

# QCD: a gauge theory for strong interactions.

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ABSTRACT: I briefly review some of the perturbative methods, standard perturbation theory, heavy quark effective theory, chiral perturbation theory, large  $N_c$  QCD, lattice QCD, used to tackle the theory at different regimes. I also discuss some of the non-perturbative results obtained within lattice QCD

## 1. Apologies

These proceedings have major defects. First, as for the references, I only quote the ones I used for preparing them. These are mostly review articles and the reader will find there more extensive citations. Second, I present a survey of perturbative methods for QCD, including at the end a part concerning non-perturbative numerical results. The details on each of the methods discussed are necessarily limited: they should be intended as a guide through some of the keywords of each subject, written in *italic*. Anyway, I could not escape for a second time from writing them without getting into troubles with my friends Nikos Trachas and George Zoupanos, whom I thank again for the pleasant atmosphere of the school.

## 2. Introduction

The theory is characterized at microscopic level by the presence of a gluon self interaction with a strength related by symmetry to the one of the quark gluon coupling. This is due to the non abelian property of the gauge group.

In the well known case of QED, the gauge symmetry is abelian and the invariance of the action under a local phase rotation of the Fermi fields,

$$\Psi(x) \rightarrow e^{i\alpha(x)}\Psi(x) \quad (2.1)$$

is fulfilled if  $\Psi$  is minimally coupled to the gauge field:

$$\partial_\mu\Psi(x) \rightarrow (\partial_\mu - ie A_\mu)\Psi(x) \quad (2.2)$$

and if the gauge field Lagrangian is invariant under

$$A_\mu \rightarrow A_\mu + 1/e \partial_\mu\alpha(x) \quad (2.3)$$

i.e. is of the form:

$$\begin{aligned} L(A_\mu) &= -1/4 F_{\mu\nu}F^{\mu\nu} \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned} \quad (2.4)$$

The Lagrangian is quadratic in the gauge field.

In the case of an  $SU(3)$  gauge symmetry, the phase transformation affecting the color degrees of freedom ( $i_c$ ):

$$\Psi(x)^{i_c} \rightarrow (e^{i\Omega(x)})^{i_c j_c} \Psi(x)^{j_c} \quad (2.5)$$

where

$$\Omega(x)^{i_c j_c} = \sum_k T_k^{i_c j_c} \alpha(x)^k \quad (2.6)$$

with  $T_k^{i_c j_c}$  the 8 traceless and Hermitean generators of the  $SU(3)$  group.

The covariant derivative acts non-trivially in colour space:

$$D_\mu^{i_c j_c} = \partial_\mu \delta^{i_c j_c} - ie A_\mu^{i_c j_c} \quad (2.7)$$

where

$$A_\mu^{i_c j_c} = \sum_k T_k^{i_c j_c} A_\mu^k \quad (2.8)$$

There are as many gauge fields as group generators. In this case, to preserve the invariance of the fermion action, the gauge field must satisfy a transformation that consists of a shift of the field (like in the abelian case) AND of a color rotation:

$$A_\mu \rightarrow A_\mu + 1/g \partial_\mu \Omega + i[A_\mu, \Omega] \quad (2.9)$$

The gauge action invariant under such a transformation is non-quadratic and contains cubic and quartic self-interactions.

$$L(A_\mu) = -1/4 G_{\mu\nu} G^{\mu\nu} \\ G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\nu, A_\mu] \quad (2.10)$$

The gluon self interactions lead to two major consequences: i) the property of asymptotic freedom i.e. a weakly interacting theory at small distances, ii) the confinement of colour charges at large distances.

An analytic solution at all distances is still missing and various perturbative methods have been devised, with a lowest order supported in each case by specific experimental facts.

In Table 1, shown at the end, I give a summary of the perturbative methods that I will be briefly reviewing in these lectures, by recalling the corresponding expansion parameter, the basic degrees of freedom, the experimental fact supporting the approximation and the name.

### 3. Perturbative QCD

The experimental evidence that a perturbative expansion in the coupling constant (very much like in QED) might be a good approximation under specific conditions is provided by the validity of *scaling laws* for processes with large momentum transfers: adimensional physical quantities, like cross sections multiplied by suitable powers of the center of mass energy, depend upon dimensionless invariants made from EXTERNAL momenta only. A prototype of these processes is the total cross section of electron positron into

hadrons, where only a single external invariant exist, i.e. the total center of mass energy. In this case, the scaling law is very simple and states that the “adimensional cross section” is a constant

$$\sigma(e^+e^- \rightarrow \text{hadrons}) \times E_{cm}^2 = \text{constant} \quad (3.1)$$

The next case, is the cross section for the hadron inclusive deep inelastic scattering of a lepton off a hadron. Here there are two external momenta, the momentum transfer of the scattered electron,  $q$ , and the momentum of the target hadron,  $P$ . One can form two independent kinematic invariants, the momentum transfer squared,  $q^2$ , and the scalar product  $Pq$ . The dimensionless cross section (the structure function) can only depend, in the scaling limit, upon the dimensionless ratio  $-q^2/2pq = x_{Bjorken}$

In the two examples above, the validity of scaling laws suggests pointlike interactions of the photon with the hadron constituents, the charged quarks in this case, only affected by small corrections governed by the small value of the strong coupling constant experienced at large transferred momenta. The existence of a coupling “*running*” with the scale is typical of renormalizable theories. For these theories, predictions for physical quantities are free of ultraviolet divergences only when expressed in terms of renormalised coupling constant and masses, i.e. of parameters fixed by some independent experimental input at some reference physical scale  $q_0^2$ .

Different choices of the reference scale bring to different values of the renormalized coupling in such a way that the final physical predictions to all orders of perturbation theory are identical. The renormalized coupling as a function of the renormalization scale moves on a curve of “constant physics” fixed by the integration constant of the renormalization group equation:

$$\frac{d\alpha(Q^2)}{d\log(Q^2)} = \beta(\alpha(Q^2)) = \\ -b_0\alpha(Q^2)^2 + b_1\alpha(Q^2)^3 + \dots \quad (3.2)$$

The first term in the expansion has a negative sign, opposite to the one of QED and leads to a solution of the form

$$\alpha(Q^2) = 1/(b_0 \log(Q^2/\Lambda^2)) \quad (3.3)$$

where  $\Lambda$  defines uniquely the theory (apart from quark masses) and is renormalisation group invariant. With increasing  $Q^2$ , the coupling constant decreases and opens the way to a perturbative treatment of strong interactions. The identification of the large momentum transfer of the process with the renormalisation scale of the “running” coupling resums the potentially large logarithms of the ratio of the scale of the process over the renormalization scale (“renormalization scale logs”) that could make the effective expansion parameter of order one and makes the perturbative approximation truncated at a fixed order more reliable. Figures 1 and 2 show the “experimental” running of  $\alpha_S$  and the determination of its value at the  $Z_0$  mass reference scale.

The smallness of the coupling constant at small distances does not imply necessarily a good convergence of the perturbative expansion. As we have seen in the case of a bad choice of the renormalization scale, each power of the coupling in the series may appear multiplied by a logarithm of the “running scale”. The “renormalisation scale logs” are not the only potentially dangerous for the convergence of the perturbative expansion [2]. Indeed, hard processes can be divided into the two following classes:

i) totally inclusive processes, like  $e + e^- \rightarrow$  *hadrons*, are “long distance insensitive”, i.e. do not show in perturbation theory a singular dependence upon quantities related to the non perturbative aspects of the “long distance” physics of QCD,

ii) inclusive, but “long distance sensitive”, like the deep inelastic scattering, where the perturbative corrections to the lowest order contains logarithms of the running scale over the quark mass or over any long distance regulator that tames the singularities of collinear emission of massless quanta from massless quanta. These “collinear singularities” are universally factorisable and can be “renormalized” like the usual ultraviolet singularities, i.e. reabsorbed into FUNCTIONS (the parton distributions) normalized at some *factorisation* scale  $Q_0^2$ . The “long distance regulator” in the ratio over the running scale in

the collinear logarithms is then replaced by the factorisation scale  $q_0^2$ . Again, a judicious choice of the factorisation scale close to the “running scale” avoids the presence large logarithms in the perturbative series. The “running” with the factorisation scale of the collinear logarithms is governed by the DGLAP equations that need input parton densities as boundary values. For the simplest case of the “non singlet” quark densities (i.e. valence quark distributions”) the equations are of the form:

$$\frac{dq^{NS}(x, q^2)}{d \log(Q^2)} = \int_x^1 \frac{dy}{y} P^{qq}(x/y) q^{NS}(y, Q^2) \quad (3.4)$$

More complicated coupled equations hold for the singlet case. The “evolution probabilities”  $P^{qq}(z)$  can be perturbatively expanded:

$$P^{qq}(Z) = \alpha P_1^{qq}(z) + \alpha^2 P_2^{qq}(z) + \dots \quad (3.5)$$

iii) processes with EXCLUSIVE kinematics, sensitive also to “infrared logarithms” arising from incomplete cancellation among real and virtual emissions of soft gluons. As an example, when the Bjorken variable  $x$  tends to 1, the evolution probabilities contain large corrections due to the presence of *double logs* of the type

$$(\alpha \log(1-x))^N \quad (3.6)$$

These are resummed by suitable techniques generalizing the exponentiation of infrared singularities of QED.

