{z}) &= \frac{1}{\sqrt{2}}\left(1 + z\bar{z}\,, z + \bar{z}\,, -i(z - \bar{z})\,, 1 - z\bar{z}\right)\,, \\ n^\mu &= \frac{1}{\sqrt{2}}\left(1, 0, 0, -1\right)\,, \end{aligned} \tag{15}$$

and satisfying the useful relations

$$n \cdot \hat{q} = -1\,, \qquad \hat{q}_i \cdot \hat{q}_j = -\,|z_{ij}|^2\,. \tag{16}$$

This allows to easily obtain the retarded coordinates r and u from the position vector X^μ through the following Lorentz contractions,

$$r = -n \cdot X\,, \qquad u = -\hat{q} \cdot X\,. \tag{17}$$

In retarded coordinates the flat metric takes the form

$$ds^2 = \eta_{\mu\nu}\, dX^\mu\, dX^\nu = -2\, du\, dr + 2r^2\, dz\, d\bar{z}\,. \tag{18}$$

The location of future null infinity $\mathscr{I}^+$ corresponds to the limit $r \to \infty$. It is a three-dimensional null manifold covered by the coordinates $(u, z, \bar{z})$.

Let us now turn to the parametrisation of generic null rays passing through the origin of Minkowski space. The parametrisation of a null ray is simply given by

$$X^\mu(s) = s\, p^\mu\,, \tag{19}$$

where s is some affine parameter and p^μ a constant null vector. A complete parametrisation of p^μ is further given in terms of three quantities $(\omega, w, \bar{w})$ by

$$p^\mu = \omega\, \hat{q}^\mu(w, \bar{w})\,, \tag{20}$$

where $\hat{q}$ has been given in (15). The components of p^μ in retarded coordinates along the null ray (19) are straightforwardly worked out,

$$\begin{aligned} p^r &= -n \cdot p = \omega\,, \\ p^u &= -\hat{q} \cdot p = \omega\,|z(s) - w|^2\,, \\ p^z &= r^{-1}\,\partial_{\bar{z}}\hat{q} \cdot p = -\omega\, r(s)^{-1}\,(z(s) - w)\,. \end{aligned} \tag{21}$$

Using (17) and (19) we find that the null trajectory is given in retarded coordinates by

$$\begin{aligned} r(s) &= -n \cdot X(s) = s\,\omega\,, \\ u(s) &= -\hat{q}(z(s), \bar{z}(s)) \cdot X(s) = s\,\omega\,|z(s) - w|^2\,. \end{aligned} \tag{22}$$

We note that $r(s)$ is an equally good affine parameter. The trajectory $z(s)$ can be obtained by studying the cartesian components $\mu = 1, 2$ of equation (14) and comparing it with (19). For our purposes it enough to notice that in that limit $s \to \infty$, we have

$$\lim_{s\to\infty} z(s) = w\,, \tag{23}$$

and therefore

$$\lim_{s\to\infty} u(s) = 0\,, \qquad \lim_{s\to\infty} \hat{q}^\mu(z(s), \bar{z}(s)) = \omega^{-1}\,p^\mu\,. \tag{24}$$

Thus the null ray intersects $\mathscr{I}^+$ at the retarded coordinates $(r, u, z, \bar{z}) = (\infty, 0, w, \bar{w})$.

Timelike infinity. Classical trajectories of massive particles asymptote to future timelike infinity i^+, which is better described using hyperbolic coordinates $(\tau, \rho, z, \bar{z})$, related to cartesian coordinates by [57]

$$X^\mu = \frac{1}{\sqrt{2}}\frac{\tau}{\rho}\left(n^\mu + \rho^2 \hat{q}^\mu(z, \bar{z})\right)\,, \tag{25}$$

such that the flat metric takes the form

$$ds^2 = -d\tau^2 + \tau^2\left(\frac{d\rho^2}{\rho^2} + \rho^2\,dz\,d\bar{z}\right)\,. \tag{26}$$

This is a foliation of the causal future of $X^\mu = 0$ by three-dimensional hyperbolic slices of constant negative curvature (aka euclidean AdS_3) with coordinates $x^a = (\rho, z, \bar{z})$ and induced metric

$$ds^2_{\mathcal{H}} = h_{ab}\,dx^a\,dx^b = \frac{d\rho^2}{\rho^2} + \rho^2\,dz\,d\bar{z}\,. \tag{27}$$

The asymptotic hyperbolic slice $\mathcal{H}^+$ located at $\tau \to \infty$ is a resolution of timelike infinity i^+. Note that we can easily extract the coordinates (τ, ρ) from the position vector X^μ through the following Lorentz contractions,

$$\tau^2 = 2\,(n \cdot X)\,(\hat{q}(z, \bar{z}) \cdot X)\,, \qquad \rho^2 = \frac{n \cdot X}{\hat{q}(z, \bar{z}) \cdot X}\,. \tag{28}$$

