PROCEEDINGS OF SCIENCE

A dispersive treatment of $K_{\ell 4}$ decays

Peter Stoffer*
Helmholtz-Institut für Strahlen- und Kernphysik (Theory) and
Bethe Center for Theoretical Physics, University of Bonn, 53115 Bonn, Germany
E-mail: stoffer@hiskp.uni-bonn.de

Gilberto Colangelo
Albert Einstein Center for Fundamental Physics, Institute for Theoretical Physics,
University of Bern, Sidlerstrasse 5, 3012 Bern, Switzerland
E-mail: gilberto@itp.unibe.ch

Emilie Passemar
Department of Physics, Indiana University, Bloomington, IN 47405, USA
Center for Exploration of Energy and Matter, Indiana University, Bloomington, IN 47403, USA
Theory Center, Thomas Jefferson National Accelerator Facility, Newport News, VA 23606, USA
E-mail: epassema@indiana.edu

PoS(CD15)067

$K_{\ell 4}$ decays have several features of interest: they allow an accurate measurement of $\pi\pi$-scattering lengths; the decay is the best source for the determination of some low-energy constants of chiral perturbation theory (χPT); one form factor of the decay is connected to the chiral anomaly.
We present the results of our dispersive analysis of $K_{\ell 4}$ decays, which provides a resummation of $\pi\pi$- and $K\pi$-rescattering effects. The free parameters of the dispersion relation are fitted to the data of the high-statistics experiments E865 and NA48/2. The data input is corrected for additional isospin-breaking effects, which were not taken into account in the experimental analyses. By matching to χPT at NLO and NNLO, we determine the low-energy constants L_1^r, L_2^r and L_3^r. In contrast to a pure chiral treatment, the dispersion relation describes the observed curvature of one of the $K_{\ell 4}$ form factors, which we understand as an effect of rescattering beyond NNLO.

The 8th International Workshop on Chiral Dynamics, CD2015
29 June 2015 - 03 July 2015
Pisa, Italy

*Speaker.

1. Motivation

$K_{\ell 4}$, the semileptonic decay of a kaon into two pions and a lepton-neutrino pair, plays a crucial role in the context of low-energy hadron physics, because it provides almost unique information about some of the $\mathscr{O}(p^4)$ low-energy constants (LECs) of chiral perturbation theory, the low-energy effective theory of QCD [1 – 3]. The physical region of $K_{\ell 4}$ starts already at the $\pi\pi$ threshold, thus it happens at lower energies than e.g. elastic $K\pi$ scattering. Although the $K_{\ell 4}$ decay offers similar information as $K\pi$ scattering, it happens in a kinematical region where the chiral expansion is more reliable.

Besides, as the hadronic final state contains two pions, $K_{\ell 4}$ is also one of the best sources of information on $\pi\pi$ interaction [4 – 6].

On the experimental side, we are confronted with impressive precision from the high-statistics measurements of the E865 experiment at BNL [7, 8] and the NA48/2 experiment at CERN [6, 9]. The statistical errors of the S-wave of one form factor reach in both experiments the sub-percent level. Matching this precision requires a theoretical treatment beyond one-loop order in the chiral expansion. A first treatment beyond one loop, based on dispersion relations, was already done twenty years ago [10]. The full two-loop calculation became available in 2000 [11]. However, as we will show below, even at two loops χPT is not able to predict the curvature of one of the form factors.

Here, we present the results of a new dispersive treatment of $K_{\ell 4}$ decays [12, 13]. We do not solve an exact dispersion relation for this process, but an approximate form, which follows if the contribution of D- and higher waves to the discontinuities are neglected. This approximation is violated only at $\mathscr{O}(p^8)$ in the chiral counting. The effects due to $\pi\pi$ and $K\pi$ rescattering in S- and P-wave are resummed to all orders. We expect this to capture the most important contributions beyond $\mathscr{O}(p^6)$. Indeed it turns out that the dispersive description is able to reproduce the curvature of the form factor.