Figure 3 shows the agreement of the perturbative predictions including the resummation of next-to-leading logarithms, i.e. those appearing raised to a power equal to the one of the perturbative expansion minus one, for the hadronic jet inclusive cross section at the Tevatron.

iv) processes affected by “low x “ logarithms in the gluon evolution probability, affecting the behaviour of singlet structure functions at very small values ( $10^{-4}$ ) of the Bjorken variables  $x$ :

$$P_{lowx}^{gg} = \alpha(1/x + a \log(x)/x + (a \log(x))^2/x + \dots) \quad (3.7)$$

The resummation implied by the DGLAP equation, leading log in the *factorisation* scale, leads to a gluon distribution behaving at low  $x$ 's like:

$$g(x) \simeq e^{\sqrt{\log 1/x}} \quad (3.8)$$

The explicit resummation of the “low  $x$  leading logs” is done via the BFKL equation and leads to a behaviour like:

$$g(x) \simeq x^{-\lambda_g} \quad (3.9)$$

However, next-to-leading-log (NLL) corrections to the BFKL resummation are very large and render a bit questionable the convergence of the approach:

$$\lambda_g^{NLL} = \lambda_g^{LL}(1 - 3.5\alpha_S), \quad (3.10)$$

with

$$\lambda_g^{LL} = 2.65\alpha_S \quad (3.11)$$

The best fits to the Hera data are today obtained by using NLL DGLAP equations together with a steep distribution of the gluon distribution. and are shown in figure 4.

The choice of the renormalisation and of the factorisation scales is irrelevant to all orders of perturbation theory, but it may affect a finite order prediction in a significant way. Although on general grounds it should be chosen “close” to the “running scale”, its actual choice is arbitrary and only the availability of higher order corrections will reduce the sensitivity of the predictions to such a choice [1]. Three loops calculations start to be available, for example, for the hadronic width of the  $Z_0$  that can be used to extract the value of the strong coupling constant at that scale, reaching an absolute error of 0.002.

The ultimate limitation of perturbative calculations comes from *power corrections*; i.e. from contributions of the type  $(\Lambda^{QCD}/Q)^P$ .

The power  $P$  can be related to the convergence of the perturbative expansion. Indeed, the generic perturbative expansion:

$$f(\alpha_S) = f(0) + \sum_{n=0} f_n \alpha_S^{n+1} \quad (3.12)$$

is expected to diverge FACTORIALY for large  $n$ :

$$f_n \simeq n! b^n \quad (3.13)$$

The terms of the series first decrease and have a minimum at  $n = n^*$ , with  $n^* \simeq 1/\alpha b$ .

Nevertheless, the series can be “resummed” by making a Borel transform:

$$B(f(t)) = f(0)\delta(t) + \sum_n \frac{f_n t^n}{n!} \quad (3.14)$$

For the terms of the new series the factorial growth has been cancelled out and the original quantity can be recovered by the inverse transform:

$$f(\alpha_S) = \int_0^\infty dt e^{-t/\alpha_S} B(f(t)) \quad (3.15)$$

This is not possible if there are poles for  $B(f(t))$  along the integration path: the possibility of going complex and circumventing the pole leads to an ambiguity in the choice of the path, above or below the pole. The difference of the two paths, i.e. the ambiguity, just amounts to a closed circle around the pole that can be estimated by the Cauchy theorem. For example, if:

$$B(t) \simeq \frac{1}{t - t_0}, \quad (3.16)$$

the uncertainty will be proportional to:

$$\begin{aligned} e^{t_0/\alpha_S(Q^2)} \simeq \\ e^{-t_0 b_0 \log(Q^2/\Lambda^2)} = \\ (\Lambda^2/Q^2)^{t_0 b_0} \end{aligned} \quad (3.17)$$

i.e. the ambiguity of the perturbative series is a power correction and is removed only by the full inclusion of all power corrections. One then EXPECTS a power correction to be present to compensate the ambiguities of the perturbative series. By looking at the location of the poles of the Borel transform, one can predict the existence of corresponding power corrections in the cases where alternative ways to determine their presence are lacking.

In QCD, there are poles of the Borel transform on the real and positive  $t$  axis related to the behaviour of the coupling constant at low values of the virtual momentum running in loop integrals of hard processes: they are called “renormalons” and lie on the positive real axis, leading to an ambiguity of the form:

$$\int d^4x e^{iqx} \langle 0 | T(J_\mu(x)J_\nu(0)) | 0 \rangle = (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi(q^2) \quad (3.18)$$

A sketch of the derivation in the limit of large number of flavours goes through the following steps:

1) define:

$$D(q^2) = 4\pi^2 d\Pi(q^2)/d\log(q^2) \quad (3.19)$$

The diagrams leading in the above mentioned limit are the iteration of fermion bubbles in the vacuum polarisation and can be resummed through the renormalisation group running of the coupling constant, with the result:

$$\begin{aligned} D(Q^2) &\simeq \int \frac{dk^2}{k^2} \alpha(k^2) F(k^2) \\ &= \sum_{n=0} \alpha_S \int_0^\infty \frac{d\eta}{\eta} F(\eta) (-b_0^{(N_f)})^n \\ &\quad \alpha_S \log(\eta/C))^n \end{aligned} \quad (3.20)$$

where  $\eta = k^2/q^2$ ,  $\alpha_S = \alpha_S(q^2)$  and  $b_0^{(N_f)}$  is the term of the leading order beta function coefficient proportional to the number of flavours (negative with the convention of eq. 3.2). In the low  $\eta$  region  $F(\eta) \simeq A \times (k^2)^a$  and the integral over  $\eta$  can be performed with the result:

$$D \simeq \sum_{n=0}^\infty \alpha_S^{n+1} n! (b_0^{(N_f)}/a)^n \quad (3.21)$$

The Borel transform is easily calculated:

$$B(t) = \sum_n t^n (b_0^{(N_f)}/a)^n \simeq \frac{1}{a - (b_0^{(N_f)})t} \quad (3.22)$$

The derivation strictly relies on the large flavour number limit. If extended to the real case of

QCD where the dominant term in the beta function comes from the gluon vacuum polarization, the coefficient  $b_0$  is positive and from eq. 3.22, one obtains that the pole of the Borel transform lies on the positive real axis, leading to an ambiguity of the form:

$$(\Lambda^2/q^2)^a \quad (3.23)$$

with  $a = 2$  in this particular case.

From the Operator Product Expansion (OPE) of the two currents we know indeed that the leading power corrections are of the type found from the localization of the first (the closest to the real axis) renormalon pole.

The information coming from the presence of renormalons in the perturbative series is then “redundant” when there is an OPE, but may be used as an indication for the leading power corrections affecting processes that do not admit an operator product expansion analysis.

## 4. Heavy Quark Effective Theory

A second useful approximation is the expansion around the limiting case of an infinitely massive quark. The effective Lagrangian describing the leading term in the approximation exhibits additional symmetries with respect to the ordinary QCD Lagrangian [4]. The basic step is the separation of the heavy quark four momentum  $q$  into a dominant term proportional to the quark mass and into a residual one, according to:

$$q^\mu = Mv^\mu + k^\mu \quad (4.1)$$

Correspondingly, the Dirac spinor  $\psi(x)$  can be factorized as follows:

$$\psi(x) = e^{-iMvx} \tilde{\psi}(x) \quad (4.2)$$

where, in turn:

$$\tilde{\psi}(x) = \frac{1+\not{v}}{2} Q_v + \frac{1-\not{v}}{2} \chi_{-v} \quad (4.3)$$

The decomposition up to now is fully general, with  $Q_v$  and  $\chi_{-v}$  eigenstates of the velocity operator  $\not{v}$  with  $\pm$  eigenvalues, respectively.

