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TASI 2022 lectures on scattering amplitudes – an addendum

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This is a short addendum to "TASI 2014 Gauge and Gravity Amplitude Relations," 1506.00974.

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1. Introduction

Much of the material presented and discussed in my "Lectures on Scattering Amplitudes" at the 2022 version of TASI[1] are written up in some technical detail in my previous TASI lecture notes, which are on the preprint archives. Instead, for these proceedings in this addendum, I provide a quick summary and a very brief update that is meant to contain some information not included in my original lectures, along with more up-to-date references and the context of my evolving perspective engaging more directly with higher-derivative operator predictions in the context of effective field theories. The 2014 TASI lectures were aiming towards students of a scattering focused school with an idea towards introducing graphical methods useful for multi-loop calculations and introducing the duality between color and kinematics and associated double-copy. In contrast these 2022 lectures were intended to be more broadly appreciated by phenomenologists with some familiarity with Quantum Field Theory but no particular exposure to modern scattering methods and approaches.

The primary takeaway of these 2022 lectures is that even without resorting to dimension-specific variables (which can indeed drastically simplify calculations – see references below on spinor-helicity) we really can carry out the idealized (perturbative) S-matrix bootstrap program: namely constraining the predictions of a wide web of theories by symmetries and principles. As we learn from quantum field theory, the combination of quantum mechanics and Lorentz invariance is incredibly powerful. When we restrict ourselves to on-shell kinematics there are only so many building blocks we can write down. In a guided manner this approach can even be an efficient approach to calculation–indeed the higher-loop calculations in gauge and gravity theories. Along the way we can appreciate a universality of building blocks that are recycled to appear in the predictions of many theories. I elide references to primary references in these notes, but do provide references to handy lectures and reviews in the concluding section.

2. Unitarity

Loop level integrands, especially those involving external gauge vectors or gravitons can be laborious to compute using standard Feynman rules. Off-shell Feynman rules carry unphysical gauge freedom that must cancel only upon consideration of gauge-invariant quantities. On-shell unitarity methods allow an efficient verification of multiloop integrands when the integrands are organized in terms of functional maps from graphs. Efficient verification lends itself to natural construction in the following sense.

We can ask verification questions targeting, at first, individual graphs. Such maximally targeted cuts, each expressed in terms of a distinct cubic graph[2], involve kinematics with all all propagators taken on-shell – maximal cuts. The contribution associated with the cut graph should involve the sum over states of a product of trees. The tree-labels come from each vertex of the cut graph, and all physical states in the theory are allowed to cross the cuts. Any non-vanishing maximal cut must be associated with the integrand of that individual graph. Associating that information with the kinematic numerator of such graphs involves an integrand that can only be in error proportional to inverse-propagators. By releasing cut conditions systematically we gradually allow more graphs

[1]All TASI 2022 video lectures can be found by searching on [YouTube].

[2]Cubic graphs involve only three-point internal vertices.

to contribute to each cut, but any missing information can only be proportional to the inverse propagators whose cut conditions have been released – representing contact terms that can be either associated with graphs with higher-order vertices, or assigned to cubic graphs by multiplying and dividing the contact term by the relevant expanded propagators. We can continue releasing such cut conditions until we have saturated the potential loop-level powercounting of the theory. While the 2014 lectures emphasized the use of unitarity methods at the multiloop level (indeed beginning with a 2-loop example), we can straightforwardly apply exactly the same ideas and techniques to constraining graph-organized tree-level amplitudes via factorization which is what we treated in the 2022 lecture videos. I will leave much of the mechanics of unitarity based cut-construction to my earlier 2014 lecture notes, but here present my current perspective on the approach and utility of these on-shell methods when combined with symmetries and principles.