Our final analysis of $K_{\ell 4}$ decays represents an extension and a major improvement of our previous dispersive framework [14 – 16]. Instead of a single linear combination of form factors, now we describe the two form factors F and G simultaneously, including more experimental data in the fits. The new framework is valid also for non-vanishing invariant energies of the lepton pair. We apply corrections for isospin-breaking effects in the fitted data that have not been taken into account in the experimental analyses [17]. Besides a matching to one-loop χPT, we also study the matching at two-loop level.

2. Dispersion relation for $K_{\ell 4}$

2.1 Matrix element and form factors

We consider the charged decay mode

$$K^+(p) \to \pi^+(p_1)\pi^-(p_2)\ell^+(p_\ell)\nu_\ell(p_\nu), \tag{2.1}$$

where $\ell \in \{e, \mu\}$ is either an electron or a muon. Experimental data is available on the electron mode.

PoS(CD15)067

After integrating out the W boson, we end up with a Fermi type current-current interaction and the matrix element splits up into a leptonic times a hadronic part. The leptonic matrix element can be treated in a standard way. The hadronic matrix element exhibits the usual $V-A$ structure of weak interaction. Its Lorentz structure allows us to write the two contributions as

$$\left\langle \pi^+(p_1)\pi^-(p_2)\middle|V_\mu(0)\middle|K^+(p)\right\rangle = -\frac{H}{M_K^3}\varepsilon_{\mu\nu\rho\sigma}L^\nu P^\rho Q^\sigma, \tag{2.2}$$

$$\left\langle \pi^+(p_1)\pi^-(p_2)\middle|A_\mu(0)\middle|K^+(p)\right\rangle = -i\frac{1}{M_K}\left(P_\mu F + Q_\mu G + L_\mu R\right), \tag{2.3}$$

where $P = p_1 + p_2$, $Q = p_1 - p_2$, $L = p - p_1 - p_2$. The form factors F, G, H and R are dimensionless scalar functions of the usual Mandelstam variables s, t and u. In K_{e4} experiments, R is not accessible. H gets a first contribution only at $\mathcal{O}(p^4)$ due to the chiral anomaly. Here we focus on the form factors F and G.

2.2 Reconstruction theorem and integral equations

Let us for the moment regard the di-lepton invariant squared energy $s_\ell = L^2$ as a fixed parameter. Based on fixed-$s/t/u$ dispersion relations, one can derive a decomposition of the form factors into functions of only one Mandelstam variable, known as 'reconstruction theorem' [18, 19]. The derivation neglects the imaginary parts of D- and higher partial waves, an $\mathcal{O}(p^8)$ effect:

$$\begin{aligned} F(s,t,u) &= M_0(s) + \frac{u-t}{M_K^2}M_1(s) + (\text{terms involving functions of } t \text{ or } u) + \mathcal{O}(p^8), \\ G(s,t,u) &= \tilde{M}_1(s) + (\text{terms involving functions of } t \text{ or } u) + \mathcal{O}(p^8), \end{aligned} \tag{2.4}$$

where the functions of one variable M_0, ... are defined to contain only the right-hand cut of the partial waves of the form factors F and G in the three channels. E.g. the function M_0 contains the right-hand cut of the s-channel S-wave f_0 of the form factor F:

$$M_0(s) = P(s) + \frac{s^2}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\mathrm{Im} f_0(s)}{(s'-s-i\varepsilon)s'^2}, \tag{2.5}$$

where $P(s)$ is a subtraction polynomial. Eight more functions M_1, ... take care of the right-hand cuts of S- and P-waves in all channels, such that all the discontinuities are divided up into functions of a single variable. They satisfy inhomogeneous Omnès equations with the solution

$$M_0(s) = \Omega_0^0(s)\left\{\tilde{P}(s) + \frac{s^3}{\pi}\int_{4M_\pi^2}^{\Lambda^2} ds' \frac{\hat{M}_0(s')\sin\delta_0^0(s')}{|\Omega_0^0(s')|(s'-s-i\varepsilon)s'^3}\right\}, \tag{2.6}$$

where $\tilde{P}(s)$ is a new subtraction polynomial and the Omnès function is given by

$$\Omega_0^0(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\delta_0^0(s')}{(s'-s-i\varepsilon)s'}\right\}. \tag{2.7}$$

In total, 9 subtraction constants appear. We need the following elastic $\pi\pi$ and $K\pi$ phase shifts as input, which we assume to reach a multiple of π at the cut-off Λ^2:

- δ_0^0, δ_1^1: elastic $\pi\pi$-scattering phase shifts [20, 21],

- $\delta_0^{1/2}$, $\delta_1^{1/2}$, $\delta_0^{3/2}$, $\delta_1^{3/2}$: elastic $K\pi$-scattering phase shifts [22, 23].

