

From brane dynamics to a Kac-Moody invariant formulation of M-theories

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ABSTRACT: Theories of gravity coupled to forms and dilatons may admit as solutions zero binding energy configurations of intersecting closed extremal branes. In such configurations, some branes may open on host closed branes. Properties of extremal branes reveal symmetries of the underlying theory which are well known in M-theory but transcend supersymmetry. From these properties it is possible to reconstruct all actions, comprising in particular pure gravity in D dimensions, the bosonic effective actions of M-theory and of the bosonic string, which upon dimensional reduction to three dimensions are invariant under the maximally non-compact simple simply laced Lie groups \mathcal{G} . Moreover the features of extremal branes suggest the existence of a much larger symmetry, namely the ‘very-extended’ Kac-Moody algebras \mathcal{G}^{+++} . This motivates the construction of explicit non-linear realisations of all simple \mathcal{G}^{+++} , which hopefully contain new degrees of freedom such as those encountered in string theories. They are defined without a priori reference to space-time and are proposed as substitutes for original field theoretic models of gravity, forms and dilatons. From the \mathcal{G}^{+++} -invariant theories, all algebraic properties of extremal branes are recovered from exact solutions, and there are indications that space-time is hidden in the infinite symmetry structure. The transformation properties of the exact solutions, which possibly induce new solutions foreign to conventional theories, put into evidence the general group-theoretical origin of ‘dualities’ for all \mathcal{G}^{+++} . These dualities apparently do not require an underlying string theory.

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1. Introduction and conclusion

Theories of gravity coupled to forms and dilatons may admit as solutions zero binding energy configurations of intersecting closed extremal branes [1]. In such configurations, some branes may open on host closed branes [2, 3, 4]. Properties of extremal branes reveal symmetries of the underlying theory, which are well known in M-theory but transcend supersymmetry. From these properties it is possible to reconstruct all actions, comprising in particular pure gravity in D dimensions, the bosonic effective actions of M-theory and of the bosonic string, which upon dimensional reduction to three dimensions are invariant under the maximally non-compact simple simply laced Lie groups \mathcal{G} [5]. Moreover the features of extremal branes suggest the existence of a much larger symmetry, namely the ‘very-extended’ Kac-Moody algebras \mathcal{G}^{+++} . Such symmetries were first conjectured in the above mentioned particular cases [6, 7], explicit representations of their Weyl group for Kasner-type solutions were obtained for all simple \mathcal{G}^{+++} [8] and their relation to the cosmological billiards [9, 10, 11] was brought to light.

These facts motivate the construction of explicit non-linear realisations of all simple \mathcal{G}^{+++} on cosets obtained from a modified Chevalley involution [12]. The \mathcal{G}^{+++} -invariant actions are proposed as substitutes for the original field theoretic models of gravity, forms and dilatons, and hopefully contain new degrees of freedom such as those encountered in string theories. The actions are defined on a world-line *a priori unrelated to space-time*. The latter should then be deduced dynamically. Such an approach to gravity and forms, if successful, would dispose of the need of explicit diffeomorphism invariance or gauge invariance. All such information should be hidden in the *global* \mathcal{G}^{+++} invariance. Although it may seem that global symmetries cannot contain local symmetries, in particular in view of the celebrated Elitzur theorem [13], it need not be the case in view of the infinite number of generators of \mathcal{G}^{+++} . We formulate the \mathcal{G}^{+++} invariant theory recursively from a level decomposition [14, 15, 16] with respect to a subalgebra A_{D-1} where D turns out to be the space-time dimension. Our formulation is exploratory and does not pretend to be a final one. No attempt is made to cope with fermionic degrees of freedom and we limit here our investigation to the classical domain. From the \mathcal{G}^{+++} -invariant theories, all algebraic properties of extremal branes are recovered from exact solutions, and there are indications that space-time is hidden in the infinite level structure. The transformation properties of the exact solutions, which also generate Kaluza-Klein waves and KK-monopoles (Taub-NUT spaces) and possibly solutions foreign to conventional theories, put into evidence the general group-theoretical origin of ‘dualities’ for all \mathcal{G}^{+++} . These dualities apparently do not require an underlying string theory.

In Sections 2 and 3, we review the properties of intersecting extremal branes and of their opening, which are used in Section 4 to reconstruct dynamically the simple simply laced Lie algebras \mathcal{G} and provide motivation for considering the very-extended Kac-Moody algebras \mathcal{G}^{+++} . The non-linear realisations of \mathcal{G}^{+++} are presented in Section 5. Extremal branes and their duality properties are revisited in this new framework in Section 6.

2. Intersection rules for extremal branes

We begin with a generic theory in D dimensions which includes gravity, one dilaton and \mathcal{M} field strengths of arbitrary form degree p_I with $p_I \leq D/2$ and arbitrary couplings to the dilaton a_I . The action is

$$S = \frac{1}{16\pi G_N^{(D)}} \int d^D x \sqrt{-g} \left(R - \frac{1}{2} (\partial\phi)^2 - \sum_I \frac{1}{2p_I!} e^{a_I \phi} F_{p_I}^2 \right), \quad I = 1 \dots \mathcal{M}, \quad (2.1)$$

where we have not included possible Chern-Simons terms. It can be shown that they have no effect on the zero binding energy solutions considered here although such terms will be important in the following sections.

The ‘BPS’ zero binding energy configurations of closed extremal branes intersecting orthogonally are obtained by first specialising to metrics of the following diagonal form,

$$ds^2 = -e^{2p^{(1)}} d\tau^2 + \sum_{\mu=2}^{D-p} e^{2p^{(\mu)}} (dx^\mu)^2 + \sum_{\lambda=1}^p e^{2p^{(D-p+\lambda)}} (dy^\lambda)^2, \quad (2.2)$$

where y^λ label p compact coordinates. The functions $p^{(\alpha)}$ ($\alpha = 1, 2, \dots, D$) depend only on the transverse coordinates x^μ in the non-compact dimensions and allow for multi-centre solutions. We choose p so that all branes are wrapped in the compact dimensions and that no compact dimension is transverse to all branes. Thus for each brane $p \geq q_A$ where q_A is the dimension of the brane. If $q_A < p$, we take a lattice of q_A -branes in the compact directions transverse to the brane and average over them. Here q_A designates either an electrically charged q_A^e -brane with respect to a p_I -form field strength F_{p_I} , or its dual q_A^m -magnetic brane. For \mathcal{N} intersecting branes, we write

$$p^{(\alpha)} = \sum_{A=1}^{\mathcal{N}} p_A^{(\alpha)} \quad \text{and} \quad \phi = \sum_{A=1}^{\mathcal{N}} \phi_A \quad (2.3)$$

with

$$(p+3-D)p_A^{(\mu)} = p_A^{(1)} + \sum_{\lambda=1}^p p_A^{(D-p+\lambda)} \quad \mu = 2, \dots, D-p. \quad (2.4)$$

Eq.(2.4) may be interpreted as the extremality condition of the component branes [17]. For electric and magnetic brane potentials, we take

$$\text{Electric : } A_{\tau\lambda_1 \dots \lambda_{q_A^e}} = \epsilon_{\tau\lambda_1 \dots \lambda_{q_A^e}} E_A^e(\{x^\nu\}), \quad (2.5)$$

$$\text{Magnetic : } \tilde{A}_{\tau\lambda_1 \dots \lambda_{q_A^m}} = \epsilon_{\tau\lambda_1 \dots \lambda_{q_A^m}} E_A^m(\{x^\nu\}), \quad (2.6)$$

where \tilde{A} is the (magnetic) potential of the dual field strength \tilde{F} defined by

$$\sqrt{-g} e^{a_I \phi} F^{\mu_1 \dots \mu_{p_I}} = \frac{1}{(D-p_I)!} \epsilon^{\mu_1 \dots \mu_{p_I} \nu_1 \dots \nu_{D-p_I}} \tilde{F}_{\nu_1 \dots \nu_{D-p_I}}. \quad (2.7)$$

Using these ansätze, the Einstein equations and the equations for the dilaton and the forms yield

$$\begin{aligned}
 p_A^{(1)} &= -\frac{D - q_A - 3}{\Delta} \ln H_A, \\
 p_A^{(\mu)} &= \frac{q_A + 1}{\Delta} \ln H_A, \\
 p_A^{(D-p+\lambda)} &= \frac{\delta_A^{(\lambda)}}{\Delta} \ln H_A, \\
 \phi_A &= \frac{D - 2}{\Delta} \varepsilon_A a \ln H_A,
 \end{aligned} \tag{2.8}$$

where $H_A(\{x^\nu\})$ is a harmonic function related to the E_A by $H_A = \sqrt{2(D-2)/\Delta} E_A^{-1}$ and

$$\Delta = (q_A + 1)(D - q_A - 3) + \frac{1}{2} a_A^2 (D - 2). \tag{2.9}$$

In Eq.(2.8) $\delta_A^{(\lambda)} = -(D - q_A - 3)$ or $(q_A + 1)$ depending on whether y^λ is parallel or perpendicular to the q_A -brane. The factor ε_A is $+1$ for an electric brane and -1 for a magnetic one. The harmonic functions $H_A(\{x^\nu\})$ allow for parallel branes and are given (for $D - p > 3$) by

$$H_A = 1 + \sum_k \frac{Q_k}{|x^\mu - x_k^\mu|^{D-p-3}}, \tag{2.10}$$

where the x_k^μ label the positions in non-compact space-time of the branes with charge Q_k . For $\mathcal{N} > 1$, Eqs.(2.8) are restricted by algebraic conditions stemming from the equation for the non diagonal components of the curvature tensor. These are, for each pair (A, B) of distinct q -branes of dimensions (q_A, q_B) , the number of dimensions \bar{q} ($-1 \leq \bar{q} \leq q_A, q_B$) on which they intersect⁰ in terms of the total number of space-time dimensions D and of the field strength couplings to the dilaton. These *intersection rules* read [17]

$$\bar{q} + 1 = \frac{(q_A + 1)(q_B + 1)}{D - 2} - \frac{1}{2} \varepsilon_A a_A \varepsilon_B a_B. \tag{2.11}$$

Note that such zero energy binding configurations were originally considered in the context of M-theory ‘phases’ but may arise classically without supersymmetry.