By comparison of (25) with (14), we also easily infer the relation to retarded coordinates,

$$u = \frac{1}{\sqrt{2}}\frac{\tau}{\rho}\,, \qquad r = \frac{1}{\sqrt{2}}\,\tau\rho\,. \tag{29}$$

The limit $\rho \to \infty$ with τ fixed thus corresponds to the limit $r \to \infty, u \to 0$ towards the middle of $\mathscr{I}^+$ where hyperbolic slices attach. The other limit of interest is $\tau \to \infty$ at fixed ρ towards the future corner of $\mathscr{I}^+$, which corresponds to $r, u \to \infty$. Finally the limit $r \to \infty$ with u fixed but arbitrary corresponds to the limit $\rho, \tau \to \infty$ taken at the same rate.

A generic classical trajectory for a massive particle is still of the form (19), with p^μ a constant timelike vector satisfying $p^2 = -m^2$. We parametrise such momentum vectors by three numbers $(\chi, w, \bar{w})$,

$$p^\mu = \frac{m}{\sqrt{2}\chi}\left(n^\mu + \chi^2\, \hat{q}^\mu(w,\bar{w})\right) \equiv m\, \hat{p}^\mu\,, \tag{30}$$

such that

$$n \cdot \hat{p} = -\frac{\chi}{\sqrt{2}}\,, \qquad \hat{q}(z,\bar{z}) \cdot \hat{p} = -\frac{1}{\sqrt{2}\chi}\left(1 + \chi^2 |z-w|^2\right)\,. \tag{31}$$

The components of the momentum (30) in hyperbolic coordinates and along the trajectory (19) are given by

$$\begin{aligned}
\hat{p}^\rho &= \frac{1}{2\tau(s)\chi}\left(\chi^2 - \rho(s)^2 - \rho(s)^2\chi^2|z(s)-w|^2\right)\,,\\
\hat{p}^\tau &= \frac{1}{2\rho(s)\chi}\left(\chi^2 + \rho(s)^2 + \rho(s)^2\chi^2|z(s)-w|^2\right)\,,\\
\hat{p}^z &= -\frac{\chi}{\tau(s)\rho(s)}(z(s)-w)\,.
\end{aligned} \tag{32}$$

In the limit $s \to \infty$, the classical trajectory satisfies $\tau(s) \approx s$ together with

$$\lim_{s\to\infty} z(s) = w\,, \qquad \lim_{s\to\infty} \rho(s) = \chi\,. \tag{33}$$

Thus the timelike trajectory intersects the asymptotic hyperbolic slice $\mathcal{H}^+$ at the coordinates $(\tau, \rho, z, \bar{z}) = (\infty, \chi, w, \bar{w})$.

4. Soft dressings of massless states

Starting from the Wilson line representations (12)-(13) of the dressings, we aim at deriving their expressions in terms of asymptotic components of the photon field A_μ. Our method will only reliably capture the soft part of the dressing since we will only retain the leading asymptotic contribution of the photon field. This is however all we need in order to capture the FK soft dressings as indicated by equation (4).

The field A_μ can be split into a radiative part and a piece sourced by charged on-shell particles. We will disregard the latter which is known to account for scattering Coulomb phases [17]. The asymptotic expansion of the *radiation* gauge field in retarded coordinates takes the form

$$A_r = O(r^{-2}\ln r)\,, \qquad A_u = O(r^{-1}\ln r)\,, \qquad A_z = A_z^{(0)}(u,z,\bar{z}) + O(r^{-1})\,. \tag{34}$$

Further information regarding this asymptotic expansion can be found in appendix A. The asymptotic photon field admits the Fourier decomposition [18],

$$A_z^{(0)}(u,z,\bar{z}) = \frac{ie}{2\sqrt{2}\pi^2}\int_0^\infty d\omega\,\left(a_+(q)\,e^{-i\omega u} - a_-^\dagger(q)\,e^{i\omega u}\right)\,, \qquad q = \omega\hat{q}(z,\bar{z})\,, \tag{35}$$

where $a_+^\dagger(q)$ is the usual creation operator for a positive helicity photon of momentum q. In the soft limit $\omega \to 0$ the two photon polarisations are however not independent, since they can be written in terms of the Goldstone mode Φ of large U(1) transformations,

$$\begin{aligned} a_+(\omega\hat{q}(z,\bar{z})) &= \partial_z \Phi(\omega,z,\bar{z}) , \qquad a_+^\dagger(\omega\hat{q}(z,\bar{z})) = \partial_{\bar{z}} \Phi^\dagger(\omega,z,\bar{z}) , \qquad (\omega \to 0) , \\ a_-(\omega\hat{q}(z,\bar{z})) &= \partial_{\bar{z}} \Phi(\omega,z,\bar{z}) , \qquad a_-^\dagger(\omega\hat{q}(z,\bar{z})) = \partial_z \Phi^\dagger(\omega,z,\bar{z}) . \end{aligned} \tag{36}$$