With an eye towards reuse of tree-amplitudes in the service of building multi-loop amplitudes whose integrals are dimensionally regulated, I stay formally in arbitrary D dimensions. This means that little-group weight are carried by polarization vectors, $\varepsilon(k)$ for external vectors of mass k, and formal spinors $\bar{u}(k)$, $v(k)$ for external fermions. How do we sum over states with formal polarizations and spinors? We simply exploit the following completeness relations as projectors to sums over physical states:

$$\sum_{s\in\text{pols}} \varepsilon^{\mu,s}(k)\varepsilon^{\nu,\bar{s}}(-k) = \eta^{\mu\nu} - \frac{k^\mu q^\nu + q^\mu p^\nu}{k\cdot q}\,, \tag{1}$$

and fermionic state sums:

$$\sum_{s\in\text{states}} v(-l,s)\bar{u}(l,\bar{s}) = \not{l} + m\,. \tag{2}$$

where $\eta^{\mu\nu}$ is taken to be the flat spacetime metric, and q is an arbitrary reference null-vector corresponding to freedom of gauge-choice. Any gauge-independent observable such as an ordered amplitude must therefore ultimately be independent of q – so all q dependence must cancel out. When the sewing of trees closes a fermion loop this will result in a trace over the fermionic indices associated with that loop.

It is worth perhaps seeing such a state-sum on some simple examples. Consider one of the simplest gauge theories – the covariantized free scalar, i.e. a massive scalar in the adjoint minimally coupled to Yang-Mills.

$$\mathcal{L} = (D\phi)^2 + m^2\phi^2 - \frac{1}{4}\text{Tr}(\text{F}^2) \tag{3}$$

To get a feel for the state-sum, let us evaluate the following s-channel cut with all momenta taken external:

$$C_1 = \sum_{s\in\text{pols}} \mathcal{A}(1^a_\phi, 2^b_\phi, p^e_{A,s})\mathcal{A}(-p_{A^e_{\bar{s}}}, 3^c_\phi, 4^d_\phi)\,. \tag{4}$$

I use an all out-going convention to specify the momentum labels of each tree so the three-point amplitude $\mathcal{A}(1,2,p)$ means that $k_1 + k_2 + p = 0$. The subscript of each label specifies the particle type and thus the on-shell conditions. As such, in C_1, the massive scalar external momenta satisfy $k_1^2 = k_2^2 = k_3^2 = k_4^2 = m^2$, and the cut gluon momenta satisfies $p^2 = 0$. The superscript index specifies the color charge of the particle. The color-weight from Feynman rules is straight-forward, with each vertex dressed with the anti-symmetric structure constant $f^{abc} \propto \mathrm{Tr}(\mathrm{T^a}[\mathrm{T^b}, \mathrm{T^c}])$ of the gauge group. Each tree amplitude is given:

$$\mathcal{A}(1^a_\phi, 2^b_\phi, p_{A_s}{}^e) = g f^{abe}(k_1 - k_2) \cdot \varepsilon_s(p) \tag{5}$$

$$\mathcal{A}(3^c_\phi, 4^d_\phi, -p_{A_{\bar{s}}}{}^e) = g f^{ecd}(k_3 - k_4) \cdot \varepsilon_{\bar{s}}(-p)\,. \tag{6}$$

Note that these amplitudes are entirely on-shell and should be gauge invariant. On-shell three-point kinematics means that $k_i \cdot p = 0$ for every external leg, so in each case we see that gauge invariance,

$$\mathcal{A}(1,2,p)|_{\varepsilon\to p} = 0 = \mathcal{A}(3,4,-p)|_{\varepsilon\to -p}\,, \tag{7}$$

is entirely manifest.

Using Eqn. 1 we simply evaluates the sum over states in Eqn. 4 as,

$$C_1 = g^2 f^{abe} f^{ecd}(k_1 - k_2)_\mu (k_3 - k_4)_\nu \sum_{s\in\text{pols}} \varepsilon_s(p)\varepsilon_{\bar{s}}(-p) \tag{8}$$

$$= g^2 f^{abe} f^{ecd}(k_1 - k_2)_\mu (k_3 - k_4)_\nu \left(\eta^{\mu\nu} - \frac{p^\mu q^\nu + q^\mu p^\nu}{p\cdot q}\right) \tag{9}$$

$$= g^2 f^{abe} f^{ecd}(k_1 - k_2)\cdot(k_3 - k_4)\,. \tag{10}$$

This is precisely the numerator of the Feynman-rule dressed s-channel graph evaluated with on-shell conditions. Note that, following maximal cut-construction, we would assign such a functional dressing to the four-point cubic topology. If we label the s-channel cubic 4-point graph $(1^a, 2^b, 3^c, 4^d)$, then we would say:

$$N_s = N(1^a, 2^b, 3^c, 4^d) = g^2 f^{abe} f^{ecd}(k_1 - k_2)\cdot(k_3 - k_4)\,. \tag{11}$$