By inserting the decomposition above in the QCD Lagrangian:

$$\bar{\psi}(i\mathcal{D} - M)\psi \quad (4.4)$$

we get:

$$\begin{aligned} & (\bar{Q}_v + \bar{\chi}_{-v})(M(\not{p} - 1) + i\mathcal{D})(Q_v + \chi_{-v}) \\ &= \bar{Q}_v ivDQ_v - \bar{\chi}_{-v}(ivD + 2M)\chi_{-v} \\ & \quad + \bar{Q}_v i\not{D}^T \chi_{-v} + \bar{\chi}_{-v} i\not{D}^T Q_v \end{aligned} \quad (4.5)$$

where  $D_\mu^T v^\mu = 0$ . The field  $\chi_{-v}$  has a mass of  $2M$ , a quadratic action and can be integrated out to derive the effective Lagrangian for the  $Q_v$  field:

$$L_{eff} = \bar{Q}_v(ivD + i\not{D}^T \frac{1}{2M + ivD} i\not{D}^T)Q_v \quad (4.6)$$

The Heavy Quark Effective Lagrangian is obtained by expanding in inverse powers of  $M$  the expression above. In particular, to the first order, one gets:

$$L = L^0 + \frac{1}{2M}L^1 \quad (4.7)$$

where,

$$\begin{aligned} L^0 &= \bar{Q}_v ivDQ_v \\ L^1 &= -\bar{Q}_v \not{D}^T \not{D}^T Q_v \\ &= -\bar{Q}_v D_T^2 Q_v - g\bar{Q}_v \sigma_{\mu\nu} G^{\mu\nu} Q_v / 2 \end{aligned} \quad (4.8)$$

The leading term ( $L^0$ ) is diagonal in Flavour and Spin space (Spin-Flavour symmetry). The coefficient of the first term in the next-to-leading Lagrangian  $L^1$  can be shown to be “protected” to all orders by the invariance under redefinition of the “residual” momentum in eq. 4.1, while the one of the second term, proportional to the strong coupling constant  $g$ , is not.

The effective Lagrangian leads to many phenomenological predictions: I will only quote some of them.

The first refers to the mass formulae for heavy-light mesons: in particular, the independence of the leading Lagrangian from the heavy quark spin, leads to the prediction of a degeneracy among doublets of mesons with a given spin for the light quark and different values of the heavy quark spin. In the zero orbital angular momentum sector, this applies to  $D - D^*$  and  $B - B^*$  states

of total angular momentum 0 and 1 respectively. In the angular momentum 1 sector, a degeneracy is expected for the states  $D_2^* - D_1$ , where the light spin combines with the orbital angular momentum to give a partial angular momentum 3/2 and the total angular momentum is 2 and 1 respectively, and for the states where the orbital-light partial angular momentum is 1/2 and the total is 1 and 0 respectively. The latter have not been observed experimentally because, according to M. Wise, they are too wide to be found. The splitting among members of a doublet is induced by the spin dependent part of the order  $1/M$  Lagrangian, i.e. by the second term in the equation cited.

The relevant hadron matrix element responsible for the splitting is:

$$\langle \sigma^H \cdot B^l \rangle \quad (4.9)$$

where the  $B^l$  is the chromomagnetic field generated by the light quark. The matrix element above, taken between the members of a doublet (denoted by  $+/-$ ), is given by:

$$\begin{aligned} & \langle (\sigma^H \cdot B^l)_{-\pm} \rangle = \\ & \pm n^\pm / 4 * constant \end{aligned} \quad (4.10)$$

where  $n^\pm = (2S^\pm + 1)$  with  $S^\pm$  is the total spin of the state. The mass formula for an element of the doublet can be finally parametrised as:

$$M_H = M + \bar{\Lambda} - \frac{\lambda_1}{2M} \pm \frac{n^\pm \lambda_2}{2M} \quad (4.11)$$

The splitting inside a doublet depends upon the parameter  $\lambda_2$ , that only depends upon the light system partial angular momentum. From  $B - B^*$  splitting, one gets:  $\lambda_2 = 0.12 GeV^2$  and from  $D - D^*$  splitting,  $\lambda_2 = 0.10 GeV^2$ , rather consistently.

From the  $D_1 - D_2^*$  “doublet” one expects in general a different value and gets:  $\lambda_2' = 0.013 GeV^2$ .

The second application of HQET is represented by the “Isgur-Wise” function, a universal function for semileptonic  $heavy \rightarrow heavy'$  decays, crucial for the determination of the weak

interaction CKM mixing matrix. To leading order in  $1/M$ , the momentum transfer to the leptonic current in the decay  $q_\mu$  is given by:

$$q_\mu = M_H v_\mu - M'_H v'_\mu \quad (4.12)$$

and its square depends upon the “overlap” between the two four velocities:  $w = v \times v'$ .

The four velocities are then used to parametrise the possible form factors:

$$\begin{aligned} & \frac{\langle D | V_\mu | B \rangle}{\sqrt{M_B M_D}} \\ &= h_+(w)(v + v')_\mu + h_-(w)(v - v')_\mu \\ & \frac{\langle D^*(\epsilon_\mu^*) | V_\nu | B \rangle}{\sqrt{M_B M_D}} = h_v(w) \epsilon^{\nu\mu\alpha\beta} \epsilon_\mu^* v'_\alpha v_\beta \\ & \frac{\langle D^*(\epsilon_\mu^*) | A_\nu | B \rangle}{\sqrt{M_B M_D}} \\ &= h_{A_1}(w)(w + 1)\epsilon_\mu^* - h_{A_2}(\epsilon^* v)v'_\mu - h_{A_3}(\epsilon^* v)v_\mu \end{aligned} \quad (4.13)$$

The strategy is to derive the most general effective Heavy Meson interactions that preserve the symmetry induced by the heavy mass limit at quark level and to find the independent form factors surviving in this limit.

Meson effective interactions are built from the field:

$$H^{ab} = \bar{Q}^a q^b \quad (4.14)$$

with the following properties:

$$\begin{aligned} & \frac{1+\not{v}}{2} H = H \\ & * Tr(H\gamma_5) \simeq \text{pseudoscalar} \\ & * Tr(H\gamma_5) \simeq \text{pseudoscalar} \\ & Tr(H\gamma_\mu) \simeq \text{vector} \end{aligned} \quad (4.15)$$

The field  $H$  with the above properties can be parametrised as:

$$H = \frac{1 + \not{v}}{2} (\gamma_\mu P_v^\mu + \gamma_5 P_v^5) \quad (4.16)$$

where  $P_v^\mu$  and  $P_v^5$  are the vector and pseudoscalar meson fields, respectively.

As an example, we derive the constraint on the form factors describing the matrix element of a vector current among pseudoscalar heavy mesons:

$$\begin{aligned} & \langle D | V_\mu | B \rangle \\ &= \langle D | \bar{c}\gamma_\mu b | B \rangle \rightarrow Tr(\bar{H}_c^{v'} \gamma_\mu H_b^v \chi) \end{aligned} \quad (4.17)$$

where the hadronic physics is entirely parametrized by  $\chi$ , a general matrix restricted by parity to be built from:

$$1 \gamma_\mu \sigma_{\mu\nu} v_\mu v'_\mu:$$

$$\begin{aligned} \chi &= \chi_0 + \chi_1 \not{v} + \chi_2 \not{v}' \\ & \quad + \chi_3 \not{v} \not{v}' \end{aligned} \quad (4.18)$$

However, the states  $\bar{H}_c^{v'}$  and  $H_b^v$  are eigenstates of  $v'$  and  $v$ , respectively and only a single form factor survives, with the Lorentz structure that can be obtained by selecting inside  $H$  the pseudoscalar part:

$$\begin{aligned} & Tr(\bar{H}_c^{v'} \gamma_\mu H_b^v \chi) \\ & \simeq Tr(\gamma_5 \frac{1+\not{v}'}{2} \gamma_\mu \frac{1+\not{v}}{2} \gamma_5) \\ & \simeq \frac{v_\mu + v'_\mu}{2} \end{aligned} \quad (4.19)$$

With the same procedure, one obtains the relations:

$$\begin{aligned} h_+(w) &= h_v(w) = h_{A_1}(w) = h_{A_3}(w) = \xi(w) \\ h_-(w) &= h_{A_2}(w) = 0 \end{aligned} \quad (4.20)$$