The inhomogeneities in the Omnès problem are given by the differences of the functions M_0, ... and the corresponding partial wave, e.g. $\hat{M}_0(s) = f_0(s) - M_0(s)$. These 'hat functions' contain the left-hand cut of the partial wave and we compute them by projecting out the partial wave of the decomposed form factor (2.4). The inhomogeneities $\hat{M}_0$, ... are then given as angular averages of all the functions M_0, ... Hence, we face a set of coupled integral equations: the functions M_0, ... are defined by dispersion integrals involving the inhomogeneities $\hat{M}_0$, ..., which are again defined as angular integrals of the functions M_0, ... This system can be solved by iteration.

2.3 Numerical solution of the dispersion relation

We note that the integral equations are linear in the subtraction constants. Therefore, for each subtraction constant we construct a basis solution, which we obtain by solving numerically the integral equations in an iterative procedure. The final result is a linear combination of these basis solutions.

We determine the subtraction constants using three sources of information: first, we fit the experimental data on the form factors F and G from the high-statistics experiments NA48/2 [6, 9] and E865 [7, 8]. Secondly, we use as an additional constraint the well-known soft-pion theorem [24–27], which establishes the following relations between F, G and f_+, the $K_{\ell 3}$ vector form factor:

$$\begin{aligned}
&F(M_\pi^2, M_K^2, M_\pi^2 + s_\ell) - G(M_\pi^2, M_K^2, M_\pi^2 + s_\ell) = \mathcal{O}(M_\pi^2), \\
&F(M_\pi^2, M_\pi^2 + s_\ell, M_K^2) + G(M_\pi^2, M_\pi^2 + s_\ell, M_K^2) = \frac{\sqrt{2}M_K}{F_\pi} f_+(M_\pi^2 + s_\ell) + \mathcal{O}(M_\pi^2).
\end{aligned} \tag{2.8}$$

Finally, we fix the subtraction constants that are not well determined by the data with chiral input.

3. Results

3.1 Fits to data

We perform a fit of the dispersion relation to both, the E865 [7, 8] and NA48/2 data sets [6, 9], corrected for additional isospin-breaking effects that were not taken into account in the experimental analyses [17]. Recently, a two-dimensional data set on the S-wave of F has become available (addendum to [9]): in this set, not only a single bin but up to 10 bins are used in s_ℓ-direction. If we allow for varying values of the di-lepton invariant squared energy s_ℓ, the subtraction constants become functions of this parameter and the functions M_0, ... depend on two variables, e.g. $M_0(s, s_\ell)$. We perform our fits in the two-dimensional (s, s_ℓ)-plane using the full available data sets on the S- and P-waves of the form factors, given by

$$\begin{aligned}
F_s(s, s_\ell) &= \left(M_0(s, s_\ell) + \hat{M}_0(s, s_\ell)\right) e^{-i\delta_0^0(s)}, \\
\tilde{F}_p(s, s_\ell) &= \left(M_1(s, s_\ell) + \hat{M}_1(s, s_\ell)\right) e^{-i\delta_1^1(s)}, \\
G_p(s, s_\ell) &= \left(\tilde{M}_1(s, s_\ell) + \hat{\tilde{M}}_1(s, s_\ell)\right) e^{-i\delta_1^1(s)}.
\end{aligned} \tag{3.1}$$

Figure 1 shows the fit results for the S-wave of the form factor F. The two-dimensional phase space is projected on the s-axis and only the data sets with a single bin in s_ℓ-direction are plotted. The dispersive description reproduces beautifully the observed curvature of the form factor F_s. Note that χPT alone is not able to describe this curvature, which can be understood as a higher-order effect of $\pi\pi$ rescattering, fully taken into account in the dispersive Omnès representation of the form factor.