3. The opening of branes

We now analyse the breaking of closed extremal branes into open branes terminating on closed ones. We consider the above BPS configurations in the special case when \bar{q} has the same dimension as the potential boundary of one of the two constituent branes, i.e $q_A - 1$ or $q_B - 1$, and study its possible opening. Such opening requires the addition of Chern-Simons terms to the action Eq.(2.1) and may enlarge the brane content of the theory. We shall see in Section 4 that, under some conditions, such openings fully determine the theory

⁰ \bar{q} must be integer and the case $\bar{q} = -1$ is relevant. It can be interpreted in terms of instantons in the Euclidean, in which case the time coordinate need not be longitudinal to all branes.

and relate brane dynamics to the existence of a symmetry. The presentation given in this section is a generalisation of the one performed in the context of M-theory [2, 3, 4].

Let us review how extended objects carrying a conserved charge can be opened. The main obstacle towards opening of branes is charge conservation. Generically, the charge density of a q -brane is measured by performing an integral of the relevant field strength on a $(D - q - 2)$ -dimensional sphere S^{D-q-2} surrounding the brane in its transverse space,

$$Q_q \propto \int_{S^{D-q-2}} *F_{q+2}. \quad (3.1)$$

If the brane is open, we can slide the S^{D-q-2} off a loose end and shrink it to zero size. This would imply the vanishing of the charge and hence a violation of charge conservation. This conclusion is avoided if, in the above process, the S^{D-q-2} necessarily goes through a region in which the equation

$$d * F_{q+2} = 0 \quad (3.2)$$

no longer holds. This is the case when the open brane ends on some other one.

In the framework of M-theory the source terms needed in Eq.(3.2) to ensure charge conservation for the open branes originate from two requirements whose interplay leads to a consistent picture. On the one hand there are space-time Chern-Simons type terms in supergravity which allow for charge conservation for well defined pairing of open and ‘host’ branes [3]. On the other hand the world-volume effective actions [2] for the branes of M-theory relate world-volume fields and pullbacks of space-time fields, and gauge invariance [18] for open branes ending on the ‘host’ brane implies that the end of the open branes acts as a source for the world-volume field living on the closed ‘host’ brane.

In reference [4] a systematic study in M-theory of all the zero binding energy configurations Eq.(2.11) corresponding to $\bar{q} = q_A - 1$ (with $q_A \leq q_B$) was performed. It was shown that in all cases it was possible to open the q_A -brane along its intersection with the q_B -brane. The crucial ingredient was the presence of the appropriate Chern-Simons terms in the supergravity Lagrangians for each case.

Here we propose to reverse the logic. Starting with a Lagrangian of type Eq.(2.1) and having zero binding energy configurations between branes we will ask that, if $\bar{q} = q_A - 1$, the corresponding q_A -brane open on the q_B -brane. This will *determine* the form of the Chern-Simons terms one has to add to Eq.(2.1) and will also in some cases require the *introduction of new field strength forms* F_{n_I} . One then proceeds iteratively.

We illustrate the role of the Chern-Simons term (see also [3, 4]) by taking as example a theory with only one n -form F_{q^e+2} and dilaton coupling in Eq.(2.1) such that the intersection rule between the electric q^e -brane and the magnetic q^m -brane ($q^m = D - q^e - 4$) is $\bar{q} = q^e - 1$.

We modify the Eq.(3.2) for F_{q^e+2} in order to be able to allow the opening of q^e on q^m by the addition to the action Eq.(2.1) of the Chern-Simons term

$$\int A_{q^e+1} \wedge F_{q^e+2} \wedge F_{D-2q^e-3}, \quad (3.3)$$

and of a standard kinetic energy term for the new¹ field strength F_{D-2q^e-3} . Its dilaton coupling is chosen such that the intersection rules Eq.(2.11) give integer intersection dimension between the new extremal electric $(D - 2q^e - 5)$ -brane and its dual magnetic $(2q^e + 1)$ -brane. Charge conservation now reads

$$d * F_{q^e+2} = F_{q^e+2} \wedge F_{D-2q^e-3} + Q^e \delta_{D-q^e-1}. \quad (3.4)$$

Here wedge products are defined up to signs and numerical factors. In the r.h.s. of Eq.(3.4) the first term comes from the variation of the Chern-Simons term and the second one is the q^e -brane charge density. δ_{D-q^e-1} is the Dirac delta function in the directions transverse to the q^e -brane. We introduce here an explicit source term for the electric brane since, to study its opening, we want to extend to the branes themselves the validity of the usual closed brane solution. Such term is required because the equations of motion from which the intersecting brane solutions were derived do not contain any source term and are therefore valid only outside the sources.

It is straightforward to show that Eq.(3.4) renders opening consistent with charge conservation and that new branes arising from F_{D-2q^e-3} have to be added. From Eqs.(3.3) and (3.4), the addition of such branes can be determined in a picturesque way. Namely, when a brane opens on a closed ‘host’ brane, the boundary appears from the world-volume point of view as a charged object under a world-volume field strength living in the closed brane. The world-volume Hodge dual of this object is the boundary of an other brane which can also be consistently opened on the same closed ‘host’ brane. The field strength associated to this new brane is precisely the one appearing in the Chern-Simons ensuring the consistency of the opening of the brane we started with.

In brief, having a theory of type Eq.(2.1) in which some zero binding energy configurations give potential boundaries, it is possible to complete it in a well-defined way by adding Chern-Simons terms², and when necessary new form field strengths (hence new branes) in order to ensure consistency of brane opening with charge conservation. In section 4, such dynamical requirement will be at work to reconstruct purely from brane considerations theories whose dimensional reduction is characterised by a coset symmetry.

4. From brane dynamics to \mathcal{G} and \mathcal{G}^{+++}

4.1 Intersection rules and dimensional reduction

We perform the dimensional reduction to three dimensions of a generic theory described by an action Eq.(2.1), possibly with the addition of Chern-Simons terms and discuss the possible emergence of a coset symmetry. We relate the onset of such symmetry to the intersection rules for BPS configurations between the extremal branes of the theory considered.

¹If $D = 3q^e + 5$ there is no need to introduce a new field strength as the Chern-Simons terms can be build with the field strength F_{q^e+2} .

²The coefficients of the Chern-Simons terms Eq.(3.3) are not fixed in this qualitative discussion. Their precise values are important when one will discuss the potential symmetries. In the framework of M-theory they are usually fixed by supersymmetry. Nevertheless it is possible to fix them or at least quantise them generically using only consistency arguments without appealing to supersymmetry, see for instance [19, 20].

Starting with the theory defined in D dimensions, we compactify to $D - 1$ dimensions while remaining in the Einstein frame with the standard 1/2 kinetic term normalisation for the new scalar. We use the notation ϕ_2 for the scalar appearing in the first step of the dimensional reduction and rename ϕ_1 the dilaton ϕ already present in the uncompactified theory defined by Eq.(2.1). The compactified coordinate is x^{D-1} , the uncompactified coordinates are x^μ where $\mu = 0 \dots D - 2$, and

$$\beta_D = \sqrt{\frac{1}{2(D-1)(D-2)}}. \quad (4.1)$$

The gravitation part of the action Eq.(2.1) becomes

$$\int d^D x \sqrt{-g_D} R_D = \int d^{D-1} x \sqrt{-g_{D-1}} \left(R_{D-1} - \frac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2 - \frac{1}{4} e^{-2(D-2)\beta_{D-1}\phi_2} F_{\mu\nu} F^{\mu\nu} \right). \quad (4.2)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

For each n_I -form field strength F_{n_I} in Eq.(2.1) we get after reduction

$$\int d^D x \sqrt{-g_D} \frac{1}{2n_I!} e^{a_I \phi_1} F_{n_I}^2 = \int d^{D-1} x \sqrt{-g_{D-1}} \left(\frac{1}{2n_I!} e^{a_I \phi_1 - 2(n_I-1)\beta_{D-1}\phi_2} F_{n_I}^{\prime 2} + \frac{1}{2(n_I-1)!} e^{a_I \phi_1 + 2(D-1-n_I)\beta_{D-1}\phi_2} F_{n_I-1}^2 \right), \quad (4.3)$$

where

$$F'_{\mu_1 \dots \mu_n} = F_{\mu_1 \dots \mu_n} - n F_{[\mu_1 \dots \mu_{n-1}} A_{\mu_n]}. \quad (4.4)$$

We can repeat this procedure step by step to obtain the theory on a p -torus. One has then obviously p scalars ϕ_j with $j = 2 \dots p+1$ parametrising the radii of the torus, coming from the diagonal components of the metric in the compact dimensions. Additional scalars denoted $\chi_{\vec{\alpha}}$ arise from several origins. They come from potentials A_μ^k which arise when reducing gravity from $D+1-k$ to $D-k$ and also from the potentials associated to the F_{n_I} (when $p \geq n_I - 1$) with indices in the compact dimensions. In addition the n -form field strengths give additional scalars when $p = D - n - 1$ by dualising them. In particular when we reach $D = 3$ the $F_{\mu\nu}^k$ (with $k = 1 \dots D - 3$) coming from the gravity part of the action (i.e. the graviphotons) can be dualised to scalars, and we are left with only scalars. The action takes then the form

$$S = \int d^3 x \sqrt{-g_3} \left(R_3 - \frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - \frac{1}{2} \sum_{\vec{\alpha}} e^{\sqrt{2}\vec{\alpha} \cdot \vec{\phi}} \partial_\mu \chi_{\vec{\alpha}} \partial^\mu \chi_{\vec{\alpha}} + \dots \right), \quad (4.5)$$

where $\vec{\phi} = (\phi_{D-2}, \dots, \phi_1)$, the $\vec{\alpha}$ are constant $(D-2)$ -vectors³ characterising each $\chi_{\vec{\alpha}}$. If we start in D dimensions without dilaton the vectors are of course $(D-3)$ -dimensional as ϕ_1 is absent in that case. The ellipsis in Eq.(4.5) stands for terms of order higher

³The normalisation factor $\sqrt{2}$ has been chosen for convenience. It will eventually correspond in the simply laced case to the standard normalisation of the roots, namely $\vec{\alpha} \cdot \vec{\alpha} = 2$.

than quadratic in the $\chi_{\vec{\alpha}}$ scalars. They come from the modification of the field strengths Eq.(4.4) in the dimensional reduction process and also from possible Chern-Simons terms in the uncompactified theory.