Note, in making this assignment, we are no longer assuming maximal cut conditions, that $(k_1 + k_2)^2 = 0$. The other two channels would follow from simple relabeling of this graph's dressings: $N_t = N(4^d, 1^a, 2^b, 3^c)$, and $N_u = N(3^c, 1^a, 4^d, 2^b)$, with the putative full amplitude given:

$$\mathcal{A} = \frac{N_s}{(k_1+k_2)^2} + \frac{N_t}{(k_2+k_3)^2} + \frac{N_u}{(k_1+k_3)^2}\,. \tag{12}$$

In this case there are no more cut conditions to release – so we have recovered all information about this amplitude that is available via unitarity cuts. For this theory we actually have the amplitude, but we could, in principle, be missing contact terms – four-field operators entirely unconstrained by factorization probes. Indeed the coefficients of contact terms, including the coefficients of local higher derivative operators are generically informed in unitarity based amplitude bootstraps by additional considerations: principles like positivity, symmetry properties, soft behavior, power-counting, and gauge invariance.

Now let us consider a massless Dirac fermion ψ, in the fundamental of some $SU(N)$ gauge group, minimally coupled to glue,

$$\mathcal{L} = \bar{\psi}(i\not{D})\psi - \frac{1}{4}\mathrm{Tr}(\mathrm{F}^2)\,. \tag{13}$$

To illustrate the state projector for fermions it's convenient to consider again a cut of a four-point amplitude, $\mathcal{A}(1^i_f, \bar{2}^{\bar{k}}_{\bar{f}}, 3^a_A, 4^b_A)$, to two three-point amplitudes, in this case with a fermion crossing the cut

$$C_2 = \sum_{s\,\in\text{states}} \mathcal{A}(1^i_f, 3^a_A, -\bar{p}^{\bar{j}}_{\bar{s}})\mathcal{A}(p^j_s, 4^b_A, \bar{2}^{\bar{k}}_{\bar{f}}) \tag{14}$$

$$= \sum_{s\,\in\text{states}} (g T^a_{i\bar{j}}\, \bar{u}_1 \not{\epsilon}_3 v_{-p,\bar{s}})(g T^b_{j\bar{k}}\, \bar{u}_{p,s} \not{\epsilon}_4 v_2) \tag{15}$$

$$= g^2 T^a_{i\bar{j}} T^b_{j\bar{k}} \bar{u}_1 \not{\epsilon}_3 (\not{k}_1 + \not{k}_3) \not{\epsilon}_4 v_2\,. \tag{16}$$

Again we recover precisely the Feynman-rule dressed numerator of the cut graph evaluated under cut-conditions. This is again manifestly gauge-invariant under such restricted cut-kinematics.

The upside of such maximal cut construction is that the entire calculation at every stage is expressible in a set of entirely gauge-invariant observables. Have we evaded the factorial complexity of the number of Feynman graphs (including contact terms) that one must construct traditionally? By no means – for brute construction we must maximally consider exactly the number of cuts as equivalent to those allowed by Feynman rules, again including contact graphs. The advantage as described above is simply in the compactness of resulting expressions, the systematic verifiability, and the ability to bootstrap – recycling lower-order in coupling prediction to build higher-order in coupling prediction. Indeed we can interpret the above approach as an efficient means of executing a Feynman-rule based calculation.