We formally write the Goldstone mode as a soft field,

$$\Phi(u,z,\bar{z}) = \frac{ie}{2\sqrt{2}\pi^2} \int_0^\infty d\omega\, f(\omega) \left(\Phi(\omega,z,\bar{z}) e^{-i\omega u} - \Phi^\dagger(\omega,z,\bar{z}) e^{i\omega u} \right) , \tag{37}$$

such that

$$A_z^{(0)}(u,z,\bar{z})\big|_{\text{soft}} = \partial_z \Phi(u,z,\bar{z}) . \tag{38}$$

Furthermore one can construct holomorphic and antiholomorphic soft currents $S_z = \partial_z \Phi$ and $S_{\bar{z}} = \partial_{\bar{z}} \Phi$, since away from operator insertions we have [53]

$$\partial_{\bar{z}} \partial_z \Phi = 0 . \tag{39}$$

In other words the Goldstone mode can be decomposed into holomorphic and antiholomorphic components,

$$\Phi(u,z,\bar{z}) = \phi(u,z) + \bar{\phi}(u,\bar{z}) . \tag{40}$$

The relations (36) then allow to write the soft components of the conformal fields (8) in terms of the Goldstone mode and its velocity at the retarded time $u = 0$,

$$\mathcal{S}\big|_{\text{soft}} = \Phi\big|_{u=0} , \qquad \partial_z \mathcal{Y}^z\big|_{\text{soft}} = \frac{1}{2} \partial_u \phi\big|_{u=0} , \qquad \partial_{\bar{z}} \mathcal{Y}^{\bar{z}}\big|_{\text{soft}} = \frac{1}{2} \partial_u \bar{\phi}\big|_{u=0} . \tag{41}$$

This will come in handy when comparing our results to the conformally soft dressings (6)-(7).

Leading soft dressing. We have now introduced all the ingredients needed to evaluate the Wilson line dressing of a charged momentum state,

$$\exp\left[-iQ \int A \right] |p,J\rangle . \tag{42}$$

Therefore the relevant quantity to compute is the line integral along the null ray (19),

$$\int A = \int_0^\infty ds\, p^\mu A_\mu(x(s)) . \tag{43}$$

We will now make use of the asymptotic expansion (34) of the radiation field A_μ near $\mathscr{I}^+$. Since we are only interested in the leading soft contribution to the integral (43), it will be sufficient to consider the leading term of this asymptotic expansion.

Plugging (34) together with (38), the leading soft contribution is therefore given by

$$\int_0^\infty ds\, p^\mu A_\mu(x(s))\Big|_{\text{soft}} = \int_0^\infty ds\, \left(p^z \partial_z \Phi + p^{\bar{z}} \partial_{\bar{z}} \Phi \right) = \int_0^\infty ds\, \frac{d}{ds} \Phi , \tag{44}$$

where we have used

$$\frac{d}{ds}\Phi(u(s),z(s),\bar{z}(s)) = p^z\partial_z\Phi + p^{\bar{z}}\partial_{\bar{z}}\Phi + p^u\partial_u\Phi \approx p^z\partial_z\Phi + p^{\bar{z}}\partial_{\bar{z}}\Phi\,. \tag{45}$$

In the last equality we have discarded $\partial_u\Phi$ since it is subleading in the soft expansion ($\partial_u \sim \omega$). We thus have

$$\int A\Big|_{\text{soft}} = \Phi(0,w,\bar{w}) - \Phi_0\,. \tag{46}$$

Just like in the general discussion (11), $\Phi_0 \equiv \Phi(0,z_0,\bar{z}_0)$ is a phase common to all dressings irrespective of the insertion point $(w,\bar{w})$ on the celestial sphere and we can safely disregard it. Hence we find that the leading soft dressing of is simply given by

$$\tilde{W}_0 = W_0\big|_{\text{soft}} = e^{-iQ\,\Phi(0,w,\bar{w})}\,, \tag{47}$$

in agreement with the soft part of the conformally soft dressing (6) through the identification (41). Note that the above expression had been considered previously to account for virtual IR divergences of the S-matrix in the approach of celestial holography [53]. Here we obtained it directly from the Wilson line dressing.