The action Eq.(2.1) dimensionally reduced to three dimensions has a \mathcal{G}/\mathcal{H} symmetry if the vectors $\vec{\alpha}$ obtained from the compactification can be identified with the positive roots of a group \mathcal{G} and if, when necessary, some precise Chern-Simons terms are added in the uncompactified theory [21]. The requirement that the $\vec{\alpha}$ correspond to positive roots is thus a necessary condition to uncover a symmetry [7].

Recall first the well-known dimensional reduction of pure gravity (see Eq.(4.2)) down to three dimensions, which leads to $\mathcal{G} = SL(D - 2)$ whose algebra is A_{D-3} . The scalars corresponding to the simple roots of A_{D-3} are of two kinds.

There are first $D - 4$ scalars which are the components A_{D-k-1}^k of the potentials coming from $g_{D-k,D-k-1}$, $k = 1 \dots D - 4$. These are obtained by performing the “fastest” reduction on the potentials A_{μ}^k (see Eq.(4.2)) to obtain a scalar going from $D - k$ to $D - k - 1$ when compactifying on T^{k+1} . The corresponding simple roots $\vec{\alpha}_k^g$ are given by

$$\vec{\alpha}_k^g = \sqrt{2}(\underbrace{0, \dots, 0}_{D-4-k \text{ terms}}, (D - k - 3)\beta_{D-k-1}, -(D - k - 1)\beta_{D-k}, \underbrace{0, \dots, 0}_{k-1 \text{ terms}}; 0),$$

$$k = 1 \dots D - 4. \tag{4.6}$$

They define a subalgebra A_{D-4} . We have indeed

$$\vec{\alpha}_k^g \cdot \vec{\alpha}_l^g = \begin{cases} 2 & k = l \\ -1 & |k - l| = 1 \\ 0 & |k - l| \geq 2 \end{cases} \tag{4.7}$$

Reading from right to left, the first component of $\vec{\alpha}_k^g$ associated to the dilaton ϕ in the original uncompactified theory Eq.(2.1) is always zero. The corresponding Dynkin diagram with $D - 4$ nodes, which from now on we will refer to as *the gravity line*, is depicted in Fig.1.

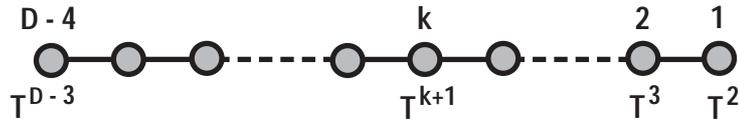


Fig.1. The gravity line.

Dynkin diagram of A_{D-4} generated by the dimensional reduction to 3 dimensions.

The remaining scalar corresponding to the missing simple root leading to the full A_{D-3} comes from dualising in three dimensions the first vector (graviphoton) that arises in the stepwise procedure namely the vector appearing already in $D - 1$ dimensions. The corresponding simple root is

$$\vec{\alpha}^{gp} = \sqrt{2}(\beta_3, \dots, \beta_{D-3}, \beta_{D-2}, (D - 2)\beta_{D-1}; 0). \tag{4.8}$$

Note that this simple root $\vec{\alpha}^{gp}$ has a non-vanishing scalar product with $\vec{\alpha}_1^g$ (i.e with the simple root already appearing when compactifying on T^2). One has indeed $\vec{\alpha}_k^g \cdot \vec{\alpha}^{gp} = -\delta_{k,1}$. Consequently it attaches itself to the right of the gravity line. The other $\frac{1}{2}(D-4)(D-3)$ scalars coming from the reduction of gravity down to three dimensions give all the positive roots of A_{D-3} .

We now turn to theories with forms given by Eq.(2.1) and consider a n_A -form F_{n_A} with dilaton coupling a_A (and $n_A \leq D/2$). Let us consider the first scalar arising from the n_A -form upon dimensional reduction up to $p = n_A - 1$. The vector $\vec{\alpha}_{n_A}^e$ associated to this scalar⁴ will from now on be called *the would-be electric root*. It is given by

$$\vec{\alpha}_{n_A}^e = \left(\underbrace{0, \dots, 0}_{D-n_A-2 \text{ terms}}, \underbrace{b_{(n_A,D)} \beta_{D-n_A+1}, \dots, b_{(n_A,D)} \beta_{D-2}, b_{(n_A,D)} \beta_{D-1}}_{n_A-1 \text{ terms}}; \frac{a_A}{\sqrt{2}} \right), \quad (4.9)$$

with

$$b_{(n_A,D)} = \sqrt{2}(D - n_A - 1). \quad (4.10)$$

First, we compute the scalar product of the would-be electric root with the gravity line⁵. Using Eq.(4.6) and Eq.(4.9) we find, for any D and dilaton coupling a_A ,

$$\vec{\alpha}_k^g \cdot \vec{\alpha}_{n_A}^e = -\delta_{k,n_A-1}. \quad (4.11)$$

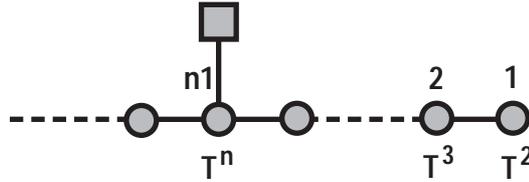


Fig.2. Would-be Dynkin diagram.

The would-be electric root associated to F_n is represented by a shaded square.

Using the scalar product Eq.(4.11) of the would-be electric root with the gravity line, we can draw a would-be Dynkin diagram where the would-be electric root associated to F_{n_A} is connected to the $(n_A - 1)^{th}$ node of the gravity line as depicted in Fig.2.

We then evaluate the length of the would-be electric root using Eq.(4.9). We find

$$\begin{aligned} \vec{\alpha}_{n_A}^e \cdot \vec{\alpha}_{n_A}^e &= \frac{(n_A - 1)(D - n_A - 1)}{(D - 2)} + \frac{a_A^2}{2}, \\ &= \frac{(q_A^e + 1)(q_A^m + 1)}{(D - 2)} + \frac{a_A^2}{2}, \\ &= \bar{q}^{(e_A, m_A)} + 1. \end{aligned} \quad (4.12)$$

⁴The other scalars obtained by further dimensional reduction give $\vec{\alpha}$ -vectors that are linear combinations with positive integer coefficients of $\vec{\alpha}_{n_A}^e$ and of the $\vec{\alpha}_k^g$.

⁵The scalar product of the would-be electric root with the graviphoton is one for any D and a_A . This implies that when forms are present in a theory with symmetry, the graviphoton is never a simple root. Consequently we focus on the gravity line.

We thus see that the square length of the would-be electric root associated to F_{n_A} can be written in terms of the intersection rule equation Eq.(2.11) giving the intersection between the electric and the magnetic brane charged under F_{n_A} .

From now on we will restrict ourselves to simply laced groups. In our normalisation all their roots are of square length two. In order for the would-be root to be a root, one must have $\vec{\alpha}_{n_A}^e \cdot \vec{\alpha}_{n_A}^e = 2$. Consequently *the existence of a BPS configuration in the original theory, consisting of an electric extremal p -brane ($p \geq 1$) and its magnetic dual whose intersection is $\bar{q}^{(e_A, m_A)} = 1$, is a necessary condition to have after dimensional reduction an enhanced simply laced Lie group symmetry.*

4.2 Dynamical reconstruction of Lie groups

Our starting point is a theory given by Eq.(2.1) in D dimensions with only one n_A -form field strength F_{n_A} and its dilaton coupling a_A . We will fix the dilaton coupling such that there exists a zero-binding energy configuration between the electric q_A^e -brane ($q_A^e = n_A - 2$) and the magnetic q_A^m -brane with $\bar{q}^{(e_A, m_A)} = 1$. As explained in the previous section this is a necessary condition in order to find a new symmetry. Once the dilaton coupling of the form is fixed, we require that, when the dimensionality of an intersection permits opening, the latter is consistent with charge conservation. Namely we impose that, if $\bar{q}^{(e_A, m_A)} = q_A^e - 1$, the electric brane open on the magnetic brane. As explained in Section 3, this requires the introduction of a specific Chern-Simons term in the action, which may contain a new form field strength F_{n_B} . The dilaton coupling a_B of the new form field strength is then again fixed modulo its sign by the necessary condition $\bar{q}^{(e_B, m_B)} = 1$. The intersection rules between the extremal branes corresponding to the different forms can be calculated and this fixes the relative signs of the dilaton couplings. We can then check if new openings are possible. If it is the case, we iterate the procedure until consistency of all the openings are ensured. In this way, we are able to reconstruct all the maximally oxidised theories corresponding to the simply laced groups \mathcal{G} [5], that is all actions Eq.(2.1) which lead upon dimensional reduction to such group \mathcal{G} and which are not dimensional reduction from an action in higher dimension [22]. This leads to the following conclusion: *The existence of BPS configurations with $\bar{q} = 1$ between any electric extremal p -brane ($p \geq 1$) and its magnetic dual, along with the requirement of consistency of brane opening in the original uncompactified theory (characterised by at most one dilaton), is a necessary and sufficient condition to have a theory whose dimensional reduction down to three dimensions has a simple simply laced group \mathcal{G} symmetry.*