The reduction of the form factors for the vector matrix element can also be obtained from the conservation of the vector current between the different flavours in the infinite quark mass limit: It follows that:

$$q_\mu \langle V_\mu \rangle = 0 \quad (4.21)$$

i.e. :

$$\begin{aligned} 0 &= (v_\mu - v'_\mu) \times \\ & (h_+(w)(v + v')_\mu + h_-(w)(v - v')_\mu) \end{aligned} \quad (4.22)$$

that leads to:  $h_-(w) = 0$ . Fermion charge conservation also leads to the normalisation condition for the Isgur-Wise function at zero recoil, i.e. at  $w=1$ , in the infinite quark mass limit:

$$\xi(1) = 1 \quad (4.23)$$

## 5. Chiral perturbation theory

The expansion parameter of this approximation is  $\frac{E_\pi^2}{(4\pi f_\pi)^2}$  where  $E_\pi$  is the energy of a light pseudoscalar. Indeed, the method relies on the construction of an effective theory among pseudoscalars that embeds the spontaneous breaking of chiral symmetry of QCD. In the QCD Lagrangian, at zero quark mass, both vector and axial phase transformations for the fermion field:

$$\begin{aligned}\delta\psi &= i\alpha\psi \\ \delta\psi &= i\alpha\gamma_5\psi\end{aligned}\quad (5.1)$$

are exact symmetries, at classical level. In a theory with  $N_f$  flavours, the symmetry extends to  $SU(N_f)_{left} \times SU(N_f)_{right} \times U(1)_{vector}$ , (the  $U(1)_{axial}$  is broken by the anomaly at quantum level) and it is realised “a’ la Goldstone”, i.e. is spontaneously broken to  $U(N_f)_{vector}$ .

The effective theory of light pseudoscalars for  $N_f = 2$  can be constructed starting from a linear sigma model for a set of scalar fields organised in a matrix as follows:

$$\Phi = \sigma + i\vec{\tau} \cdot \vec{\pi} \quad (5.2)$$

with the following most general renormalisable Lagrangian:

$$\begin{aligned}L &= Tr(\partial_\mu\Phi^*\partial_\mu\Phi) \\ &- \mu^2/2Tr(\Phi^*\Phi) - \lambda/48Tr((\Phi^*\Phi)^2)\end{aligned}\quad (5.3)$$

The Lagrangian is invariant under the transformation:

$$\phi \rightarrow V_R\Phi V_L^+ \quad (5.4)$$

where  $V_{L,R}$  are unitary  $2 \times 2$  matrices. footnote ?? The model exhibits an  $SU(2)_{left} \times SU(2)_{right}$  symmetry. Note that the  $\phi$  field transforms like an interpolating field for a “meson” pseudoscalar made of  $q_{left} - q_{right}$  pair. The Lagrangian written in terms of  $\sigma$  and  $\pi$  fields, reads:

$$\begin{aligned}L &= ((\partial_\mu\sigma)^2 + (\partial_\mu\pi)^2)/2 \\ &- \mu^2(\sigma^2 + \pi^2) - \lambda(\sigma^2 + \pi^2)^2/4!\end{aligned}\quad (5.5)$$

The “potential” is a function of  $\rho^2 = \sigma^2 + \pi^2$ : the symmetry of the model can also be seen as an  $O(4)$  invariance in the for the vector  $(\sigma, \pi)$ . There is a total analogy with the Higgs sector of the standard model, and, as in that case, the spontaneous breaking of the symmetry depends upon the location of the minimum of the potential. A spontaneously broken solution is obtained when the mass term is negative and the location of the minimum is at:

$$\rho = \sqrt{-12\mu^2/\lambda} \quad (5.6)$$

The “unbroken” solution has a minimum at the origin that is obviously invariant :

$$0 \rightarrow V_R 0 V_L^+ = 0 \quad (5.7)$$

the residual invariance of the broken solution depends upon the direction of the breaking: by choosing it proportional to the identity in the notation of equation 5.2, i.e. in the direction of the  $\sigma$  field, one gets:

$$\begin{aligned}\rho \cdot 1 &\rightarrow \rho \cdot V_R \cdot 1 \cdot V_L^+ \\ &= \rho \cdot V_R \cdot V_L^+\end{aligned}\quad (5.8)$$

If,  $V_R = V_L$ , the minimum is invariant. There is a residual symmetry left unbroken represented by the  $SU(2)_{vector}$  transformations, like in the QCD case.

The physical set of states is found by expanding around the new minimum, i.e. by shifting the  $\sigma$  field:

$$\sigma \equiv S + \rho \quad (5.9)$$

The Lagrangian in the shifted fields has now a positive mass term for the  $S$  field and a zero mass term for the  $\vec{\pi}$  fields: a Lagrangian for the  $\vec{\pi}$  fields alone is obtained by sending the  $S$  field mass to infinity while keeping the position of the minimum constant. This limit transforms the “double well” section of the potential into a “double delta function” potential, with a fixed modulus for the  $O(4)$  vector:

$$\sigma^2 + \vec{\pi}^2 = \rho^2 \quad (5.10)$$

and the model becomes the non-linear sigma model.



By measuring the fields in “ $\rho$  units, the condition above simply states that the original  $\Phi$  matrix is an  $SU(2)$  matrix. The field  $S$  is now a function of the  $\vec{\pi}$  fields and the matrix  $\Phi$  is more conveniently reparametrised as:

$$\Phi = e^{i\vec{\tau}\cdot\vec{\pi}/\rho} \quad (5.11)$$

The expression for  $\Phi$ , expanded to the second order in powers of  $\vec{\pi}$ , reduces to the original one with  $S$  expressed, to the same order, as a function of  $\vec{\pi}$ .

The non linear Lagrangian contains only a kinetic part: the potential enforces the constraint between  $S$  and  $\vec{\pi}$  fields already taken into account:

$$L = \rho^2 Tr(\partial_\mu \Phi^\dagger \partial_\mu \Phi)/4 \quad (5.12)$$

One can extract the Noether axial current:

$$\vec{J}_\mu^5 = \frac{\delta S}{\delta(\partial_\mu \Phi)} \delta^5 \Phi \simeq \rho \partial_\mu \vec{\pi} \quad (5.13)$$

in the one pion sector.

Its matrix element between a pion and the vacuum is

$$\begin{aligned} & \langle 0 | \vec{J}_\mu^5 | \pi \rangle = \\ \rho \langle 0 | \partial_\mu \vec{\pi} | \pi(q) \rangle & = \rho q_\mu \end{aligned} \quad (5.14)$$

By comparing with the well known PCAC relation, we identify  $\rho$  with  $f_\pi$  and fix the only unknown constant of the effective Lagrangian for massless pseudoscalar interactions. Indeed, by expanding to higher orders in the “pion” fields, one get multifield interacting terms. The first is a four pion interaction with strength:

$$\partial_\mu \vec{\pi}^2 \partial^\mu \vec{\pi}^2 / \rho^2 \quad (5.15)$$

Proportional to  $E_\pi^2/f_\pi^2$ , i.e. to the expansion parameter of the  $\chi p T$  approach.

The theory is NOT renormalizable ( the price of decoupling the  $S$  field..), very much like the standard model without the Higgs field. Loop corrections need the presence of higher dimensional operators with proper coefficients to cancel the ultraviolet divergences.

In principle one should include all counterterms ( an infinite number) needed to keep the

predictions finite. However, these terms are organized in increasing powers of the expansion parameter and can be neglected if the energy of the pions much smaller than  $4\pi f_\pi$ , under the assumption that the finite part of the counterterms is not anomalously large.

The next order ( $O(E^4)$ ), where there are 12 new operators entering the Lagrangian with coefficients related to experimental quantities, still retains some degree of predictivity.

Interesting extensions of the low effective Lagrangian approach are being developed for vector mesons and for baryons.

## 6. The large $N_c$ expansion

The approximation consists in considering the theory in the limit of large number of colours where only a subset of diagrams survive, preserving the main non perturbative aspects of the theory at finite colour number ( three...) [5]. The diagrammatic simplification was first noticed by G.’t Hooft and it is best seen by adopting his notation for describing fermions and gluons.

The first possess only one colour index and their propagation in a diagram is indicated by a line with an arrow distinguishing the fermion from the antifermion case. The second belong to the adjoint representation, have an upper and a lower index and are represented by two parallel lines with opposite arrows. To each ordinary Feynman diagram corresponds a “color” diagram constructed according to the notation described. The color lines can be open or closed. In the latter case, a color index is summed and a factor  $N_c$  is gained for the diagram. As an example, the lowest order correction due to a quark loop to the photon vacuum polarisation contains a single closed colour line ( the fermion loop) and is of order  $N_c$ .