The P-waves of F and G, which are fitted simultaneously with the S-wave, are shown in figure 2.

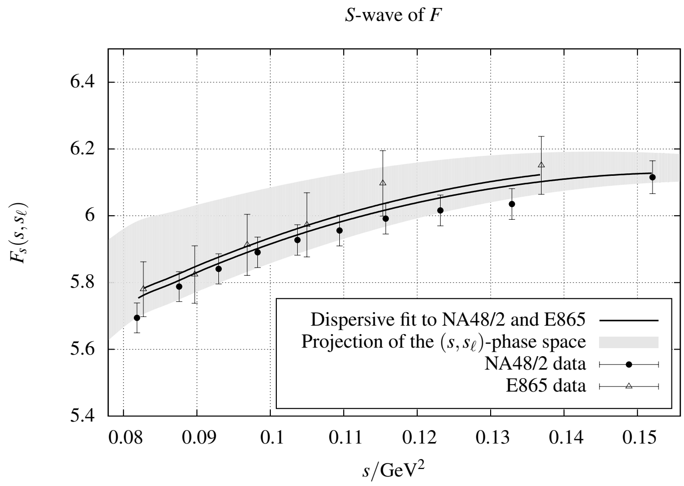

Figure 1: Fit result for the S-wave of the form factor F. The dispersive description reproduces beautifully the curvature of the form factor. The (s,s_ℓ)-phase space is projected on the s-axis, the plotted lines correspond to splines through the (s,s_ℓ)-values of the data sets with a single bin in s_ℓ-direction.

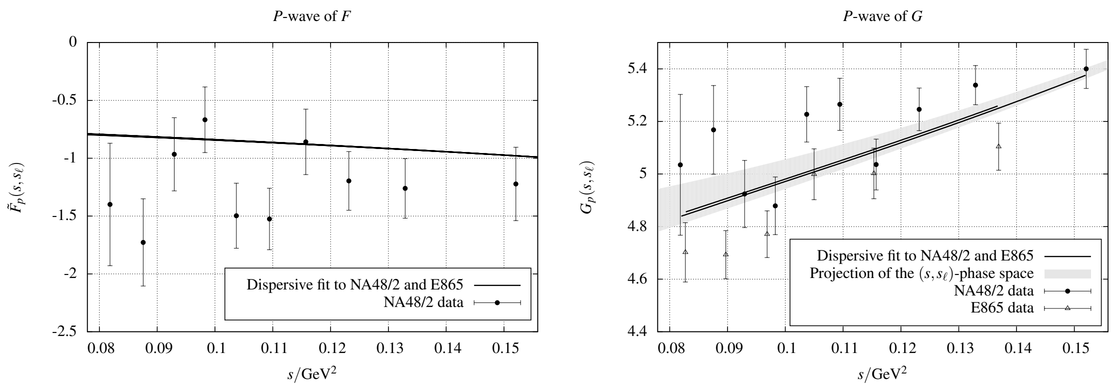

Figure 2: Fit results for the P-waves of the form factors F and G. The (s,s_ℓ)-phase space is again projected on the s-axis.

3.2 Matching to χPT

We perform the matching to χPT directly on the level of the subtraction constants. This means that we decompose the chiral expression at NLO or NNLO according to the reconstruction theorem and write the functions $M_0, \ldots$ in a chirally expanded Omnès form. This allows us to directly identify the subtraction polynomials $\tilde{P}(s)$ in (2.6) with chiral expressions. Such a procedure separates the resummation of rescattering effects from the chiral matching. Note also that we subtract all the functions $M_0, \ldots$ at zero energy.

By matching the dispersion relation to χPT, we are able to determine the LECs L_1^r, L_2^r and L_3^r. Using the information on the s_ℓ-dependence of the form factors, also L_9^r can be extracted, though the present experimental data does not allow a precise determination.

If we perform the matching at two-loop level, many NNLO LECs C_i^r enter the matching equations. We compare different input values for the C_i^r [28 – 30] and assign a 50% uncertainty to the contribution of the C_i^r to the subtraction constants. This C_i^r contribution is then fitted as well, using constraints on the chiral convergence of the subtraction constants. We find that the C_i^r input values of the BE14 global fit [30] lead to the best chiral convergence and a good χ^2 of the whole fit.