Using Eq.(4.12) the condition $\bar{q}^{(e_B, m_B)} = 1$ yields up to a sign the dilaton coupling

$$a_n^2(D) = 2 \frac{(1-q)D + (q^2 + 4q - 1)}{D - 2}. \quad (4.13)$$

with

$$a_n^2 \geq 0. \quad (4.14)$$

The procedure outlined above gives the following relations between q, D , the Chern-Simons terms, and the Lie group symmetry of the theory reduced to three dimensions exhibited in Table I

q	D	a_{q+2}^2	Openings	$C.S.$	\mathcal{G}
0	–	$2(D-1)/(D-2)$	–	–	A_{D-1}
1	–	$8/(D-2)$	–	–	D_{D-2}
2	11	0	$2 \cap 5 = 1$	$A_3 \wedge F_4 \wedge F_4$	E_8
2	10	$1/4$	$2 \cap 4 = 1, \dots$	$A_3 \wedge F_4 \wedge H_3, \dots$	E_8
2	9	$4/7$	$2 \cap 3 = 1$	$A_3 \wedge F_4 \wedge F_2$	E_7
2	8	1	$2 \cap \bar{2} = 1$	$A_3 \wedge F_4 \wedge F_1$	E_6
3	10	0	–	–	E_7

 Table I : Reconstruction from branes of the simple simply laced Lie groups \mathcal{G} .

We see indeed that all simple simply laced Lie groups (in their maximally non compact form) are recovered from the intersecting extremal brane solutions and their openings from actions of the type Eq.(2.1) with Chern-Simons terms. The action which upon dimensional reduction to three dimensions exhibit these Lie group symmetry are fully determined.

4.3 From \mathcal{G} to \mathcal{G}^{+++}

Taking into account the intersection rules Eq.(2.11), one verifies that Eq.(2.8) implies

$$\sum_{\alpha=1}^D (p^{(\alpha)})^2 - \frac{1}{2} \left(\sum_{\alpha=1}^D p^{(\alpha)} \right)^2 + \frac{1}{2} \phi^2 = \frac{D-2}{\Delta} \sum_A \ln^2 H_A(x^a), \quad (4.15)$$

and we recall that the extremal branes in the intersecting brane configurations satisfy the extremality conditions Eq.(2.4). Eqs. (2.4) and (4.15) indicate that the solutions Eq.(2.8) carry a group-theoretical significance in the triple (or very-) Kac-Moody extension \mathcal{G}^{+++} of \mathcal{G} , for all simple groups \mathcal{G}^{+++} (simply laced or not) as we now show.

We first recall how all simple Lie algebra \mathcal{G} can be embedded in a very-extended Kac-Moody algebra \mathcal{G}^{+++} . The simple roots of \mathcal{G}^{+++} are given by adding two nodes to the gravity line of the Dynkin diagram of the affine extension \mathcal{G}^+ of \mathcal{G} , thus increasing by three the rank of \mathcal{G} [23]. The resulting Dynkin diagrams for \mathcal{G}^{+++} are shown in Fig.3. As in the case of Lie algebras, the Dynkin diagrams yield the Cartan matrix of \mathcal{G}^{+++} which is then entirely determined from the Serre relations.

The group $SL(D)$ defined by this triple extended gravity line can be extended to the full deformation group $GL(D)$ whose algebra, generated by D^2 generators K_b^a , $a, b = 1, \dots, D$, is a subalgebra of \mathcal{G}^{+++} . The K_b^a satisfy the following commutation relations

$$[K_b^a, K_d^c] = \delta_b^c K_d^a - \delta_d^a K_b^c. \quad (4.16)$$

One considers the Cartan subalgebra of \mathcal{G}^{+++} generated by the K_a^a and $s = r - D$ abelian generators R_u where r is the rank of \mathcal{G}^{+++} . We write the corresponding abelian group element $\mathcal{V}_{\text{abelian}}$ as

$$\mathcal{V}_{\text{abelian}} = \exp\left(-\sum_{a=1}^D p^{(a)} K_a^a - \sum_{u=1}^s \phi^u R_u\right). \quad (4.17)$$

and tentatively identify $p^{(a)}$ and ϕ^u in Eq.(4.17) with (the log of) the diagonal vielbein in a triangular gauge⁶ $e_\alpha^a(x)$, $\alpha \leq a$ and ϕ^u with dilaton fields $\phi^u(x)$ in any action \tilde{S} describing a maximally oxidised theory⁷. The actions \tilde{S} generalise to all simple Lie groups the ones reconstructed in Section 3 for simply laced groups. It takes the form

$$\tilde{S} = \frac{1}{16\pi G_N^{(D)}} \int d^D x \sqrt{-g} \left[R - \frac{1}{2} \sum_{u=1}^s (\partial\phi^u)^2 - \frac{1}{2} \sum_I \frac{1}{p_I!} \exp\left(\sum_{u=1}^s a_I^u \phi^u\right) F_{p_I}^2 \right] + C.S., \quad (4.18)$$

where $C.S.$ represents Chern-Simons terms that are required for some groups \mathcal{G} .

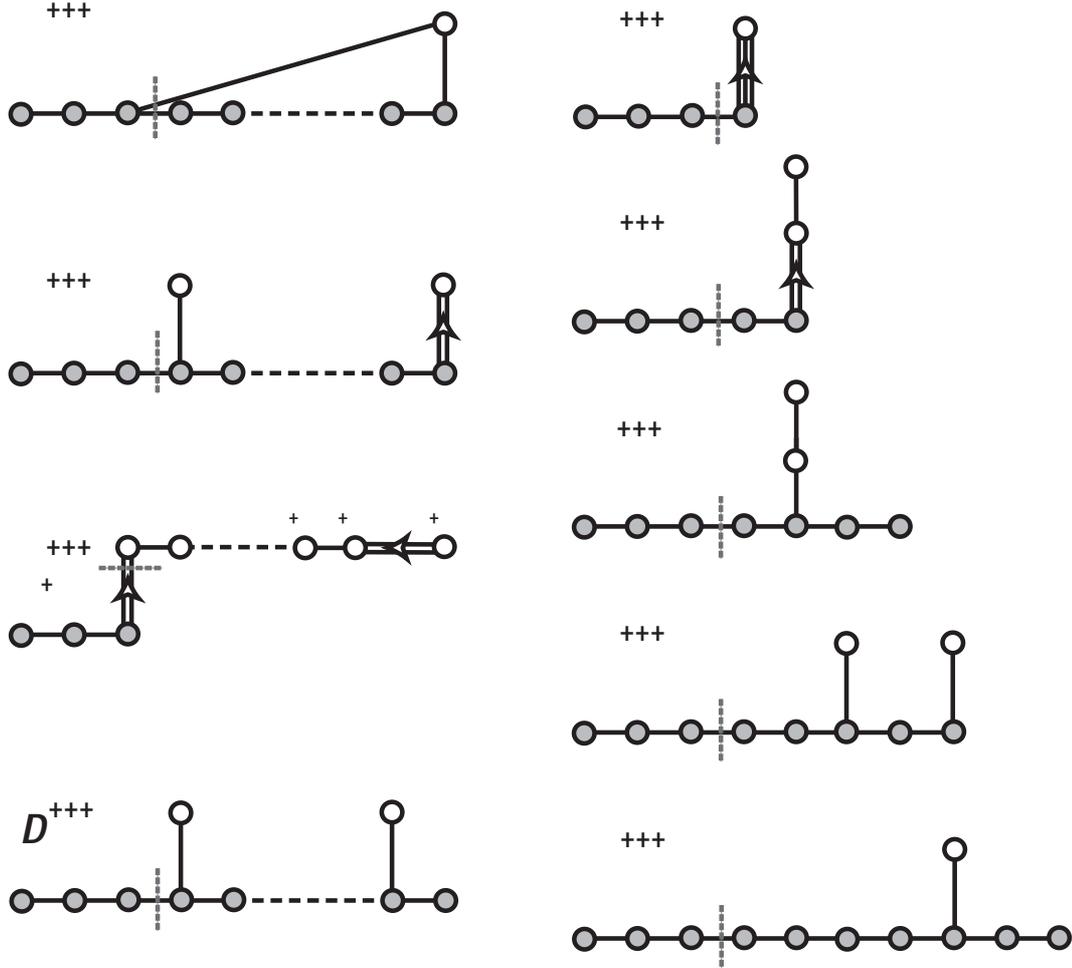


Fig.3. Dynkin diagram of \mathcal{G}^{+++} .

The nodes of the gravity line are shaded. The Dynkin diagram of \mathcal{G} is that part of the diagram of \mathcal{G}^{+++} which sits on the right of the dashed line. The first three nodes define the Kac-Moody extensions.

⁶We can indifferently label the $p^{(a)}$ by a curved or a flat index, as it is uniquely defined by the diagonal vielbein in the triangular gauge. The position of this index as a subscript or superscript is a matter of convention and has no tensor significance.

⁷A geometrical motivation of this identification is given in [8] and an alternate motivation has been proposed in [24] by extending $GL(D)$ to $IGL(D)$. In this work the justification of the identification follows from the analysis of Section 5.