To the next order, the first non trivial in the strong coupling constant, one can exchange a gluon inside the quark loop generating a color diagrams with two closed loops, i.e. of order  $N_c^2 g^2$ , where I have also included the order in the coupling constant for future use.

At order  $g^4$  there are three possible diagrams: two with the exchange of two gluons, one *planar* and another *non-planar* where the two gluon lines

cannot be drawn in a plane without crossing them, an a third diagram with two quark loops connected by two gluons. The corresponding “colour” diagrams have three colour loops, one colour loop and two colour loops, respectively. Their relevance is then very different in the large  $N_c$  limit. Indeed, in this limit, the expansion is governed by the “effective coupling”  $g_N^2 \equiv N_c g^2$  and the limit should be taken while such a coupling is kept fixed. The three diagrams in the new coupling are of order:  $N_c g_N^4$ ,  $N_c g_N^4/N_c^2$  and  $N_c g_N^4/N_c$  respectively. Notice that the extra fermion loop, the third diagram, costs an extra inverse power of  $N_c$ . This appears also in the one-loop expression for the beta function of the  $g_N^2$  coupling, where the fermion contribution to the vacuum polarisation is suppressed by an inverse power of  $N_c$ .

Qualitative important consequences of the fermion loops suppression in the large  $N_c$  limit are the prediction of infinitely narrow  $q\bar{q}$  resonances: their decay needs the formation of an extra  $q\bar{q}$  pair. Experimentally, such a suppression seems at work in the  $\Phi$  decay that occurs preferentially in the  $KK$  channel, unfavoured by the restricted phase space, instead of the  $\pi\pi$  channel ( the Zweig rule). The latter would require an extra fermion loop.

An interesting recent application of large  $N_c$  suppression is the estimate of the relevance of finite parts of the counterterms of the effective Chiral Lagrangian. The leading lagrangian contains only a single flavour trace ( remember that the group is a flavour group) and among all diagrams, there is a leading subset in the large  $N_c$  limit with a single fermion loop. The situation is different for the next-to-leading lagrangian: there are terms with a single flavour trace, like:

$$L_3 \times Tr(\partial_\mu \Phi^+ \partial_\mu \Phi \partial_\nu \Phi^+ \partial_\nu \Phi) \quad (6.1)$$

and terms with two flavour traces, like:

$$L_2 \times Tr(\partial_\mu \Phi^+ \partial_\mu \Phi) Tr(\partial_\nu \Phi^+ \partial_\nu \Phi) \quad (6.2)$$

In the second case, the two traces require a second fermion loop and a suppression is expected. In Table 2, I report the experimental

determinations of the finite parts of the counterterms of the order  $E^4$  Lagrangian compared to their relative relevance in the large  $N_c$  limit. The average value of the group of  $O(1)$  is about 0.4 and the one of the group of  $O(N_c)$  is about 3.3, showing again the qualitative agreement with large  $N_c$  predictions.

Unfortunately, although the approximation seems a very promising one, no real quantitative progress has been achieved along this line for QCD, while it has been successfully exploited in various 2-dimensional models.

$L_i$	value	order
$2L_1 - L_2$	$-0.6 \pm 0.5$	1
$L_4$	$-0.3 \pm 0.5$	1
$L_6$	$-0.2 \pm 0.3$	1
$L_7$	$-0.4 \pm 0.2$	1
$L_2$	$1.4 \pm 0.3$	$N_c$
$L_3$	$-3.5 \pm 1.1$	$N_c$
$L_5$	$1.4 \pm 0.5$	$N_c$
$L_8$	$0.9 \pm 0.3$	$N_c$
$L_9$	$6.9 \pm 0.7$	$N_c$
$L_{10}$	$-5.5 \pm 0.7$	$N_c$

**Table 2:** Large N order of finite counterterms of the  $O(p^4)$   $\chi p T$  Lagrangian

## 7. Lattice QCD

The basic approximation of lattice QCD is the discretisation of the space time: quarks and gluons live on a 4-d crystal. In particular, fermions live on lattice sites. A fermion field  $\psi^A(i, j, k, l)$  is a function of four discrete coordinates [6] [7]. The theory on the lattice is the Euclidean extension of the original Minkowski theory, where the time has been rotated to the imaginary axis, but the original continuous  $O(4)$  symmetry of the theory has been replaced by a discrete hypercubic symmetry. The main appeal of the lattice formulation of the theory, originally proposed by K.Wilson, is that gauge invariance can be maintained EXACT. The explicit construction can be found by “undoing” the limits of ordinary dif-

ferential calculus, i.e. by introducing the “finite difference derivatives:

$$\partial_\mu \psi \rightarrow 1/2(\nabla_\mu + \nabla_\mu^*)\psi \quad (7.1)$$

where

$$\begin{aligned} \nabla_\mu \psi(x) &\equiv \psi(x + \mu) - \psi(x) \\ \nabla_\mu^* \psi(x) &\equiv \psi(x) - \psi(x - \mu) \end{aligned} \quad (7.2)$$

The request of LOCAL gauge invariance reads:

$$\psi^A(x) \rightarrow G^{AB}(x)\psi^B(x) \quad (7.3)$$

where  $G^{AB}(x)$  is a unitary  $SU(3)$  matrix, is violated (as in the continuum) by the finite difference derivative:  $\psi(x)$  and  $\psi(x + \mu)$  transform differently under a LOCAL gauge rotation, they are at different sites, and cannot be combined together in the finite difference derivative. Again, the solution is the introduction of a field  $U_\mu$  to match the gauge phase rotation of neighbouring fermions. If:

$$U_\mu(x) \rightarrow G(x)U_\mu(x)G(x + \mu) \quad (7.4)$$

then :

$$U_\mu(x)\psi(x + \mu) \rightarrow G(x)U_\mu(x)\psi(x + \mu) \quad (7.5)$$

A kinetic gauge invariant lattice action for the fermions can be constructed:

$$L_{fermions} = \frac{1}{2}\psi^\dagger(x)(\nabla(U) + \nabla^*(U))\psi(x) \quad (7.6)$$

where

$$\begin{aligned} \nabla_\mu(U)\psi(x) &\equiv U_\mu\psi(x + \mu) - \psi(x) \\ \nabla_\mu^*(U)\psi(x) &\equiv \psi(x) - U_\mu^+\psi(x - \mu) \end{aligned} \quad (7.7)$$

The gauge fields are represented by the group elements, instead of the group generators, they “live” on the links connecting neighbouring lattice sites ( they are also named “links”). A gauge invariant interaction among them is the colour trace of any closed circuit, the “Wilson loops”. It is straightforward to check that the cyclic property of the trace ensures the invariance, under the transformation of eq. 7.5, of the Wilson loops.

The simplest circuit that can be formed is the elementary *plaquette*  $PL$ , i.e. a square around four neighbouring corners. The Wilson action for the gauge fields is just the sum of all possible plaquettes.

The “naive continuum limit” of the action is obtained by expanding in powers of the lattice spacing, traditionally called “ $a$ ”. For this, we need the expression relating the gauge link to the ordinary gluon fields of the continuum theory:

$$U_\mu = \exp(i \int_x^{x+\mu} dx_\mu A^\mu) \quad (7.8)$$

The plaquette is easily handled in the abelian case: the result is the exponential of the curl of the gauge field around the elementary square:

$$\begin{aligned} \text{Plaquette} &\simeq \exp(i \int_{\text{perimeter}} dx_\mu A^\mu) \\ &= \exp(i \int_{\text{surface}} d\sigma_{\mu\nu} F^{\mu\nu}) \\ &\simeq 1 - a^4 F^{\mu\nu} F_{\mu\nu} \dots \end{aligned} \quad (7.9)$$

up to terms of higher order in the lattice spacing or that vanish by the antisymmetry of the gauge field tensor  $F_{\mu\nu}$ .

The Dirac term is also reduced to the usual continuum expression, by neglecting higher orders in the lattice spacing:

$$\begin{aligned} U_\mu\psi(x + \mu) &\sim (1 + iaA_\mu)\psi(x) + a\partial_\mu\psi \\ &\rightarrow \psi(x) + a(\partial_\mu + iA_\mu)\psi(x) + \dots \end{aligned} \quad (7.10)$$

Notice that the coupling constant has disappeared in the covariant derivative. This comes from a redefinition of the gauge fields that now incorporate the coupling constant. As a consequence, the pure gauge action appears DIVIDED by the coupling constant square.