We can supplement such brute construction with insightful guesses. We call such guesses ansatze, and we can invoke a variety of physical principles and symmetries to constrain such ansatze so that even at relatively high loop order in symmetric theories the size of the calculation can still be manageable. This gains tremendous power from the notion of spanning cuts. If cut A is describable by imposing additional cut conditions on cut B, than we say cut B spans cut A. If cut B is satisfied by a set of mappings from graphs to dressings, than every cut spanned by B will also be satisfied by that set of mappings. So verification of a dressing can proceed on a very small number of cuts – a minimal set of spanning cuts. This means that once an ansatz has been constrained on easy to preform near-maximal cuts, only a few large cuts are necessary to verify that we have reproduced the amplitude required for the theory. This is what allows maximally supersymmetric gauge and gravitational calculations to ascend to the loop order they have – at four points the five loop correction has been completed for the maximally supersymmetric gravitational theory at four-points, and six loops for the maximally supersymmetric gauge theory.

3. Duality between Color and Kinematics and the Double Copy

In 2008, Zvi Bern, Henrik Johansson, and I (BCJ) discovered a particularly powerful set of constraints for gauge theories at tree-level, which we generalized to multi-loop integrands in 2010. For many theories in the adjoint we can require the kinematic numerator weights of graphs to act in concordance or duality with the color weights – obeying the same algebraic relations between distinct graphs. The covariantized simple scalar four-point amplitude discussed above is given in such a representation. We can write $N(1^a, 2^b, 3^c, 4^d) = c(1^a, 2^b, 3^c, 4^d)n(1, 2, 3, 4)$, where

$$n(1, 2, 3, 4) = g^2(k_1 - k_2) \cdot (k_3 - k_4) \tag{17}$$

$$c(1^a, 2^b, 3^c, 4^d) = f^{abe} f^{ecd} \,. \tag{18}$$

Note the antisymmetry of the color-factors is mirrored in $n(1, 2, 3, 4) = -n(2, 1, 3, 4)$. More surprisingly and far reaching is the fact that the color Jacobi relations are also mirrored:

$$c_s = c_t + c_u \tag{19}$$

$$n_s = n_t + n_u \,. \tag{20}$$

Here I followed the same relabeling conventions of the N_i above.

The adjoint color-factors for all distinct $(2n - 5)!!$ cubic graphs contributing to any n-point amplitudes can be reduced via Jacobi to the color-weights of $(n - 2)!$ graphs – the so called half-ladder graphs where the two-outermost legs are held fixed (say legs 1 and n) and the rest are permuted. The same reduction holds for the kinematic graphs. But when we make our graphs functional via an ansatze we are dressing every half-ladder topology with the same function – simply with permuted arguments. This means that we only need to give a single ansatz to the half-ladder at any multiplicity, and color-dual kinematic relations propagate it to all graph topologies. This is called making manifest the duality between color and kinematics. This not only drastically reduces the number of graphs we must dress with an ansatz, but allows the resulting amplitude to participate in double-copy construction – recycling kinematic building blocks to generate predictions in a wide web of theories from a relatively small number of distinct primary theories.

Let me introduce the following setup for discussing adjoint double-copy theories. Consider a gauge theory scattering amplitude expressed in terms of a sum over all distinct external labels permutations of cubic-graphs Γ_3:

$$\mathcal{A} = \int \sum_{g \in \Gamma_3} \frac{1}{S_g} \frac{n_g c_g}{d_g} \,. \tag{21}$$

Here again n_g are called the kinematic numerator weights and contain all kinematic dependence outside of the denominator weights d_g which encode the standard propagator structure, and the c_g are the graphs dressed with structure constants f^{abc} at each vertex. The S_g represent the internal symmetry factors of the graphs. The integral is over all internal loop momenta. It is worth emphasizing that as a result of Jacobi relations the c_g are not independent. That means that individual n_g are not required by any means to be gauge-invariant. In general gauge invariance for vector theories will occur in conspiracy between graphs as a result of the algebraic identities obeyed by the c_g.