Subleading soft dressing. We similarly compute the subleading soft dressing starting from the generalised Wilson line,

$$\exp\left[-\frac{iQ}{8}\int ds\, F_{\mu\nu}(x(s))J^{\mu\nu}\right]|p,J\rangle\,. \tag{48}$$

The asymptotic expansion (34) together with (38) imply that the only nonvanishing soft components of the field strength are

$$F_{uA}\big|_{\text{soft}} = \partial_u\partial_A\Phi\,, \tag{49}$$

such that

$$F^{\mu\nu}\big|_{\text{soft}}\,J_{\mu\nu} = -2r^{-2}\left(F_{uz}\,J_{r\bar{z}} + F_{u\bar{z}}\,J_{rz}\right) = -2r^{-1}\left(F_{uz}\,\partial_{\bar{z}}\hat{q}^\nu + F_{u\bar{z}}\,\partial_z\hat{q}^\nu\right)\hat{q}^\mu J_{\mu\nu}\,. \tag{50}$$

At this point it is convenient to express the momentum state parametrised by (20) in terms of conformal primary states [44],

$$|p,J\rangle = \int_{1-i\infty}^{1+i\infty} d\Delta\,\omega^{-\Delta}\,|\Delta,J,w,\bar{w}\rangle\,, \qquad p^\mu = \omega\,\hat{q}^\mu(w,\bar{w})\,, \tag{51}$$

and exploit the action of the Lorentz generators [55]

$$\hat{q}^\mu\partial_{\bar{z}}\hat{q}^\nu J_{\mu\nu}\,|\Delta,J,w,\bar{w}\rangle = -2i\left(2h(w-z) + (w-z)^2\partial_w\right)|\Delta,J,w,\bar{w}\rangle\,, \tag{52}$$

where the conformal weights are defined as

$$(h,\bar{h}) = \frac{1}{2}(\Delta+J,\Delta-J)\,. \tag{53}$$

On a conformal primary state $|\Delta, J, w, \bar{w}\rangle$, the action of (50) therefore gives

$$\begin{aligned} -iF^{\mu\nu}\big|_{\text{soft}} J_{\mu\nu} &= 4r^{-1} F_{uz} \left(2h(w-z) + (w-z)^2 \partial_w\right) + \text{c.c.} \\ &= 4\omega^{-1} \left(2hp^z \partial_z \partial_u \Phi + p^z \partial_z \partial_u \Phi (w-z) \partial_w + \text{c.c.}\right) \\ &= 4\omega^{-1} \left(2hp^z \partial_z \partial_u \Phi + p^z \partial_z [\partial_u \Phi (w-z)] \partial_w + p^z \partial_u \Phi\, \partial_w + \text{c.c.}\right) . \end{aligned} \tag{54}$$

Using the decomposition (40) together with the relations (41), we can now rewrite this as a total derivative term,

$$\begin{aligned} -iF^{\mu\nu}\big|_{\text{soft}} J_{\mu\nu} &= 8\omega^{-1} p^z \partial_z \left[2h\partial_z \mathcal{Y}^z + \partial_z \mathcal{Y}^z (w-z)\partial_w + \mathcal{Y}^z\, \partial_w\right]\big|_{\text{soft}} + \text{c.c.} \\ &= 8\omega^{-1} \frac{d}{ds} \left(2h\partial_z \mathcal{Y}^z + \partial_z \mathcal{Y}^z (w-z)\partial_w + \mathcal{Y}^z\, \partial_w\right)\big|_{\text{soft}} + \text{c.c.}\,, \end{aligned} \tag{55}$$

such that

$$W_1\big|_{\text{soft}} |p, J\rangle = \tilde{W}_1 |p, J\rangle = \int_{1-i\infty}^{1+i\infty} d\Delta\, \omega^{-\Delta}\, \tilde{W}_1^{\text{conf}} |\Delta, J, w, \bar{w}\rangle\,, \tag{56}$$

with

$$\tilde{W}_1^{\text{conf}} = \exp\left[\frac{Q}{\omega}\left(2h\,\partial_w \mathcal{Y}^w + \mathcal{Y}^w \partial_w + 2\bar{h}\,\partial_{\bar{w}} \mathcal{Y}^{\bar{w}} + \mathcal{Y}^{\bar{w}} \partial_{\bar{w}}\right)\big|_{\text{soft}}\right], \tag{57}$$

in agreement with (7) as given in [55].

We have thus recovered the soft contributions to the leading and subleading conformal dressings associated with charged massless states, starting from the generalised Wilson line operators.

5. Soft dressings of massive states

Similarly to the case of massless fields worked out in the previous section, the determination of the soft dressings associated with charged massive states requires a control over the asymptotics of the radiation field A_μ near timelike infinity i^+. We will make use of the following expansion in the large τ limit,

$$A_\tau = O(\tau^{-2})\,, \qquad A_a = \partial_a \Phi_{\mathcal{H}} + O(\tau^{-1})\,, \tag{58}$$

where $\Phi_{\mathcal{H}}$ is would-be pure gauge mode satisfying

$$\Box_{\mathcal{H}} \Phi_{\mathcal{H}} = 0\,. \tag{59}$$

Note that another scalar mode is in principle allowed by the asymptotic equations of motion in A_ρ at order $O(\tau^0)$. However this mode is not produced by the standard free photon field operator and we therefore disregard it. Further details regarding this asymptotic expansion can be found in appendix B. Solutions to (59) are fully determined in terms of the boundary value $\Phi^+(z,\bar{z}) = \lim_{\rho\to\infty} \Phi_{\mathcal{H}}(\rho, z, \bar{z})$ [22, 53],

$$\Phi_{\mathcal{H}}(\rho, z, \bar{z}) = \int d^2 w\, K_2(\rho, z, \bar{z}; w, \bar{w})\, \Phi^+(w, \bar{w})\,, \tag{60}$$

where K_2 is a bulk-boundary propagator given in (95).