The properly normalized lattice action is then:

$$L_{gauge} = \beta \sum_{\text{plaquettes}} PL \quad (7.11)$$

where  $\beta \equiv 2N_{\text{colours}}/g^2$ .

The above limit is called “naive” because it refers to a classical continuum limit. In the quantum theory, the lattice spacing acts as an ultraviolet cutoff ( the fields cannot vary at distances

smaller than the lattice spacing) and the continuum limit is affected by divergences if taken naively without a proper renormalization procedure.

Quantum averages are obtained from functional integrals of the type:

$$\frac{\langle \phi_1 \phi_2 \dots \phi_n \rangle = \int D\phi e^{(-S_E(\phi))} \phi_1 \phi_2 \dots \phi_n}{\int D\phi e^{(-S_E(\phi))}} \quad (7.12)$$

where the real exponent instead of a complex one is a consequence of the Euclidean rotation.

Quantum averages appear like statistical averages on a 4-d system in the canonical formulation. The usual sum over states of statistical mechanics is replaced by the functional integral that on a discretized space-time is a functional sum over possible gauge and fermion fields “configurations”.

Analytic estimates for the quantum averages can be obtained in the language of stat-mech, in the “high temperature approximation”, i.e. in the limit when  $\beta_{\text{Boltzmann}} \rightarrow 0$ . From the expression for the gauge action of eq. 7.11, this limit corresponds to the “strong coupling expansion”, i.e.  $g^2 \rightarrow \infty$ . In normal perturbation theory, one solves exactly the free part of quantum correlations and expands the rest in powers of the coupling constant, in this case the whole exponential is expanded in a power series in  $\beta$ .

This approximation reproduces an important feature of QCD, colour confinement, and is taken as a good starting point of the lattice QCD approach, with some caveats to be seen.

A confinement test can be constructed from the connected correlation of two plaquettes, when the relative distance becomes very large. Each plaquette is a “gluon source” that can be decomposed into a sum of terms with a given number of gluons by expanding to an arbitrary order the expression for  $U_\mu$  as a function of  $A_\mu$ , starting from a 2 gluon state. If confinement is ABSENT, massless gluons are the good degrees of freedom also at large distances, if it is PRESENT, only colour singlets survive at large distances, with a non zero mass.

The plaquette correlation provides a test of the mass of the states surviving at large distances.

$$\begin{aligned} & \sum_{\vec{x}} \langle 0 | \text{PL}_{(\vec{0},0)} \text{PL}_{(\vec{x},T)} | 0 \rangle = \\ & \sum_n \sum_{\vec{x}} (\langle 0 | \text{PL}_{(\vec{0},0)} | n \rangle)^2 e^{ip_n x} \\ & \rightarrow (\text{Euclidean}) \sum_n |C_n|^2 e^{-m_n T} \quad (7.13) \end{aligned}$$

For large time separation, only the lowest energy state with the quantum numbers of the source survives. A free multigluon state has zero lowest energy, therefore the existence of a “mass gap”, i.e. of a non zero mass state provides evidence for colour confinement.

In the strong coupling approximation, lattice QCD confines. Technically, this is a consequence of exact gauge invariance of the lattice functional measure and of the action. This implies that the link integral:

$$\begin{aligned} & \int DU_\mu U_\mu \rightarrow (\text{on the lattice}) \\ & \int \prod_{i,\mu} dU_\mu(i) U_\mu(i) = 0 \quad (7.14) \end{aligned}$$

should be zero. Indeed, its matrix elements can be arbitrarily changed with a gauge transformation and the only invariant value is zero.

This implies that after expanding the whole action, the non zero integrals always imply that a link must appear with its hermitian conjugate (or powers of  $U - U^+$  pairs).

The connected correlation of the two plaquettes at large time separation:

$$\begin{aligned} & \langle \text{PL}_1 \text{PL}_2 \rangle = \\ & \frac{\prod_{i,\mu} \int dU_\mu^i \exp(-\beta \sum \text{plaquettes}) \text{PL}_1 \text{PL}_2}{\prod_{i,\mu} \int dU_\mu^i \exp(-\beta \sum \text{plaquettes})} \\ & \rightarrow (\text{strong coupling}) \\ & \frac{\prod_{i,\mu} \int dU_\mu^i \sum (-\beta \sum \text{plaquettes})^n / n! \text{PL}_1 \text{PL}_2}{\prod_{i,\mu} \int dU_\mu^i \sum (-\beta \sum \text{plaquettes})^n / n!} \quad (7.15) \end{aligned}$$

Remembering that links must overlap pairwise, the leading term is the one with a “tube” with the two plaquettes in the correlation as end-caps. Up to irrelevant constants, one gets for the time dependence of the correlation:

$$\begin{aligned} & \langle \text{PL}_1 \text{PL}_2 \rangle \\ & \simeq (4\beta)^T \simeq e^{-T \log(4/\beta)} \end{aligned} \quad (7.16)$$

i.e. a “mass gap “  $M \sim \log(4/\beta)$ .

In the same strong coupling limit, “meson” correlations, i.e. the correlations to leading order of two point-like currents made of a  $q - \bar{q}$  pair, are obtained from a sum of paths where the  $q - \bar{q}$  pair do not separate at all: a non zero “internal” area between the path of the quark and the one of the antiquark would be penalized by a factor:  $\beta^{\text{area}}$ .

Overall, the strong coupling expansion provides a good starting picture of confined quarks and gluons. However, such a limit of the theory is very unphysical and might be very misleading: also “compact QED” ( a version of QED on the lattice) exhibits confinement in this limit, while electric charges are well propagating to large distances in reality!

The point is that the strong coupling limit is very far from the “continuum limit” of the theory, where the lattice cutoff is small enough to produce negligible effects.

The continuum limit can be seen from a stat-mech point of view as the approach to a second order ( or higher) phase transition. Indeed, in LATTICE units, the correlation length increases toward the continuum limit ( the basic length measure is shrinking) like in a phase transition. The transition point is obtained at the “critical temperature” of the system, i.e. at a critical coupling. From the point of view of field theory, PHYSICAL units are kept fixed and the lattice spacing goes to zero or equivalently the UV cutoff diverges. In order to approach a finite theory, the renormalized one, the (bare) coupling must be properly tuned to keep physical quantities constant.

In both views, the continuum limit occurs at a critical value of the coupling. Such a value is known from the asymptotic freedom property of the theory, where the behaviour of the bare coupling upon the cutoff, at small distances, i.e. large values of the momentum cutoff, can be extracted from a perturbative calculation. The infinite cutoff limit ( the continuum limit) is obtained at zero bare coupling, i.e. at  $\beta \rightarrow \infty$ . The

knowledge of the  $\Lambda$  parameter of QCD, translated into the particular renormalization scheme represented by the lattice regularization, provides an estimate of the cutoff in physical units, given the value of the bare coupling. The strong coupling approximation holds at most when  $g^2(a) \simeq 10$  that corresponds to a lattice spacing of a Fermi or slightly below. The QCD dynamics is not “resolved” by such an approximation and the issue is whether the important property of confinement persists to higher values of  $\beta$ , i.e. for relatively small values of the bare coupling where:

$$a^{-1} \gg \Lambda_{QCD} \quad (7.17)$$

Entering the “weak (bare) coupling region” in a non perturbative way can only be achieved numerically. The discretization on the lattice and therefore the reduction of the theory to a finite number of degrees of freedom is a prerequisite for a numerical estimate.

Numerical simulations are done through the following steps.

1) one generates a set of “equilibrium” configurations, distributed according the Boltzmann functional measure. As an example, one a) starts from a random configuration, b) updates the field values **LOCALLY** until the equilibrium distribution is reached by applying an iterative algorithm that leads from a field value to the next one after comparing the action in the two cases, as follows:

$$\phi \rightarrow \phi' \text{ and } S_{old} \rightarrow S_{new} \quad (7.18)$$

if  $\Delta S \equiv S_{new} - S_{old} < 0$  the new value that decreases the action is accepted, otherwise it is accepted with a conditional probability

$$p \equiv e^{-\Delta S} \quad (7.19)$$

The first choice correspond to the search of the absolute minimum, i.e. to the classical limit, the second option takes into account quantum fluctuations around the minimum.

By iterating this algorithm, one finally obtains, after some “thermalization time”, a set of equilibrium configurations.

2) One makes the functional integral (sum) by averaging the observables, functions of the gauge fields, over the set of equilibrium configurations collected.

Fermions are a special case. 1) they cannot be associated with C numbers ( they are non-commuting Grassman variables), 2) their correlations can be worked out analytically, thanks to the quadratic form of the action.