If we have also identified a set of kinematic dressing $\tilde{n}_g$ that obey the same algebraic relations as arbitrary adjoint c_g, then we can replace the c_g with $\tilde{n}_g$ without violating the gauge invariance of the full amplitude:

$$\mathcal{A}^{n \otimes \tilde{n}} = \int \sum_{g \in \Gamma_3} \frac{1}{S_g} \frac{n_g \tilde{n}_g}{d_g} \,. \tag{22}$$

If both n and $\tilde{n}$ were kinematic weights of vector gauge theories than the resulting double-copy constructed amplitude will be a gravitational theory. The linearized gauge invariance of the vector theories is promoted through the double-copy to linearized diffeomorphism invariance.

Are all gauge theories adjoint color-dual? The answer perhaps not so surprisingly is no but for pure vectors to find an example we must consider higher-derivative operators.

4. EFT, the double-copy, and the emergence of string theory

Consider the higher-derivative operator $Tr(F^4)$ compatible with supersymmetry. The four-point amplitude associated with this operator insertion goes as:

$$\mathcal{A}^{F^4} = d^{abcd}\mathcal{A}^{\mathrm{BI}} \tag{23}$$

where $\mathcal{A}^{\mathrm{BI}}$ is the four-point permutation-invariant Born-Infeld photon amplitude, and d^{abcd} is the completely permutation invariant normalized sum over all distinct four-point color-traces. While this may be color-dual in a permutation-invariant sense, it is not color-dual in any adjoint sense. Intriguingly the amplitudes in this theory can be written as an adjoint double-copy because Born-Infeld amplitudes are given as an adjoint double-copy between the kinematics of Yang-Mills amplitudes and NLSM pion amplitudes.

This just begins to scratch the surface of the web of theories, but there is an important hint that provides the key to recovering both open and closed string theory amplitudes as field theory adjoint double-copies. Instead of rigidly considering double-copy only between kinematic weights n, allow mixed color and kinematic functions z_g that obey anti-symmetry and Jacobi. For example, consider the following function:

$$z_s^{(2)} = z(1^a, 2^b, 3^c, 4^d) = d^{abcd} n^{\pi}(1,2,3,4) \tag{24}$$

where

$$n^{\pi}(1,2,3,4) = (k_1+k_2)^2\left((k_1+k_3)^2 - (k_1+k_4)^2\right)/3 \tag{25}$$

Define $z_t^{(2)}$ and $z_u^{(2)}$ by the same relabeling as above. We see that $z_s^{(2)} = z_t^{(2)} + z_u^{(2)}$ – the permutation invariant color-weight d^{abcd} doesn't affect the algebraic relations obeyed by the kinematics. It's not hard to see that:

$$d^{abcd}\mathcal{A}^{\mathrm{NLSM}} = \frac{c_s z_s}{(k_1+k_2)^2} + \frac{c_t z_t}{(k_2+k_3)^2} + \frac{c_u z_u}{(k_1+k_3)^2}\,. \tag{26}$$

The construction of $\mathcal{A}^{F^4}$ as a double-copy follows from replacing the c_g on the RHS with the kinematic weights of Yang-Mills.

We can consider amplitudes in a very general bi-colored adjoint color-dual scalar theory with mass scale $1/\alpha'$ and all order in α' color-dual corrections:

$$\mathcal{Z} = \sum_{g\in\Gamma_3}\sum_{i=0}^{\infty} \alpha'^i W_i \frac{z_g^{(i)} c_g}{d_g}\,. \tag{27}$$

Here the c_g are in the adjoint, but the z_g are allowed to depend on arbitrary color-traces as well as kinematics – the only constraint is that they satisfy adjoint color-kinematics duality and the amplitudes support factorization. With the appropriate Wilson coefficients W_i, these amplitudes are known as Z-theory amplitudes and double-copy with supersymmetric Yang-Mills to build the tree-level amplitudes of the open superstring to all multiplicity.