Leading dressing. The determination of the leading soft dressing proceeds by evaluating the following Wilson line integral along the timelike trajectory (19) and (30). Using the falloffs (58), its soft contribution reduces to

$$\int dx^\mu A_\mu\Big|_{\text{soft}} = \int ds\, p^a \partial_a \Phi_{\mathcal{H}} = \int ds\, \frac{d}{ds}\Phi_{\mathcal{H}} = \Phi_{\mathcal{H}}(\chi, w, \bar{w}) - \Phi_{\mathcal{H}}(\rho_0, z_0, \bar{z}_0)\,. \tag{61}$$

Again the second term is phase common to all dressings, and we can discard it. The leading soft conformal dressing of a massive particle is thus given by

$$\tilde{W}_0 = W_0\Big|_{\text{soft}} = e^{-iQ\,\Phi_{\mathcal{H}}(\chi,w,\bar{w})}\,, \tag{62}$$

where $\Phi_{\mathcal{H}}$ is determined by (60). In this way we recover the result of the literature [53, 54].

Subleading soft dressing. To determine the subleading soft dressing, we would need to evaluate the soft contribution to the line integral of $F_{\mu\nu}J^{\mu\nu}$. From the falloffs (58), we see that the field strength vanishes near i^+,

$$F_{\tau a} = O(\tau^{-2})\,, \qquad F_{ab} = O(\tau^{-1})\,. \tag{63}$$

Given that the leading soft dressing comes from the order $O(\tau^0)$ in the gauge potential, we can expect the subleading dressing to be associated with the subleading order $O(\tau^{-1})$. Asymptotically the time coordinate τ coincides with the Minkowskian time t conjugated to the energy ω. Therefore given a quantity $g(t)$ in the time-domain and its Fourier transform $g(\omega)$, we also have

$$\lim_{\tau\to\infty} g(\tau) = \lim_{\tau\to\infty}\int_0^\infty d\omega\, e^{-i\omega\tau} g(\omega)\,. \tag{64}$$

The soft expansion of $g(\omega)$ in the limit $\omega \to 0$,

$$g(\omega) = \sum_{n=0} g_n\,\omega^n\,, \tag{65}$$

then maps to an expansion at large time $\tau \to \infty$ through the Laplace transform

$$\int_0^\infty d\omega\, e^{-i\omega\tau}\,\omega^n = \frac{n!}{(i\tau)^{n+1}}\,. \tag{66}$$

Thus higher orders in the soft expansion naturally map to higher orders the late-time asymptotic expansion towards i^+.

The explicit evaluation of the subleading soft dressing thus requires a detailed study of the Maxwell field near i^+ at subleading order $O(\tau^{-1})$ in the large-τ expansion. This can be done along the lines of appendix B. In particular one needs to ensure consistency of the solution space near i^+ with that considered at $\mathscr{I}^+$ in appendix A. This however goes beyond the scope of the present article. Note that a similar systematic study of the solution space of general relativity near spatial infinity i^0 and its relation with the solution space near $\mathscr{I}^+$ is discussed in [58]. As with the leading soft dressing, one can expect the subleading soft dressing of a massive state to resemble that of a massless state, modulo convolution by appropriate AdS_3 bulk-boundary propagators. We leave this to future endeavors.

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A. Asymptotic expansion near $\mathscr{I}^+$

Here we discuss the asymptotics of the gauge field A_μ in retarded coordinates (18), in close parallel to the analysis performed [52]. Although in the main body of the text we restrict our attention to radiative solutions ($j_\mu = 0$), we keep the discussion general here.

Maxwell equations $\partial_\mu(\sqrt{-g}F^{\mu\nu}) = e^2\sqrt{-g}\, j^\nu$ take the form

$$\begin{aligned} -r^2\partial_u F_{ru} + \partial_z F_{\bar{z}u} + \partial_{\bar{z}} F_{zu} &= e^2 r^2 j_u\,, \\ -\partial_r\left(r^2 F_{ur}\right) + \partial_z F_{\bar{z}r} + \partial_{\bar{z}} F_{zr} &= e^2 r^2 j_r\,, \\ -r^2\partial_u F_{rz} - r^2\partial_r F_{uz} + \partial_z F_{\bar{z}z} &= e^2 r^2 j_z\,, \end{aligned} \tag{67}$$

while the Lorenz gauge condition $\partial_\mu(\sqrt{-g}A^\mu) = 0$ reads

$$-r^2\partial_u A_r - \partial_r\left(r^2 A_u\right) + \partial_z A_{\bar{z}} + \partial_{\bar{z}} A_z = 0\,. \tag{68}$$