In table 1, we show the results of the matching at NLO and NNLO for the low-energy constants L_1^r, L_2^r and L_3^r. For comparison, we also quote the values of the BE14 global fit [30].

Table 1: Results for the LECs ($\mu = 770$ MeV).

	$10^3 \cdot L_1^r$	$10^3 \cdot L_2^r$	$10^3 \cdot L_3^r$
Dispersive treatment, NLO matching	0.51(6)	0.89(9)	−2.82(12)
Dispersive treatment, NNLO matching	0.69(18)	0.63(13)	−2.63(46)
BE14 global fit [30]	0.53(6)	0.81(4)	−3.07(20)

At NLO, a large contribution to the uncertainties comes from the high-energy behaviour of the phase shifts, either from the $\pi\pi$ phases in the case of L_1^r and L_2^r or the $K\pi$ phases in the case of L_3^r. At NNLO, the largest uncertainty is due to the fitted contribution of the C_i^r.

4. Conclusions

We have presented a dispersive representation of $K_{\ell 4}$ decays that provides a model independent parametrisation valid up to and including $\mathcal{O}(p^6)$.[1] The dispersion relation is based on unitarity, analyticity and crossing. It includes a full resummation of $\pi\pi$- and $K\pi$-rescattering effects. The dispersion relation is parametrised by subtraction constants that we determine by fitting experimental data and by using the soft-pion theorem as well as chiral input.

In contrast to a pure chiral description, the dispersion relation describes perfectly the experimentally observed curvature of the S-wave of the form factor F, which we interpret as a result of significant $\pi\pi$-rescattering effects. This is yet another case in which high-precision data clearly call for effects which go even beyond NNLO in χPT.

By using the matching equations to χPT we have extracted the values of the low-energy constants L_1^r, L_2^r and L_3^r. The correction from NLO to NNLO, when matching the chiral and dispersive representations and fitting the latter to the data are smaller than the corrections from NLO to NNLO

[1] In [31, 32] the reconstruction theorem is used for a similar dispersive description of the $K_{\ell 4}$ form factors in order to study isospin-breaking effects in the phases at two loops.

observed in direct χPT fits. Constraints on the chiral convergence of the subtraction constants allow us to reduce the dependence on the input values for the C_i^r. Still, the poorly known values of the C_i^r are responsible for the larger uncertainties in the matching at NNLO.

The two-dimensional NA48/2 data set for the S-wave of F, which shows both the s- as well as the s_ℓ-dependence, has allowed us to extract a value for L_9^r [13], which is roughly compatible with previous determinations. In accuracy, however, it cannot compete yet, as it reflects the low precision in the measurement of the s_ℓ-dependence of F.

Acknowledgements

The speaker (PS) thanks the conference committee for the organisation of the Chiral Dynamics 2015 conference. We are very grateful to J. Bijnens, B. Bloch-Devaux, G. Ecker, J. Gasser, I. Jemos, B. Kubis, S. Lanz, H. Leutwyler, S. Pislak, P. Truöl and A. van der Schaaf for valuable discussions and diverse support.

Financial support by the Swiss National Science Foundation, the DFG (CRC 16, "Subnuclear Structure of Matter") and the U.S. Department of Energy (contract DEAC05-06OR23177) is gratefully acknowledged.

References

[1] S. Weinberg, Phys. Rev. **166**, 1568 (1968).

[2] J. Gasser and H. Leutwyler, Annals Phys. **158**, 142 (1984).

[3] J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465 (1985).

[4] E. P. Shabalin, J. Exp. Theor. Phys. (USSR) **44**, 765 (1963), [Sov. Phys. JETP **17**, 517 (1963)].

[5] N. Cabibbo and A. Maksymowicz, Phys. Rev. **137**, B438 (1965), [Erratum-ibid. **168**, 1926 (1968)].

[6] J. Batley *et al.* (NA48-2 Collaboration), Eur. Phys. J. **C70**, 635 (2010).

[7] S. Pislak *et al.* (BNL-E865 Collaboration), Phys. Rev. Lett. **87**, 221801 (2001), [Erratum-ibid. **105**, 019901 (2010)], [arXiv:hep-ex/0106071].