Fermion correlations, according to the Wick theorem, are sums of products of two point correlations given by the “propagators”, i.e. the inverse of the quadratic operator in the action. Besides, their integration produces a “fermion determinant”, i.e. the determinant of the same quadratic operator raised to a power equal to the number of flavours . The quadratic fermion operator is a function of the gauge configuration and its determinant represents a NON-LOCAL function of the links, i.e. contains Wilson loops of arbitrary length and complexity and not just the elementary plaquette of the pure gauge action. They arise from the wondering on the lattice of virtual fermions. The updating of a single link in the lattice action including the fermion determinant requires a calculation that involves most of the links on the lattice and not just the neighbouring ones of the plaquettes containing the updated link. Numerically, this represents a formidable task and has delayed for a long time simulations including the effects of virtual fermion loops.

The last part of these lectures is devoted to a summary of some of the important results in lattice QCD along its history.

Strong coupling lattice QCD was invented by K.Wilson in '74. He wanted to connect the unphysical strong coupling region to the physical weak coupling region through a renormalisation group approach in configuration space, successfully exploited in the analysis of phase transitions in statistical mechanics. The idea is to define an effective action for the theory by integrating the high frequency modes of the fields. In figure 5 there is a sketch of the “blocking” procedure, leading from the original variables, the small circles, to the blocked ones, the big circles, defined on a coarser lattice with a lattice spacing larger, in the figure, by a factor two than the original one. The basic ingredient of the transformation is the coupling of the new block variables to the site ones, defined by:

$$T(U_{local}, U_{block}) \quad (7.20)$$

The integration over the old , local variables, produces an effective action among the block ones:

$$e^{-S_{effective}(U_{block})} = \int dU_{local} e^{-S(U_{local})+T(U_{local}, U_{block})} \quad (7.21)$$

If the transformation kernel  $T$  satisfies:

$$\int dU_{block} e^{T(U_{local}, U_{block})} = 1 \quad (7.22)$$

the partition function in the blocked variables is identical to the one in the local variables. The integration of the local variables has tinned the degrees of freedom of the theory, but the “block” theory has the same cutoff effects of the original one. In the new variables, the lattice spacing is larger ( twice the old one) and the physical correlation length in units of the NEW lattice spacing is SMALLER than the old one:

The expected effective action “moves” in the coupling constant space away from the critical point ( small  $g$ ) and toward the strong coupling region.

The effective action itself cannot be expected to have the simple form of the original one. In general many couplings are generated beside the “elementary plaquette” coupling and one is forced to truncate the effective action to a finite set of coupling, reducing the precision of the blocking transformation.

The strategy of the project was to integrate numerically and iteratively local variables along the “renormalised trajectory” to move from large  $\beta$  ( small  $g$ ) to small  $\beta$  ( large  $g$ ) until one would reach the region of large couplings where an analytic strong coupling expansion would converge in a reliable way.

The project failed mainly because of the limited computer resources available at that time that required wild approximations for the form of the effective action with a consequent loss of precision of the method.

In '79, M.Creutz showed numerically that the confinement property valid in the strong coupling region indeed does extend to the weak cou-

pling region by calculating the potential between two static quarks parametrised as follows:

$$V(R) \simeq \alpha(R)/R + kR \quad (7.23)$$

i.e. as the sum of a Coulomb part and of a part linearly rising with the separation, the confining potential, governed by the “string tension”  $k$ .

He made simulations showing the persistence of the string tension up to values of  $\beta \sim 5.4 - 5.7$  corresponding to a lattice resolution of about  $.25Fm$ . The latter estimate was obtained by comparing the value of the string tension in lattice units extracted from the simulations with the phenomenological value ( extracted from a simple model for resonances) of about half a GeV.

From the 80’s have started the simulations aiming to calculate the hadron spectrum. The first, done by Hamber and Parisi, used a  $5^3 \times 10$  lattice !

Since, there has been an increase in the computing power by about a factor  $10^6$ , i.e. from machines with a peak speed in the Megaflop range ( a Million of Floating Point Operations per second), to present machines close to the Teraflop range. This increase in computing power is much less dramatic when translated into an improvement of the lattice resolution. The point is that the CPU power scales with the number of points per side  $N$  like  $N^{4+\epsilon}$  where  $\epsilon$  is of order one in the most favourable calculations like those involving gluons only, but bigger for those involving fermions. This leads to an improvement in the resolution by about a factor 20 at the best.

A better resolution is not the only obstacle to precise numerical estimates. As already mentioned, the simulations involving virtual fermions, i.e. including in the gauge action the effective part due to integration over fermion loops, are more demanding by about a factor 100 than those where fermion loops have been “manually” suppressed ( the “quenching” approximation). Furthermore, the light quark mass values cannot be set equal to the physical ones of a few MeV without stepping into severe finite volume effects that increase exponentially, for a fixed physical volume, when the pion mass goes to zero. Most of the results are obtained by chiral extrapolations

of calculations made at sizeable quark masses, for example of the order of a 100 MeV, when the total lattice size does not exceed a couple of Fermi.

The onset of lattice artifacts, with a given number of lattice points, depends upon the form of the action that is not unique. They are of order  $a^2$  for the pure gauge part of the Wilson action, but of order  $a$  for the Wilson fermion action, where a problem not mentioned in this rapid survey, the fermion doubling, is solved by explicitly breaking the chirality on the lattice.

Indeed, there are two approaches to use the freedom in the choice of the action to deal with acceptable lattice artefacts. The first is the one already presented and due to Wilson. The action along the renormalised trajectory is “perfect”, i.e. embodies the CONTINUUM properties, but is very complicated to determine. The second approach, originally suggested by Symanzik, consists in a systematic “improvement” of the action. The lattice action can be expanded in a power series in the lattice spacing:

$$L_{lattice} = L_{continuum}^0 + aL_{continuum}^1 + a^2L_{continuum}^2 + \dots \quad (7.24)$$

and the approach aims to eliminate lattice artefacts up to a finite order in the lattice spacing. For example, the  $L_{continuum}^1$  part can be eliminated by a single “counterterms” added to the lattice action with a coupling non-perturbatively tuned. However, physical observables involving composite operators in general also require an additional operator improvement.

I will present a selection of results for the spectrum, quenched and unquenched, for the  $\eta'$ , for some quantities “running” with the scale, like  $\alpha_S(Q^2)$  or the first moment of valence quark distribution  $\int dx x q(x, Q^2)$ . There many other subject where there is a lot of activity in the lattice community, and in particular on the calculation of hadronic matrix elements of the weak Hamiltonian, on the study of the high temperature properties of the theory, on bound states with exotic quantum numbers and on the definition of an exact chiral symmetry on the lattice that would allow a non-perturbative study of the standard model. Figure 6 shows the octet baryon

quenched spectrum as a function of the pseudoscalar mass squared both normalized by the  $K^*$  mass at a fixed lattice spacing slightly below  $0.1\text{Fermi}$ , at  $\beta = 6.2$ , and for the standard and the improved cases. The same plots for the decuplet is in figure 7 where the partial cancellation of lattice artifacts brings into a better agreement with experimental data represented by the crosses. The results for unquenched simulations are given in figure 8 and 9. In the first, showing the spectrum, the lowest value of the pion mass reached is higher than in quenched simulations given the much higher computing demand of these calculations. Figure 9 shows the consistency of the lattice spacing extracted from different physical quantities, when extrapolated to the chiral limit. Indeed, in the unquenched case, the quark mass enters in the effective gauge action through the fermion determinant, changes the action and therefore the value of the lattice cut-off. Notice that the  $\beta$  values of unquenched simulations are much lower than those of quenched. If one takes the lattice spacing as function of the bare coupling from perturbation theory:

$$a = 1/\Lambda_{QCD} e^{-1/(b_0 g^2(a))} \quad (7.25)$$

to be the same in the quenched and unquenched case, one gets:

$$\frac{g_{quenched}^2}{g_{unquenched}^2} \sim \frac{b_0^{unquenched}}{b_0^{quenched}} \quad (7.26)$$

and, since  $b_0^{unquenched} < b_0^{quenched}$ , one gets  $g_{unquenched}^2 < g_{quenched}^2$ .

The effects of unquenching are often barely visible, at present values of dynamical quark masses. Figure 10 show the unquenched effective quark potential, where at large enough separation the possibility of creating quark-antiquark pairs should break the string tension, i.e. the potential should flatten out. The horizontal line shows where this should happen but the data do not show a clear indication of the phenomenon. Precise quenched calculations now available show the inadequacy of the quenched approximation in a few cases, like the pseudoscalar meson constants that are larger than their experimental values.