Given any symmetrisable Kac-Moody algebra there exists, up to a numerical factor, a unique scalar product defined on the algebra that is invariant under the adjoint action of the algebra [25]. For a finite-dimensional simple Lie algebra this is just the Killing form which can be expressed as the trace of the generators in any finite-dimensional representation. In the Cartan subalgebra described in the $L_i = \{K^a_a, R_u\}$ basis it is [8]

$$\langle K^a_a, K^b_b \rangle = \delta_{ab} - \frac{1}{2}\Xi_{ab} \quad ; \quad \langle R_u, R_v \rangle = \frac{1}{2}\delta_{uv} \quad ; \quad \langle K^a_a, R_v \rangle = 0, \quad (4.19)$$

where Ξ is the matrix with all entries equal to one. The matrix $\mathbf{G}_{ij} = \langle L_i, L_j \rangle$ is invariant under the group of Weyl transformations (and of outer automorphisms) of \mathcal{G}^{+++} . Hence the quadratic form

$$\sum_{\alpha=1}^D (p^{(\alpha)})^2 - \frac{1}{2} \left(\sum_{\alpha=1}^D p^{(\alpha)} \right)^2 + \frac{1}{2} \sum_{u=1}^s (\phi^u)^2 \quad (4.20)$$

is invariant under the Weyl reflections in \mathcal{G}^{+++} .

Eq.(4.20) is identical with the left hand side of Eq.(4.15). In addition, one can prove that the extremality condition Eq.(2.4) defines a Weyl preserving embedding of \mathcal{G}^{p+1} in \mathcal{G}^{+++} where \mathcal{G}^{p+1} is the symmetry group resulting from dimensional reduction of the action Eq.(4.18) to $(D - p - 1)$ dimensions where the compactification is performed on a spatial p -torus and on time. These features would provide a strong indication of the existence of a hidden \mathcal{G}^{+++} symmetry of the theory defined by Eq.(4.18) or by some generalisation thereof, if the right hand side of Eq.(4.15) would share such group-theoretical significance. In the next section, we show, by construction of an explicit \mathcal{G}^{+++} invariant action, that this is indeed the case.

5. Non-linear realisation of \mathcal{G}^{+++}

This section is based on reference [12].

5.1 The temporal involution and the coset space $\mathcal{G}^{+++}/K^{+++}$

We want ultimately \mathcal{G}^{+++} to fully characterise the symmetry of the action \tilde{S} defined by Eq.(4.18), or of some more general theory. As an enlargement of the *global* symmetry group \mathcal{G} arising in the dimensional reduction of \tilde{S} , \mathcal{G}^{+++} should also define a *global* symmetry. This poses a dilemma. The metric and the form field strengths in \tilde{S} are genuine space-time fields and \tilde{S} is invariant under the *local* diffeomorphism and gauge groups. How could a global symmetry encompass a local symmetry? The present analysis is an attempt to solve this dilemma by taking advantage of the infinite dimensionality of the algebra \mathcal{G}^{+++} . More precisely we replace the action \tilde{S} by an action \mathcal{S} explicitly invariant under the global \mathcal{G}^{+++} symmetry. \mathcal{S} contains an infinite number of objects that are tensors under $SL(D)$. These comprise a symmetric tensor $g_{\mu\nu}$, scalars ϕ^u and $(p_I - 1)$ -form potentials $A_{\mu_1\mu_2\dots\mu_{p_I-1}}$ which can be interpreted as the corresponding fields occurring in Eq.(4.18) taken at a *fixed* space time point. Their motion in space-time, as well as those of possible additional fields, is expected to take place through an infinite number of field derivatives at this point, encoded in other objects in \mathcal{S} .

The positive (negative) step operators in the A_{D-1} subalgebra are, from Eq.(4.16), the K^a_b with $b > a$ ($b < a$). They define the level zero step operators of the \mathcal{G}^{+++} adjoint representation. The positive (negative) levels of the adjoint representation of \mathcal{G}^{+++} are defined as follows. One takes a set of q non-negative (non-positive) integers, excluding q zeros, where q is the number of simple roots of \mathcal{G}^{+++} not contained in the gravity line. The q integers count the number of times each such root appears in the decomposition of the adjoint representation of \mathcal{G}^{+++} into irreducible representations of A_{D-1} . Positive (negative) levels contain only positive (negative) roots and the number of irreducible representations of A_{D-1} at each level is finite. All step operators of level greater than zero may be written as irreducible tensors $R_{d_1 \dots d_s}^{c_1 \dots c_r}$ of the A_{D-1} subalgebra of \mathcal{G}^{+++} , namely

$$[K^a_b, R_{d_1 \dots d_s}^{c_1 \dots c_r}] = \delta_b^{c_1} R_{d_1 \dots d_s}^{a \dots c_r} + \dots + \delta_b^{c_r} R_{d_1 \dots d_s}^{c_1 \dots a} - \delta_{d_1}^a R_{b \dots d_s}^{c_1 \dots c_r} - \dots - \delta_{d_s}^a R_{d_1 \dots b}^{c_1 \dots c_r}. \quad (5.1)$$

The commutators of all positive step operators are generated by the commutators of step operators corresponding to simple roots. At level zero these ‘simple step operators’ are, from Eq.(4.16), the K^a_{a+1} ($a = 1, 2, \dots, D-1$). In general, when s dilatons are present in the action \tilde{S} , the rank of \mathcal{G}^{+++} is $D + s$. The s abelian generators R_u of its subgroup $GL(D) \times U(1)^s$ have non vanishing commutators with the tensor step operators $R^{a_1 a_2 \dots a_r}$ associated to electric or magnetic simple roots. In dimensional reduction, these arise from n -form potentials where $n = p_I - 1$ for an electric root and $n = D - p_I - 1$ for a magnetic one. We can read off their commutation relation, in the normalisation given for the dilaton in Eq.(4.18), namely [8]

$$[R_u, R^{a_1 a_2 \dots a_r}] = -\varepsilon \frac{a_1^u}{2} R^{a_1 a_2 \dots a_r}, \quad (5.2)$$

where $\varepsilon = +1$ for an electric root and -1 for a magnetic one.

To switch from positive K^a_b ($b > a$) step operators to negative ones it suffices to interchange upper and lower indices. At higher levels, the negative of the tensor $R_{d_1 \dots d_s}^{c_1 \dots c_r}$ is similarly a tensor $\bar{R}_{c_1 \dots c_r}^{d_1 \dots d_s}$. The invariant scalar product for step operators is given by

$$\langle K^b_a, K^d_c \rangle = \delta_c^b \delta_a^d \quad a > b, d > c \quad ; \quad \langle R_{b_1 \dots b_s}^{a_1 \dots a_r}, \bar{R}_{d_1 \dots d_r}^{c_1 \dots c_s} \rangle = \delta_{b_1}^{c_1} \dots \delta_{b_s}^{c_s} \delta_{d_1}^{a_1} \dots \delta_{d_r}^{a_r}. \quad (5.3)$$

Iterative procedures to compute the step operators at any level can be devised. They build, together with the Cartan generators K^a_a ($a = 1, 2, \dots, D$) and R_u ($u = 1, 2, \dots, s$), the full content of the adjoint representation of \mathcal{G}^{+++} .

The metric $g_{\mu\nu}$ at a fixed space time point parametrises the coset $GL(D)/SO(D-1, 1)$. To construct a \mathcal{G}^{+++} invariant action \mathcal{S} containing such a tensor, we shall build a non linear realisation of \mathcal{G}^{+++} in a coset space $\mathcal{G}^{+++}/K^{+++}$ where the subgroup K^{+++} contains the Lorentz group $SO(D-1, 1)$. We use a recursive construction based on the level decomposition of \mathcal{G}^{+++} . As at each level the $SO(D-1, 1)$ invariance must be realised for a finite number of generators, we cannot use the Chevalley involution to build the coset $\mathcal{G}^{+++}/K^{+++}$. Rather we shall use a ‘temporal’ involution from which the required non-compact generators of K^{+++} can be selected.

The ‘temporal’ involution ω is defined in the following way. For the generators of the Cartan subalgebra we take, as in the Cartan involution,

$$K^a_a \xrightarrow{\omega} -K^a_a \quad R_u \xrightarrow{\omega} -R_u, \quad (5.4)$$

while for step operators we take

$$R_{d_1 \dots d_s}^{c_1 \dots c_r} \xrightarrow{\omega} -\eta \bar{R}_{c_1 \dots c_r}^{d_1 \dots d_s}, \quad (5.5)$$

with $\eta = (-1)^{n_t}$ where n_t is the number of time indices in $R_{d_1 \dots d_s}^{c_1 \dots c_r}$. All commutation relations in \mathcal{G}^{+++} are preserved under the mapping Eqs.(5.4) and (5.5), and so is its bilinear form. This mapping constitutes an involution that we label the temporal involution. We define the subgroup K^{+++} of \mathcal{G}^{+++} as the subgroup invariant under this involution. Its generators are

$$R_{d_1 \dots d_s}^{c_1 \dots c_r} - \eta \bar{R}_{c_1 \dots c_r}^{d_1 \dots d_s}. \quad (5.6)$$

K^{+++} contains the Lorentz group $SO(D-1, 1)$ and all generators with $\eta = -1$ are non-compact.

5.2 Non-linear realisation of \mathcal{G}^{+++} in $\mathcal{G}^{+++}/K^{+++}$

We will follow a similar line of thought as the one developed in reference [14] in the context of E_8^{+++} . Consider a group element \mathcal{V} built out of Cartan and positive step operators in \mathcal{G}^{+++} . It takes the form

$$\mathcal{V} = \exp\left(\sum_{a \geq b} h_b^a K_a^b - \sum_{u=1}^s \phi^u R_u\right) \exp\left(\sum_{r!s!} \frac{1}{r!s!} A_{b_1 \dots b_s}^{a_1 \dots a_r} R_{a_1 \dots a_r}^{b_1 \dots b_s} + \dots\right). \quad (5.7)$$

We have written it so that the first exponential contains only level zero operators (i.e. the Cartan and the level zero positive step operators) and the second one contains the positive step operators of level strictly greater than zero. The tensors $h_b^a, \phi^u, A_{b_1 \dots b_s}^{a_1 \dots a_r}$, bear a priori no relation with the metric, the dilaton and the potentials of the p_I form field strengths F_{p_I} entering the action Eq.(4.18). However we shall see that a dictionary can be established relating the tensors which appear at low levels with the fields occurring in Eq.(4.18) at a *fixed* space-time point. For higher levels the dictionary between group parameters and space-time fields should arise from the analysis of the dynamics encoded in the \mathcal{G}^{+++} invariant action \mathcal{S} below.