We assume the following asymptotic expansion for the gauge field,[3]

$$\begin{aligned} A_r &= \sum_{n=2} r^{-n} A_r^{(n)} + \sum_{m=2} r^{-m}\ln r\,\tilde{A}_r^{(m)}\,, \\ A_u &= \sum_{n=2} r^{-n} A_u^{(n)} + \sum_{m=1} r^{-m}\ln r\,\tilde{A}_u^{(m)}\,, \\ A_z &= \sum_{n=0} r^{-n} A_z^{(n)} + \sum_{m=1} r^{-m}\ln r\,\tilde{A}_z^{(m)}\,, \end{aligned} \tag{69}$$

and for the matter current,

$$j_u = r^{-2} j_u^{(2)} + O(r^{-3})\,, \qquad j_z = r^{-2} j_z^{(2)} + O(r^{-3})\,, \qquad j_r = O(r^{-3})\,. \tag{70}$$

Under these assumptions the gauge condition (68) imposes

$$\begin{aligned} \partial_u \tilde{A}_r^{(2)} &= -\tilde{A}_u^{(1)}\,, \\ \partial_u A_r^{(2)} &= -\tilde{A}_u^{(1)} + \partial_{\bar{z}} A_z^{(0)} + \partial_z A_{\bar{z}}^{(0)}\,, \end{aligned} \tag{71}$$

[3] A nonzero $A_u^{(1)}$ can always be set to zero by a residual gauge transformation $A_\mu \mapsto A_\mu + \partial_\mu \varepsilon$ with

$$\varepsilon(r,u,z,\bar{z}) = r^{-1}\varepsilon^{(1)}(u,z,\bar{z}) + O(r^{-2})\,, \qquad \partial_u \varepsilon^{(1)} = A_u^{(1)}\,.$$

while Maxwell equations additionally imply

$$\begin{aligned} -2\partial_u \tilde{A}_u^{(1)} &= e^2 j_u^{(2)}\,, \\ \partial_u \tilde{A}_z^{(2)} - \partial_z \tilde{A}_u^{(1)} &= 0\,, \\ 2(\partial_z\partial_{\bar{z}} A_z^{(0)} + \partial_u A_z^{(1)}) - 2\partial_u \tilde{A}_z^{(1)} &= e^2 j_z^{(2)}\,. \end{aligned} \tag{72}$$

Although the coordinate system (18) is slightly different than the one used in [52], the equations take exactly the same form.

B. Asymptotic expansion near i^+

In this appendix we work out the asymptotics of the gauge field A_μ close to future timelike infinity i^+. This is done by adopting the hyperbolic coordinates $(\tau, x^a) = (\tau, \rho, z, \bar{z})$ and working in the limit $\tau \to \infty$. See [22, 59, 60] for relevant earlier work.

In the hyperbolic slicing (26), Maxwell equations take the form

$$\begin{aligned} h^{ab} D_a F_{b\tau} &= e^2 \tau^2 j_\tau\,, \\ -\tau\,\partial_\tau(\tau F_{\tau b}) + h^{ca} D_c F_{ab} &= e^2\tau^2 j_b\,, \end{aligned} \tag{73}$$

where D_a is the Levi-Civita connection associated with the three-dimensional metric h_{ab} and where $F_{\tau a}\,, F_{ab}$ are viewed as (1,0)- and (2,0)-tensors on $\mathcal{H}$, respectively. On the other hand the Lorenz gauge condition reads

$$-\,\partial_\tau(\tau^3 A_\tau) + \tau\, h^{ab} D_a A_b = 0\,. \tag{74}$$

We start by assuming the standard falloffs for the gauge field

$$A_\tau = \tau^{-1}\,\bar{A}_\tau + O(\tau^{-2})\,, \qquad A_a = \bar{A}_a + O(\tau^{-1})\,, \tag{75}$$

such that the asymptotics of the field strength are

$$F_{a\tau} = \tau^{-1}\, D_a \bar{A}_\tau + O(\tau^{-2})\,, \qquad F_{ab} = \bar{F}_{ab} + O(\tau^{-1})\,, \qquad \bar{F}_{ab} \equiv D_a \bar{A}_b - D_b \bar{A}_a\,, \tag{76}$$

For the matter current on the other hand, we assume

$$j_\tau = \tau^{-3}\,\bar{j}_\tau + O(\tau^{-2})\,, \qquad j_a = \tau^{-2}\,\bar{j}_a + O(\tau^{-3})\,, \tag{77}$$

At leading order Maxwell equations simply yield

$$D^2 \bar{A}_\tau = e^2\,\bar{j}_\tau\,, \qquad D^a \bar{F}_{ab} = e^2\,\bar{j}_b\,, \tag{78}$$

while the Lorenz condition yields

$$D_a \bar{A}^a = 2\bar{A}_\tau\,. \tag{79}$$

For radiative solutions we set $j_\mu = 0$, and we will see that consistency with the falloffs at $\mathscr{I}^+$ in that case requires $\bar{A}_\tau = 0$. We are then effectively left with a lower dimensional Maxwell theory on $\mathcal{H}$ in Lorenz gauge.