It turns out the construction of the $z_g^{(i)}$ in $\mathcal{Z}$, at least at low multiplicity, is not as daunting a task as it might seem. One can identify functional compositions for algebraic relations such that if x_g and y_g both satisfy algebraic relations, a $z_g = \text{comp}_{\text{alg}}(x, y)_g$ will obey such algebraic relations. This allows one to start with an algebraic weight linear in Mandelstam invariants, and via composition climb to the UV. Four points provides a very simple example. Let's define at four-points,

$$\text{comp}_{\text{adj}}(x, y)_s = x_t y_t - x_u y_u \tag{28}$$

It is not hard to see that we can build the pion numerator from the composition between covariatized scalar numerators given above,

$$n_s^\pi \propto n_t^2 - n_u^2\,. \tag{29}$$

One could continue climbing the ladder to higher mass dimension scalar adjoint numerator weights, but there a fantastic surprise awaits us – all higher mass-dimension adjoint scalar numerators can be expressed in terms of powers of the two scalar permutation invariants $\sigma_2 = s^2 + t^2 + u^2$ and $\sigma_3 = stu$ and a basis of n_g and n^π. As we span all distinct color-traces at four-points with d^{abcd} and c_s, c_t, and c_u – one can span all $z^{(i)}$ to any mass dimension i with a very small number of building blocks indeed. What physical principles beyond unitarity and good UV behavior constrain the Wilson coefficients W_i appearing in the open superstring remains very much an open question.

5. Further reading

- For a pedagogic discussion of graph-based methods in amplitudes, including unitarity methods, color-kinematics and the double copy, see my earlier 2014 TASI lecture notes [1].
- For a gentle (non-technical) introduction to double-copy and applications to particle-physics, cosmology, and gravitational wave astrophysics, written for the broader particle physics community, please see the Snowmass white-paper [2].
- For applications of similar and related ideas to a Cosmological Bootstrap, there is a gentle introduction to the ideas and literature via Snowmass in ref. [3].
- For a slightly more technical invitation to learn more about double-copy and amplitudes and the web of color-dual theories, I recommend the SAGEX review [4]. Indeed the entire SAGEX compilation [5] provides an excellent overview of many research directions in the amplitudes community.
- In 2019 we presented a comprehensive, at the time, technical review of double-copy and it's applications in ref. [6]. In this review we work out some low-multiplicity bootstrapping of NLSM (Chiral Lagrangian pions) and YM in Chapter 3.
- For lecture notes in the form of a mathematica notebook, please see my Amplitudes 2017 Summer School lectures [3] of [7]. Uses some public graph manipulation code I have on github

[3]Video for the lectures are linked to at https://prettyquestions.com/lectures/2017_Amplitudes_School_Lectures/.

based more or less on how I talk about things in my TASI 2014 lectures. This goes into some details on cut verification, cut construction, and double-copy to gravity amplitudes at 2-loops.

- For bootstrapping massive scalar QCD in arbitrary representations with color-kinematics see ref. [8] and extracting out pure Einstein-Hilbert gravity mediated massive scalars via double-copy and projection [9]. Similar bootstrap for actual QCD with quarks through 4-point 1-loop in the fundamental given in ref. [10]. This later reference provides some nice pedagogic discussion of functional fermionic ansatze.
- For bootstrapping (higher)-spin massive amplitudes using massive spinor helicity, including imposing Majorana vs Dirac conditions, see Henrik Johanson and Alex Ochirov's nice paper [11].
- For detailed discussion of color-dual construction of higher-derivative EFT operator predictions in gauge and gravity theories I would take a look at refs. [12–16].

For broader context and discussion specializing to four-dimensions, I would absolutely recommend the following:

- For bootstrapping massless theories with spinor helicity at 3-points see Clifford Cheung's TASI notes [17] and and Paolo Benincasa and Freddy Cachazo's seminal paper [18]. It's an excellent exercise to carry out this exercise in D-dimensions at three-points both for massive and massless matter.
- Definitely check out Lance Dixon teaching amplitudes methods and ideas to phenomenologists, in TASI-1995 [19] – which is where I learned spinor helicity, and TASI-2013 [20]. I believe Lance lives in mostly minus convention, so it's good to compare with Henriette Elvang and Yu-tin Huang's awesome textbook [21] which is in mostly-plus.

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