A differential motion in the coset $\mathcal{G}^{+++}/K^{+++}$ can be constructed from Eq.(5.7). Define

$$dv = d\mathcal{V}\mathcal{V}^{-1} \quad d\tilde{v} = -\omega dv \quad ; \quad dv_{sym} = \frac{1}{2}(dv + d\tilde{v}). \quad (5.8)$$

As dv and $d\tilde{v}$ are differentials in the Lie algebra, dv_{sym} contains only the Cartan generators and the combinations of step operators $R_{d_1 \dots d_s}^{c_1 \dots c_r} + \eta \bar{R}_{c_1 \dots c_r}^{d_1 \dots d_s}$.

To construct the action \mathcal{S} we wish to map a manifold \mathcal{M} into \mathcal{G}^{+++} . We do not want to take for \mathcal{M} a space-time manifold, as this might require the explicit introduction of local symmetries which we hope to be hidden in the infinite algebra of \mathcal{G}^{+++} . We shall take for \mathcal{M} a one-dimensional world-line in ξ -space, i.e. $dv_{sym} = dv_{sym}(\xi) = d\mathcal{V}(\xi)\mathcal{V}^{-1}(\xi)$, where $\mathcal{V}(\xi)$ are the group parameters appearing in Eq.(5.7) that are now fields dependent on the variable ξ . Here no connection is imposed a priori between ξ -space and space-time.

A reparametrisation invariant action is then

$$\mathcal{S} = \int d\xi \frac{1}{n(\xi)} \left\langle \left(\frac{dv_{sym}(\xi)}{d\xi} \right)^2 \right\rangle, \quad (5.9)$$

where $n(\xi)$ is an arbitrary lapse function ensuring reparametrisation invariance on the world-line. The ‘trace’ $\langle \rangle$ is given by Eqs.(4.19) and (5.3). It ensures the invariance of the non-linear action \mathcal{S} defined on the coset space $\mathcal{G}^{+++}/K^{+++}$ under global \mathcal{G}^{+++} transformations.

We now compute the level zero of the action Eq.(5.9), that is the terms generated by K_a^b ($a \geq b$) and R_u in \mathcal{V} . From Eqs.(5.7), (5.8) and (4.16), one obtains the contribution of the level zero to $v_{sym}(\xi)$,

$$\frac{dv_{sym}^0(\xi)}{d\xi} = -\frac{1}{2} \sum_{a \geq b} [e^h(\frac{de^{-h}}{d\xi})]_b^a (K_a^b + \eta K_b^a) - \sum_{u=1}^s \frac{d\phi^u}{d\xi} R_u, \quad (5.10)$$

where h is triangular matrix with elements h_b^a . We now evaluate $\langle (dv_{sym}^0(\xi)/d\xi)^2 \rangle$ using Eqs.(4.19) and (5.3). The η -symbols defining the temporal involution allow the raising or lowering of the a, b indices of the ξ -fields multiplying the negative step operator in $\langle dv_{sym}^0/d\xi, dv_{sym}^0/d\xi \rangle$ with the Minkowskian metric η_{ab} . This ensures that this expression is a Lorentz scalar. The Lorentz invariant action at level zero, $\mathcal{S}^{(0)}$, is

$$\begin{aligned} \mathcal{S}^{(0)} &= \int d\xi \frac{1}{n(\xi)} \langle (\frac{dv_{sym}^0(\xi)}{d\xi})^2 \rangle, \\ &= \frac{1}{2} \int d\xi \frac{1}{n(\xi)} \left\{ [e^h(\frac{de^{-h}}{d\xi})]_b^a [e^h(\frac{de^{-h}}{d\xi})]_a^T b + [e^h(\frac{de^{-h}}{d\xi})]_b^a [e^h(\frac{de^{-h}}{d\xi})]_a^b \right. \\ &\quad \left. - ([e^h(\frac{de^{-h}}{d\xi})]_a^a)^2 + \sum_{u=1}^s (\frac{d\phi^u}{d\xi})^2 \right\}, \end{aligned} \quad (5.11)$$

where the summation is performed over Lorentz indices. Note that the lower indices of e^{-h} and the upper indices of e^h cannot be lowered or raised by the Lorentz metric. To avoid confusion we label these indices with greek letters, namely we define ‘vielbein’

$$e_\mu^a = (e^{-h})_\mu^a \quad e_b^\mu = (e^h)_b^\mu \quad ; \quad g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}. \quad (5.12)$$

Although we have not yet introduced a space-time, we shall name the a indices flat and the μ indices curved. As a result of the temporal involution and of the scalar product $\langle \rangle$ in \mathcal{G}^{+++} , the flat-index tensors have been endowed with a Lorentz metric while curved-index tensors define a metric in $GL(D)/SO(D-1, 1)$. Hence, for any ξ , we are allowed to identify $g_{\mu\nu}(\xi)$ in Eq.(5.12) as the metric tensor in \tilde{S} , Eq.(4.18), at a fixed space-time point.

Using Eq.(5.12), one can rewrite the action Eq.(5.11) as

$$\mathcal{S}^{(0)} = \frac{1}{2} \int d\xi \frac{1}{n(\xi)} \left[\frac{1}{2} (g^{\mu\nu} g^{\sigma\tau} - \frac{1}{2} g^{\mu\sigma} g^{\nu\tau}) \frac{dg_{\mu\sigma}}{d\xi} \frac{dg_{\nu\tau}}{d\xi} + \sum_{u=1}^s \frac{d\phi^u}{d\xi} \frac{d\phi^u}{d\xi} \right]. \quad (5.13)$$

At higher levels, the tensors multiplying the step operators couple nonlinearly to the level zero objects and between themselves. The coupling to the metric and to ϕ^u can be formally written down for all levels, but the self-coupling of the $A_{b_1 \dots b_s}^{a_1 \dots a_r}$ depend specifically on the group \mathcal{G} .

Consider a general A_{D-1} tensor $A_{a_1 \dots a_r}^{b_1 \dots b_s}$ parametrising a normalised step operator $R_{b_1 \dots b_s}^{a_1 \dots a_r}$. The commutation relations of $R_{b_1 \dots b_s}^{a_1 \dots a_r}$ with the K_a^b are given by the tensor transformations as in Eq.(4.16) and (5.1) and those with R_u have the form

$$[R_u, R_{b_1 \dots b_s}^{a_1 \dots a_r}] = \lambda_u R_{b_1 \dots b_s}^{a_1 \dots a_r}. \quad (5.14)$$

Here $\lambda_u = \sum \lambda_{u,i}$ where the $\lambda_{u,i}$ are the scale parameters of the simple step operators entering the multiple commutators defining $R_{b_1 \dots b_s}^{a_1 \dots a_r}$. This property follows from the Jacobi identity. Identifying for simple step operators $\lambda_{u,i}$ with $-\varepsilon a_i^u/2$ in Eq.(5.2) we may identify for any ξ the $\phi^u(\xi)$ in \mathcal{S} , Eq.(5.9), with the dilatons fields in \tilde{S} , Eq.(4.18), at a fixed space time point. The particular $A_{a_1 \dots a_r}^{b_1 \dots b_s}(\xi)$ multiplying the step operators belonging to the subgroup \mathcal{G} can be similarly identified to the corresponding potential forms in \tilde{S} along with their duals.

It is straightforward to compute the contribution $dv^{(A)}$ to dv of a given tensor when commutators of the $R_{b_1 \dots b_s}^{a_1 \dots a_r}$ between themselves are disregarded. One gets

$$dv^{(A)} = \frac{1}{r!s!} dA_{\mu_1 \dots \mu_r}^{\nu_1 \dots \nu_s} \exp\left(-\sum_{u=1}^s \lambda^u \phi^u\right) e_{a_1}^{\mu_1} \dots e_{a_r}^{\mu_r} e_{\nu_1}^{b_1} \dots e_{\nu_s}^{b_s} R_{b_1 \dots b_s}^{a_1 \dots a_r}. \quad (5.15)$$

The contribution $\mathcal{S}_0^{(A)}$ of $v^{(A)}$ to the action \mathcal{S} is computed as previously and one gets

$$\mathcal{S}_0^{(A)} = \frac{1}{2} \int d\xi \frac{1}{n(\xi)} \left[\frac{1}{r!s!} \exp\left(-\sum_{u=1}^s 2\lambda^u \phi^u\right) \frac{dA_{\mu_1 \dots \mu_r}^{\nu_1 \dots \nu_s}}{d\xi} g^{\mu_1 \mu'_1} \dots g^{\mu_r \mu'_r} g_{\nu_1 \nu'_1} \dots g_{\nu_s \nu'_s} \frac{dA_{\mu'_1 \dots \mu'_r}^{\nu'_1 \dots \nu'_s}}{d\xi} \right]. \quad (5.16)$$

The full action can only be approached in a recursive way. In $\mathcal{S}_0^{(A)}$, one must replace derivatives by non linear generalisations to take into account the non vanishing commutators between tensor step operators. We represent such terms by ‘covariant’ derivatives symbol $D/D\xi$ [14, 12]. Their evaluation is group dependent. Formally the full action \mathcal{S} is

$$\mathcal{S} = \mathcal{S}^{(0)} + \sum_A \mathcal{S}^{(A)}, \quad (5.17)$$

$$\mathcal{S}^{(A)} = \frac{1}{2} \int d\xi \frac{1}{n(\xi)} \left[\frac{1}{r!s!} \exp\left(-\sum_{u=1}^s 2\lambda^u \phi^u\right) \frac{DA_{\mu_1 \dots \mu_r}^{\nu_1 \dots \nu_s}}{d\xi} g^{\mu_1 \mu'_1} \dots g^{\mu_r \mu'_r} g_{\nu_1 \nu'_1} \dots g_{\nu_s \nu'_s} \frac{DA_{\mu'_1 \dots \mu'_r}^{\nu'_1 \dots \nu'_s}}{d\xi} \right]$$

where the sum on A is a summation over all tensors appearing at all positive levels in the decomposition of \mathcal{G}^{+++} into irreducible representations of A_{D-1} .