The asymptotic equations (78)-(79) are equations on the euclidean hyperboloid $\mathcal{H}$. As is familiar from the AdS/CFT correspondence, we can fully characterize their solutions in terms of a 'Fefferman–Graham' (FG) expansion at large ρ. For ease of notation we will use the following diagonal form of the euclidean metric,

$$ds^2_{\mathcal{H}} = \frac{d\rho^2}{\rho^2} + \rho^2\, \delta_{ij}\, dx^i\, dx^j\,, \tag{80}$$

instead of the off-diagonal one (27). Once the FG expansion of the solutions to (78)-(79) are worked out, we can look at their consistency with the falloffs assumed at $\mathscr{I}^+$. Indeed the regime $\tau \gg \rho \gg 1$ corresponds to the regime $r \gg u \gg 1$, i.e., to the future corner of null infinity. By explicit coordinate transformation (29), we have

$$\begin{aligned} A_u &= \frac{1}{\sqrt{2}}\left(\rho A_\tau - \tau^{-1}\rho^2 A_\rho\right) = \frac{\tau^{-1}}{\sqrt{2}}\left(\rho \bar{A}_\tau - \rho^2 \bar{A}_\rho\right) + O(\tau^{-2})\,,\\ A_r &= \frac{1}{\sqrt{2}}\left(\rho^{-1} A_\tau + \tau^{-1} A_\rho\right) = \frac{\tau^{-1}}{\sqrt{2}}\left(\rho^{-1} \bar{A}_\tau + \bar{A}_\rho\right) + O(\tau^{-2})\,. \end{aligned} \tag{81}$$

This will come in handy when working out the matching between quantities in the two coordinate systems.

FG expansion of $\bar{A}_\tau$. The massless equation $D^2 \bar{A}_\tau = 0$ explicitly becomes

$$\partial_\rho(\rho^3 \partial_\rho \bar{A}_\tau)) + \rho^{-1}\partial^2 \bar{A}_\tau = 0\,, \tag{82}$$

with asymptotic solution

$$\bar{A}_\tau = A_\tau^{(0)} + \rho^{-2} \ln\rho\, \tilde{A}_\tau + \rho^{-2}\, A_\tau^{(2)} + \dots\,. \tag{83}$$

The independent free data for this second order differential equation is $A_\tau^{(0)}$ and $A_\tau^{(2)}$. However regularity of the solution at $\rho = 0$ discards half of the solution space, and therefore relates them in a nonlocal way via bulk-boundary propagators.

FG expansion for $\bar{A}_a$. The equations (78)-(79) can be written

$$\partial_\rho(\rho^3 \bar{A}_\rho) + \rho^{-1}\delta^{ij}\partial_i \bar{A}_j = 2\rho\, \bar{A}_\tau\,, \tag{84}$$

and

$$\begin{aligned} &\partial_\rho(\rho\, \partial_\rho(\rho^3 \bar{A}_\rho)) + \partial^2 \bar{A}_\rho = 2\partial_\rho(\rho^2 \bar{A}_\tau)\,,\\ &\partial_\rho(\rho\, \partial_\rho \bar{A}_i)) + \rho^{-3}\partial^2 \bar{A}_i = -2\partial_i(\bar{A}_\rho - \rho^{-1}\bar{A}_\tau)\,. \end{aligned} \tag{85}$$

If $\bar{A}_\tau \neq 0$, the leading behavior of $\bar{A}_\rho$ is given by

$$\bar{A}_\rho = \rho^{-1}\, A_\tau^{(0)} + \dots\,, \tag{86}$$

such that, together with (83) and (81), we find

$$A_r = \sqrt{2}\,\tau^{-1}\rho^{-1}\, A_\tau^{(0)} + \dots = O(r^{-1}u^0)\,. \tag{87}$$

Thus our assumption at $\mathscr{I}^+$ that $A_r = O(r^{-2}\ln r)$ from appendix A requires $A_\tau^{(0)} = 0$ and thus $\bar{A}_\tau = 0$ from regularity at $\rho = 0$. In that case the FG expansion of $\bar{A}_a$ reduces to

$$\begin{aligned} \bar{A}_\rho &= \rho^{-3}\ln\rho\,\tilde{A}_\rho + \rho^{-3}A_\rho^{(3)} + \dots\,, \\ \bar{A}_i &= \ln\rho\,\tilde{A}_i + A_i^{(0)} + \dots\,. \end{aligned} \tag{88}$$