[8] S. Pislak *et al.* (BNL-E865 Collaboration), Phys. Rev. **D67**, 072004 (2003), [Erratum-ibid. **D81**, 119903 (2010)], [arXiv:hep-ex/0301040].

[9] J. Batley *et al.* (NA48/2 Collaboration), Phys. Lett. **B715**, 105 (2012), [arXiv:1206.7065 [hep-ex]], [Addendum-ibid. **B740**, 364 (2015)].

[10] J. Bijnens, G. Colangelo and J. Gasser, Nucl. Phys. **B427** (1994), [arXiv:hep-ph/9403390].

[11] G. Amoros, J. Bijnens and P. Talavera, Nucl. Phys. **B585**, 293 (2000), [arXiv:hep-ph/0003258].

[12] P. Stoffer, *Dispersive Treatments of $K_{\ell 4}$ Decays and Hadronic Light-by-Light Scattering*, PhD thesis, University of Bern, 2014, arXiv:1412.5171 [hep-ph].

[13] G. Colangelo, E. Passemar and P. Stoffer, Eur. Phys. J. **C75**, 172 (2015), [arXiv:1501.05627 [hep-ph]].

[14] P. Stoffer, *A Dispersive Treatment of $K_{\ell 4}$ Decays*, Master's thesis, University of Bern, 2010.

[15] G. Colangelo, E. Passemar and P. Stoffer, EPJ Web Conf. **37**, 05006 (2012), [arXiv:1209.0755 [hep-ph]].

[16] P. Stoffer, G. Colangelo and E. Passemar, PoS **CD12**, 058 (2013).

[17] P. Stoffer, Eur. Phys. J. **C74**, 2749 (2014), [arXiv:1312.2066 [hep-ph]].

[18] J. Stern, H. Sazdjian and N. H. Fuchs, Phys. Rev. **D47**, 3814 (1993), [arXiv:hep-ph/9301244].

[19] B. Ananthanarayan and P. Buettiker, Eur. Phys. J. **C19**, 517 (2001), [arXiv:hep-ph/0012023].

[20] B. Ananthanarayan, G. Colangelo, J. Gasser and H. Leutwyler, Phys. Rept. **353**, 207 (2001), [arXiv:hep-ph/0005297].

[21] I. Caprini, G. Colangelo and H. Leutwyler, Eur. Phys. J. **C72**, 1860 (2012), [arXiv:1111.7160 [hep-ph]].

[22] P. Buettiker, S. Descotes-Genon and B. Moussallam, Eur. Phys. J. **C33**, 409 (2004), [arXiv:hep-ph/0310283].

[23] D. Boito, R. Escribano and M. Jamin, JHEP **1009**, 031 (2010), [arXiv:1007.1858 [hep-ph]].

[24] C. Callan and S. Treiman, Phys. Rev. Lett. **16**, 153 (1966).

[25] S. Weinberg, Phys. Rev. Lett. **17**, 336 (1966).

[26] S. B. Treiman, E. Witten, R. Jackiw and B. Zumino, *Current Algebra and Anomalies* (World Scientific Publishing, 1985).

[27] V. de Alfaro, S. Fubini, G. Furlan and C. Rossetti, *Currents in Hadron Physics* (North-Holland Publishing Company, 1973).

[28] J. Bijnens and I. Jemos, Nucl. Phys. **B854**, 631 (2012), [arXiv:1103.5945 [hep-ph]].

[29] S.-Z. Jiang, Y. Zhang, C. Li and Q. Wang, Phys. Rev. **D81**, 014001 (2010), [arXiv:0907.5229 [hep-ph]].

[30] J. Bijnens and G. Ecker, Ann. Rev. Nucl. Part. Sci. **64**, 149 (2014), [arXiv:1405.6488 [hep-ph]].

[31] V. Bernard, S. Descotes-Genon and M. Knecht, Eur. Phys. J. **C73**, 2478 (2013), [arXiv:1305.3843 [hep-ph]].

[32] M. Knecht, V. Bernard and S. Descotes-Genon, PoS **CD15**, 054 (2015).