One may expand \mathcal{S} given in Eq.(5.17) in power of fields parametrising the positive step operators. Up to quadratic terms, the result $\mathcal{S}^{(Q)}$ is obtained by retaining in $v(\xi)$ terms independent or linear in these fields. Define the one-forms $[dA]$ and the moduli $p^{(a)}$ (or $p^{(\mu)}$) as in Section 4.3 by

$$\exp\left(-\sum_{u=1}^s \lambda^u \phi^u\right) \hat{e}_{a_1}^{\mu_1} \dots \hat{e}_{a_r}^{\mu_r} \hat{e}_{\nu_1}^{b_1} \dots \hat{e}_{\nu_s}^{b_s} \frac{dA_{\mu_1 \dots \mu_r}^{\nu_1 \dots \nu_s}}{d\xi} \stackrel{def}{=} \frac{[dA]_{a_1 \dots a_r}^{b_1 \dots b_s}}{d\xi}, \quad (5.18)$$

$$\hat{e}_b^\mu \frac{d\hat{e}_\mu^a}{d\xi} \stackrel{def}{=} \frac{dp^{(a)}}{d\xi}, \quad (5.19)$$

where \hat{e} means that only the diagonal vielbein are kept. We get

$$\begin{aligned} \mathcal{S}^{(Q)} = \int d\xi \frac{1}{n(\xi)} & \left[\sum_{a=1}^D \left(\frac{dp^{(a)}}{d\xi} \right)^2 - \frac{1}{2} \left(\sum_{a=1}^D \frac{dp^{(a)}}{d\xi} \right)^2 + \frac{1}{2} \sum_{u=1}^s \left(\frac{d\phi^u}{d\xi} \right)^2 \right. \\ & \left. + \frac{1}{2} (e_b{}^\mu \frac{de_\mu{}^a}{d\xi} \frac{de_{av}}{d\xi} e^{\nu b})^{(1)} + \frac{1}{2} \frac{1}{r!s!} \sum_A \frac{[dA]_{a_1 \dots a_r}{}^{b_1 \dots b_s}}{d\xi} \frac{[dA]_{b_1 \dots b_s}{}^{a_1 \dots a_r}}{d\xi} \right], \end{aligned} \quad (5.20)$$

where the superscript (1) in the vielbein term indicates that only terms quadratic in $h_b{}^a$ ($a > b$) are kept. In the next section we shall produce solutions of the action Eq.(5.20) which are *exact* solutions of the full action Eq.(5.17).

6. Extremal branes from the \mathcal{G}^{+++} invariant actions

We shall look for solutions of the equations of motion derived from \mathcal{S} and containing only one $A(\xi)$ -field, or one non-diagonal $h(\xi)$ -field, with *given indices* [12]. Such truncation is consistent with all the equations of motion and hence we may disregard all non-linearity in the step operators. Therefore it suffices for obtaining such solutions to replace the action \mathcal{S} by its simplified version Eq.(5.20).

We shall as in Eqs.(2.5) and (2.6), consider A to be an antisymmetric tensor with a time index τ and r space indices coupled to a step operator of the \mathcal{G} subalgebra. The equation of motions are

a) *The lapse constraint.*

Eq.(5.20), taking Eqs.(5.18) and (5.19) into account, reads

$$\sum_{\alpha=1}^D \left(\frac{dp^{(\alpha)}}{d\xi} \right)^2 - \frac{1}{2} \left(\sum_{\alpha=1}^D \frac{dp^{(\alpha)}}{d\xi} \right)^2 + \frac{1}{2} \left(\frac{d\phi}{d\xi} \right)^2 - \frac{1}{2} \exp[\varepsilon a \phi - 2p^{(\tau)} - 2 \sum_{\lambda=\lambda_1}^{\lambda_r} p^{(\lambda)}] \left(\frac{dA_{\tau\lambda_1 \dots \lambda_r}}{d\xi} \right)^2 = 0 \quad (6.1)$$

Here we have taken one dilaton with scaling $\lambda = -\varepsilon a/2$ in accordance with Eq.(5.2). Note that this relation is valid whether or not the magnetic root is simple, as seen in dimensional reduction. A crucial feature of this equation is the minus sign in front of the exponential. Its origin can be traced back to the temporal involution defining our coset space, hence to Lorentz invariance, because *both* magnetic and electric potentials have a time index.

b) *The equation of motion for A.*

We take the lapse $n(\xi) = 1$. One gets

$$\frac{d}{d\xi} \left(\exp[\varepsilon a \phi - 2p^{(\tau)} - 2 \sum_{\lambda=\lambda_1}^{\lambda_r} p^{(\lambda)}] \frac{dA_{\tau\lambda_1 \dots \lambda_r}}{d\xi} \right) = 0. \quad (6.2)$$

c) *The dilaton equation of motion.*

$$-\frac{d^2 \phi}{d\xi^2} - \frac{1}{2} \varepsilon a \exp[\varepsilon a \phi - 2p^{(\tau)} - 2 \sum_{\lambda=\lambda_1}^{\lambda_r} p^{(\lambda)}] \left(\frac{dA_{\tau\lambda_1 \dots \lambda_r}}{d\xi} \right)^2 = 0. \quad (6.3)$$

d) The vielbein equations of motion.

$$-2 \frac{d^2 p^{(\alpha)}}{d\xi^2} + \sum_{\beta=1}^D \frac{d^2 p^{(\beta)}}{d\xi^2} = 0 \quad \alpha \neq \tau, \lambda_i \quad (i = 1, 2, \dots, r), \quad (6.4)$$

$$-2 \frac{d^2 p^{(\alpha)}}{d\xi^2} + \sum_{\beta=1}^D \frac{d^2 p^{(\beta)}}{d\xi^2} + \exp[\varepsilon a \phi - 2p^{(\tau)} - 2 \sum_{\lambda=\lambda_1}^{\lambda_r} p^{(\lambda)}] \left(\frac{dA_{\tau\lambda_1 \dots \lambda_r}}{d\xi} \right)^2 = 0 \quad \alpha = \tau, \lambda_i. \quad (6.5)$$

We take as ansätze the solutions of the extremal brane problem but with H_A an unknown function of $H(\xi)$. Namely we pose

$$A_{\tau\lambda_1 \dots \lambda_r} = \epsilon_{\tau\lambda_1 \dots \lambda_r} \left[\frac{2(D-2)}{\Delta} \right]^{1/2} H^{-1}(\xi), \quad (6.6)$$

$$p^{(\tau)} = p^{(\lambda_i)} = -\frac{D-r-3}{\Delta} \ln H(\xi) \quad ; \quad p^{(\alpha)} = \frac{r+1}{\Delta} \ln H(\xi) \quad \alpha \neq \tau, \lambda_i. \quad (6.7)$$

$$\phi = \frac{D-2}{\Delta} \varepsilon a \ln H(\xi). \quad (6.8)$$

From these equations and from Eq.(2.9) we see that $\varepsilon a \phi - 2p^{(\tau)} - 2 \sum_{\lambda=\lambda_1}^{\lambda_r} p^{(\lambda)} = 2 \ln H(\xi)$. It then follows that the equation of motion for A , Eq.(6.2), reduces to, using Eq.(6.6),

$$\frac{d^2 H(\xi)}{d\xi^2} = 0. \quad (6.9)$$

Given this result, it is straightforward to verify that the ansätze Eqs.(6.6), (6.7) and (6.8) satisfy the dilaton and the vielbein equations of motions. The lapse constraint takes the form

$$\sum_{\alpha=1}^D (dp^{(\alpha)})^2 - \frac{1}{2} \left(\sum_{\alpha=1}^D dp^{(\alpha)} \right)^2 + \frac{1}{2} (d\phi)^2 - \frac{D-2}{\Delta} (d \ln H)^2 = 0. \quad (6.10)$$

where the differentials are taken in ξ -space. It has therefore exactly the same form in ξ -space as Eq.(4.15) has in space-time. The relations Eqs.(6.6), (6.7) and (6.8), together with Eqs.(6.9) and the lapse constraint Eq.(6.10) fully describe an exact solution of the full \mathcal{G}^{+++} invariant action \mathcal{S} defined recursively by Eq.(5.17). We now discuss the significance of this result.

The Eqs.(6.6), (6.7) and (6.8) characterise completely the algebraic structure of the extremal brane solution but do not yield its harmonic character in space-time. As the functions $A_{\tau\lambda_1 \dots \lambda_r}(\xi)$, $p^{(\tau)}(\xi)$, $p^{(\lambda_i)}(\xi)$ and $\phi(\xi)$ were interpreted in the action \mathcal{S} as functions at a fixed space-time point of the independent variable ξ , this is a consistent result. The solution $H = a + b\xi$ of Eq.(6.9) would then describe a motion in the space of solutions, for instance of branes with different charges. However the fact that we have exact solutions of the action \mathcal{S} with the correct algebraic structure of the extremal branes, means that these solutions are only indirectly related to the corresponding space-time solution. One expects that the information contained in this solution, which is of course not contained in a trivial constant space-time solution of the Einstein equation, is the required information to build

coupled equations to higher space-time derivatives encoded in higher level representations, which would then be directly related to space-time solutions.