Here $(\tilde{A}_\rho\,, A_\rho^{(3)})$ and $(\tilde{A}_i\,, A_i^{(0)})$ are pairs of free data, that are again partially related by bulk-boundary propagators due to the requirement of regularity at $\rho = 0$. Again our assumptions that $A_z = O(r^0)$ at $\mathscr{I}^+$ requires to set $\tilde{A}_i = 0$. One could naively think that regularity at $\rho = 0$ implies $A_i^{(0)} = 0$ as well, but this is not quite the case. Indeed would-be pure gauge solutions are still allowed,

$$\bar{A}_a = \partial_a \Phi_{\mathcal{H}}\,. \tag{89}$$

These trivially satisfy the reduced Maxwell equations (78), while the Lorenz gauge condition implies

$$\Box_{\mathcal{H}}\Phi_{\mathcal{H}} = 0\,, \tag{90}$$

whose solutions admit the large-ρ expansion

$$\Phi_{\mathcal{H}} = \Phi_{\mathcal{H}}^{(0)} + \rho^{-2}\ln\rho\,\tilde{\Phi}_{\mathcal{H}} + \rho^{-2}\,\Phi_{\mathcal{H}}^{(2)} + \dots\,, \qquad \tilde{\Phi}_{\mathcal{H}} = \frac{1}{2}\partial^2\Phi_{\mathcal{H}}^{(0)}\,. \tag{91}$$

Hence the corresponding gauge field reads

$$\begin{aligned} \bar{A}_\rho &= -2\rho^{-3}\ln\rho\,\tilde{\Phi}_{\mathcal{H}} + \rho^{-3}\,(\tilde{\Phi}_{\mathcal{H}} - 2\Phi_{\mathcal{H}}^{(2)}) + \dots\,, \\ \bar{A}_i &= \partial_i\Phi_{\mathcal{H}}^{(0)} + \dots\,. \end{aligned} \tag{92}$$

This forms the subset of the solutions (88) for which $\tilde{A}_i = 0$.

Equation (85) admits other solutions of $\bar{A}_\rho$ that are not pure gauge. With $\bar{A}_\tau = 0$ and making the change of variable $\psi = \rho^2\bar{A}_\rho$, the latter takes the form

$$\left(D^2 + 1\right)\psi = 0\,, \tag{93}$$

which is the equation of a massive scalar right at the Breitenlohner–Freedman bound $m^2 = -1$ [61, 62]. Regularity at $\rho = 0$ uniquely specifies it in terms of the boundary mode $\tilde{A}_\rho$,

$$\bar{A}_\rho(\rho, z, \bar{z}) = \rho^{-2}\int d^2w\,K_1(\rho, z, \bar{z}\,; w, \bar{w})\tilde{A}_\rho(w, \bar{w})\,, \tag{94}$$

where the bulk-boundary propagator for general Δ is explicitly given by [63]

$$K_\Delta(\rho, z, \bar{z}\,; w, \bar{w}) = C_\Delta\left(\frac{\rho}{1 + \rho^2|z - w|^2}\right)^\Delta\,. \tag{95}$$

Summary. By consistency with the falloff conditions at $\mathscr{I}^+$, we conclude that the radiative solution space near i^+ admits the simple asymptotic expansion

$$A_\tau = O(\tau^{-2})\,, \qquad A_a = \bar{A}_a + O(\tau^{-1})\,, \tag{96}$$

with

$$\begin{aligned}\bar{A}_\rho &= \rho^{-3}\ln\rho\,(\tilde{A}_\rho - \partial^2\Phi^{(0)}_{\mathcal{H}}) + O(\rho^{-3})\,,\\ \bar{A}_i &= \partial_i\Phi^{(0)}_{\mathcal{H}} + O(\rho^{-2}\ln\rho)\,.\end{aligned} \tag{97}$$

The functions $\Phi^{(0)}_{\mathcal{H}}(z,\bar{z})$ and $\tilde{A}_\rho(z,\bar{z})$ specify the two kinds of regular radiative solutions consistent with the aforementioned falloff conditions. In particular the solution specified by $\Phi^{(0)}_{\mathcal{H}}(z,\bar{z})$ is would-be pure gauge. Finally, the relations (81) together with the coordinate change (29) allow to map this radiative data to radiative data in retarded coordinates, namely

$$\lim_{u\to\infty} A^{(0)}_z(u,z,\bar{z}) = \partial_z\Phi^{(0)}_{\mathcal{H}}\,, \qquad \lim_{u\to\infty} \tilde{A}^{(1)}_u(u,z,\bar{z}) = \frac{1}{4}(\tilde{A}_\rho - \partial^2\Phi^{(0)}_{\mathcal{H}})\,. \tag{98}$$

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