There are indications that this may indeed be the case but it is unclear whether it is sufficient to consider the adjoint representation of \mathcal{G}^{+++} [12] or if one has to include other representations [26, 27].

The solution Eqs.(6.6), (6.7) and (6.8) satisfy, in ξ -space, the relation Eq.(2.4). This relation define an embedding of a subgroup \mathcal{G}^{p+1} of \mathcal{G}^{+++} acting on the p compact space dimensions in which the branes live and on the time dimension [5]. We shall consider the subgroup \mathcal{G}^p of \mathcal{G}^{p+1} which acts on the space dimensions only and we take $p \leq D - 4$ so that \mathcal{G}^{p+1} is a Lie group. This group is conjugate by a Weyl reflection in \mathcal{G}^{+++} of the group \mathcal{G}'^{p+1} obtained by deleting the first $D - p - 1$ nodes of the gravity line [5] and hence \mathcal{G}^p is conjugate to its subgroup \mathcal{G}'^p characterising the usual dimensional reduction of Eq.(4.18) to $D - p$ dimensions.

We shall consider the transformations mapping *one* root to another root, thereby generating solutions of the same ‘family’ as the extremal solution just described. These transformations include the Weyl group $W(\mathcal{G}^{+++})$ of \mathcal{G}^{+++} . We shall examine Weyl transforms of the extremal brane solution characterised by one positive step operator which send the positive root into a positive root. Such transformations leave invariant not only \mathcal{S} but also preserves their quadratic truncation Eq.(5.20). Hence Eq.(6.10) is invariant under the Weyl group of \mathcal{G}^{p+1} . The restriction to the Weyl group of \mathcal{G}^p selects transformed fields with one time index.

Thus $W(\mathcal{G}^p)$ leaves invariant the quadratic form

$$\sum_{a=1}^D (dp^{(a)})^2 + \frac{1}{2} \left[- \left(\sum_{i=1}^D dp^{(a)} \right)^2 + \sum_{u=1}^s (d\phi^u)^2 + (e_b{}^\mu de_\mu{}^a de_{av} e^{\nu b})^{(1)} + \frac{1}{r!s!} \sum_A [dA]_{a_1 \dots a_r}^{b_1 \dots b_s} [dA]_{b_1 \dots b_s}^{a_1 \dots a_r} \right], \quad (6.11)$$

and the embedding relation Eq.(2.4) in ξ -space. It acts on A -fields, or non-diagonal vielbein, containing one time index. The sum of the first three terms is the invariant metric of \mathcal{G}^{+++} restricted to its Cartan subgroup. Together with the embedding relation Eq.(2.4) they are left invariant under the Weyl group of \mathcal{G}^{p+1} . We now see the group theoretical relevance of the right hand side of Eq.(4.15) or better the last term of Eq.(6.10) in ξ -space. The Weyl transformations of the step operators and the additional terms in Eq.(6.11) guarantee indeed the invariance of Eq.(6.10) under $W(\mathcal{G}^p)$. For all \mathcal{G}^{+++} , the Weyl transformations generate new solutions from one extremal brane solution. We stress again that in the present approach both electric and magnetic branes are described ‘electrically’.

For M-theory, this yield the well-known duality symmetries of M-theory, including the duality transformations of branes, KK waves and KK monopoles (Taub-NUT spaces). Such transformations are however not a privilege of M-theory and occur in *all* \mathcal{G}^{+++} invariant actions. This is exemplified below, taking for definiteness the action \mathcal{S} for the group E_7^{+++} which is related to the action $\tilde{\mathcal{S}}$ with gravity coupled to a 4- and a 2- form field strength in 9 space-time dimensions. The Dynkin diagram of E_7^{+++} is depicted in Fig.3, which exhibits the two simple electric roots (10) and (9) corresponding respectively to the step operators R^{789} and R^9 which couple to the electric potentials A_{789} and A_9 .

We take as input the electric extremal 2-brane $\mathbf{e}_{(8,9)}$ in the directions (8,9) associated with the 4-form field strength whose corresponding potential is A_{189} and submit it to the non trivial Weyl reflection W_{10} associated with the electric root (10) of Fig.3. We display below, both for $\mathbf{e}_{(8,9)}$ and its transform, the moduli, i.e. the vielbein components $p^{(a)}$ and the the dilaton value ϕ , of the brane solution Eqs.(6.7) and (6.8) as a ten-dimensional vector where the last component is the dilaton. We also indicate the transform of the step operator R^{189} under the Weyl transformation. We obtain

$$\begin{aligned} (-4, 3, 3, 3, 3, 3, 3, -4, -4; 2\sqrt{7}) \frac{\ln H(\xi)}{14} \quad \mathbf{e}_{(8,9)} \quad R^{189} \quad (6.12) \\ \downarrow W_{10} \end{aligned}$$

$$(-7, 0, 0, 0, 0, 0, 7, 0, 0; 0) \frac{\ln H(\xi)}{14} \quad \mathbf{kk}_{\mathbf{e}(7)} \quad K^1_7 \quad (6.13)$$

The transformation of the 2-brane is reminiscent of a double T-duality in M-theory.

We now move the electric brane through Weyl reflections associated with roots of the gravity line to $\mathbf{e}_{(5,9)}$ and submit it to the Weyl reflection W_{10} . We now find that the brane $\mathbf{e}_{(5,9)}$ is invariant but moving it to the position $\mathbf{e}_{(5,6)}$, we get

$$\begin{aligned} (-4, 3, 3, 3, -4, -4, 3, 3, 3; 2\sqrt{7}) \frac{\ln H(\xi)}{14} \quad \mathbf{e}_{(5,6)} \quad R^{156} \quad (6.14) \\ \downarrow W_{10} \end{aligned}$$

$$(-1, 6, 6, 6, -1, -1, -1, -1, -1; 4\sqrt{7}) \frac{\ln H(\xi)}{14} \quad \mathbf{m}_{(5,6,7,8,9)} \quad R^{156789} \quad (6.15)$$

This is a magnetic 5-brane in the directions (5,6,7,8,9) associated to the 2-form field strength ! It is expressed in terms of its dual potential A_{156789} . Submit instead $\mathbf{e}_{(5,9)}$ to to the Weyl reflection W_9 associated with the electric root (9) of Fig.3. The 2-brane $\mathbf{e}_{(5,9)}$ is again invariant, but moving it to to the position $\mathbf{e}_{(5,6)}$, we now get

$$\begin{aligned} (-4, 3, 3, 3, -4, -4, 3, 3, 3; 2\sqrt{7}) \frac{\ln H(\xi)}{14} \quad \mathbf{e}_{(5,6)} \quad R^{156} \\ \downarrow W_9 \\ (-3, 4, 4, 4, -3, -3, 4, 4, -3; -2\sqrt{7}) \frac{\ln H(\xi)}{14} \quad \mathbf{m}_{(5,6,9)} \quad R^{1569} \quad (6.16) \end{aligned}$$

This is a magnetic 3-brane in the directions (5,6,9) associated to the 4-form field strength, expressed in terms of its dual potential A_{1569} .

Finally, let us submit the magnetic 5-brane $\mathbf{m}_{(5,6,7,8,9)}$ obtained in Eq.(6.15) to the Weyl reflection W_9 . One obtains

$$\begin{aligned} (-1, 6, 6, 6, -1, -1, -1, -1, -1; 4\sqrt{7}) \frac{\ln H(\xi)}{14} \quad \mathbf{m}_{(5,6,7,8,9)} \quad R^{156789} \\ \downarrow W_9 \\ (0, 7, 7, 7, 0, 0, 0, 0, -7; 0) \frac{\ln H(\xi)}{14} \quad \mathbf{kk}_{\mathbf{m}(2,3,4;9)} \quad R^{156789,9} \quad (6.17) \end{aligned}$$

Eq.(6.17) describes, as in M-theory, a purely gravitational configuration, namely a KK-monopole with transverse directions (2,3,4) and Taub-NUT direction (9) in terms of a dual gravity tensor $h_{156789,9}$.

It is also possible to generate solutions not contained, at least explicitly, in the group \mathcal{G}^p . These are very interesting solutions as they may test the significance of genuine Kac-Moody extensions of the Lie groups. Such analysis is outside the scope of the present work where we test only solutions which can straightforwardly be mapped to space time solutions of the effective actions Eq.(4.18).

The above example illustrate the analogy of M-theory duality transformations with similar transformations in all ‘M-theories’ defined by all \mathcal{G}^{+++} . One may indeed carry the same analysis for all \mathcal{G}^{+++} and exhibit for each of them the ‘duality’ transformations of the branes. As in M-theory, these dualities are symmetries in non-compact space-time. This is because \mathcal{G}^{p+1} is, as \mathcal{G}'^{p+1} , the Lie group symmetry of the action Eq.(4.18) dimensionally reduced to three dimensions (for $p = D - 4$). They differ because while the latter reduction leaves a Lorentzian non-compact space-time, the former leads to a Euclidean space-time by compactifying time. The group \mathcal{G}^p of transformations on ξ -space discussed above, is thus in one to one correspondence with the group \mathcal{G}^p of space-time transformations when time is decompactified. In particular, the functions $H(\xi)$ can thus be mapped into harmonic functions $H(\{x^\nu\})$. However as pointed out in the previous section, more work is needed to relate directly $H(\xi)$ to $H(\{x^\nu\})$, and solutions in ξ -space to solutions in space-time for all \mathcal{G}^{+++} , through translation operators hopefully induced by group generators.

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