A quantity where unquenching effects are EXPECTED to be crucial is the splitting of the  $\eta'$

mass with respect to the mass of the pseudoscalar octet, due to the anomaly. This is a fermion loop effect and is proportional to the number of flavours running in the loop. One of the first estimates of the effect was done with a trick that replaces fermion statistics with boson statistics. In this case the  $\eta'$  becomes LIGHTER for larger flavour number. The effect of virtual BOSON loops, with a Dirac operator (the "bermions") is easy to simulate by MonteCarlo ( boson fields ARE C numbers) and has been used successfully to find the sign and the order of magnitude of unquenching effects.

Running quantities are difficult to calculate, because one needs to follow their running over many orders of energy scale. This cannot be done on a single lattice that can accomodate only scales differing at most by the number of points per side. A finite volume recursive method allows to "match" simulations done at different values of the lattice spacing and of the bare coupling and to explore a large range of energy scales. These calculations have only been done in the quenched approximation so far. Figure 11 shows the non-perturbative calculation of the running coupling constant. The final precision attainable is of the order of 2 per cent at the reference  $Z_0$  mass scale.

Recently some effort has been devoted to the calculation of the non perturbative running of the average momentum carried by valence quarks. The basic ingredient, present also in the calculation of the running coupling constant, is the "step scaling function", i.e. the change in the running quantity when the scale is changed by a factor two, at various values of the scale, or equivalently of the coupling renormalized at that scale. The extrapolation to the continuum of such a function given in figure 12 shows that a common limit is reached by using different actions ( improved and not) and gives confidence on the reliability of these continuum extrapolations. The ultimate goal, still far in a quenched approximation, is to compute the first few moments of the gluon distribution that may provide an information complementary to the one extracted, mainly at low values of  $x_{Bjorken}$ , from fits to hard processes.



## 8. Conclusions

We have been rapidly surveying different analytic perturbative approaches and we have landed on a numerical non-perturbative method. Solving QCD by computer might not be aesthetically appealing, but 1) it may clarify the validity of perturbative approaches, 2) it may “join” the analytic method of the strong coupling expansion at some point 3) as a byproduct pushes the development of fast algorithms and machines, 4) it is the only method we have to investigate, without further assumptions, the non perturbative aspects of the theory.

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perturbation in	basic degrees of freedom	supported by	name
$\alpha_s$	quarks and gluons	scaling laws	perturbative QCD
$\frac{1}{M_q}$	infinitely heavy quarks	heavy mesons splittings	HQET
$(\frac{E_\pi}{4\pi f_\pi})^2$	light pseudoscalars	PCAC, low energy theorems	$\chi$ pT
$\frac{1}{N_{colors}}$	narrow resonances, planar theory	Zweig rule, sea/valence ratio	$N_c$ expansion
$\beta \equiv \frac{1}{g_{bare}^2}$	quarks and gluons on a crystal	confinement	lattice QCD

Table 1: The summary of perturbative approaches

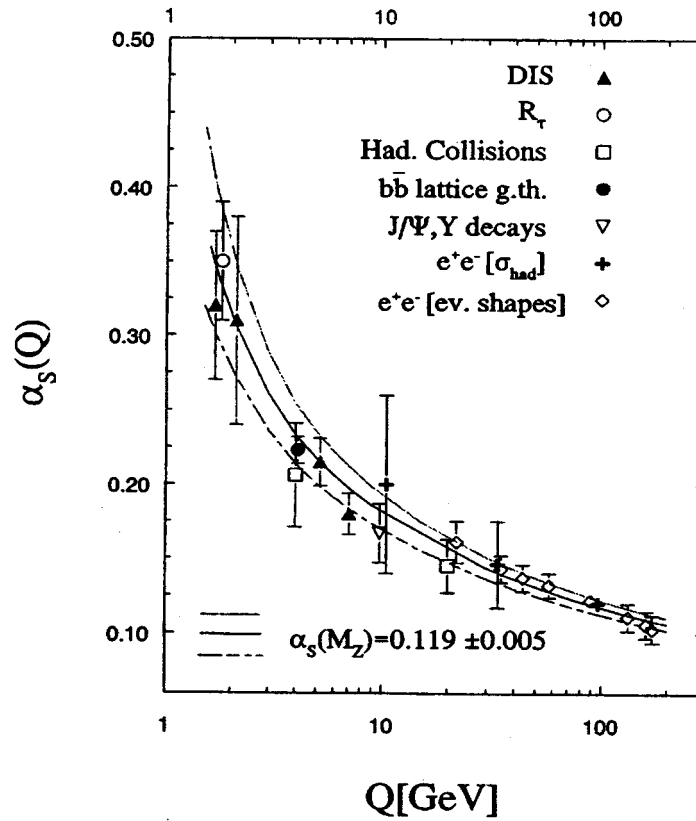


Figure 1: The running of  $\alpha_s$

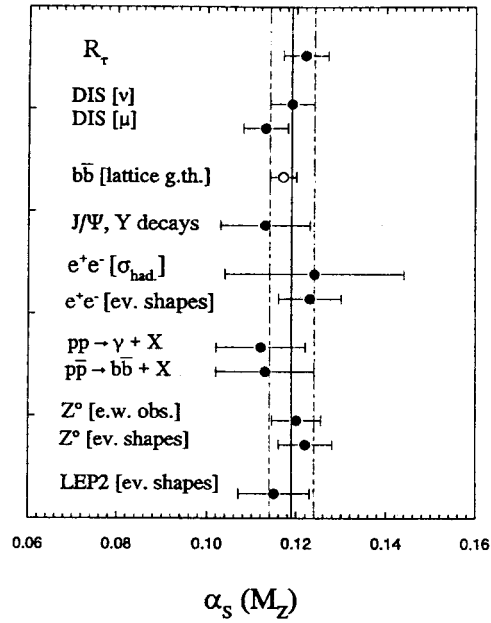


Figure 2: Determinations of  $\alpha_S$  at the  $Z_0$  mass

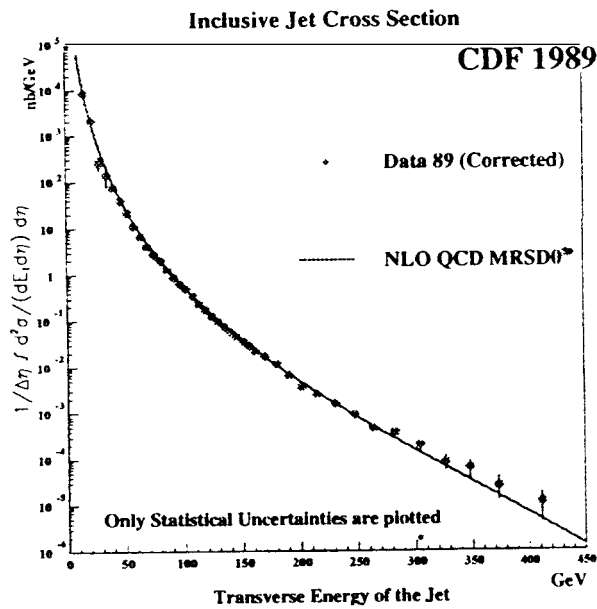
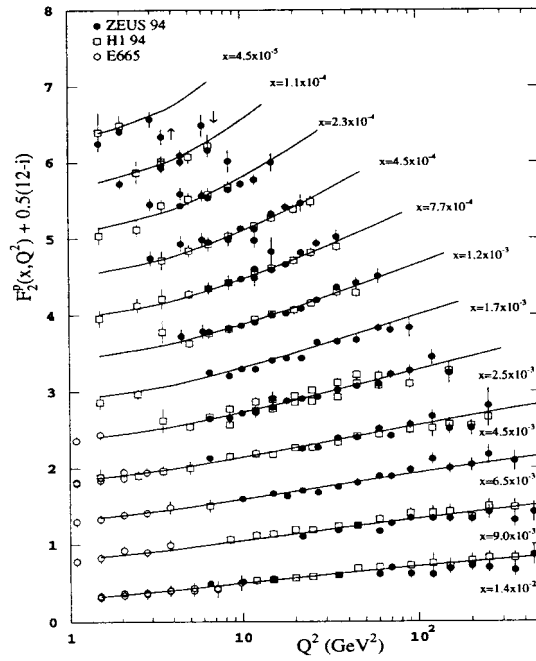
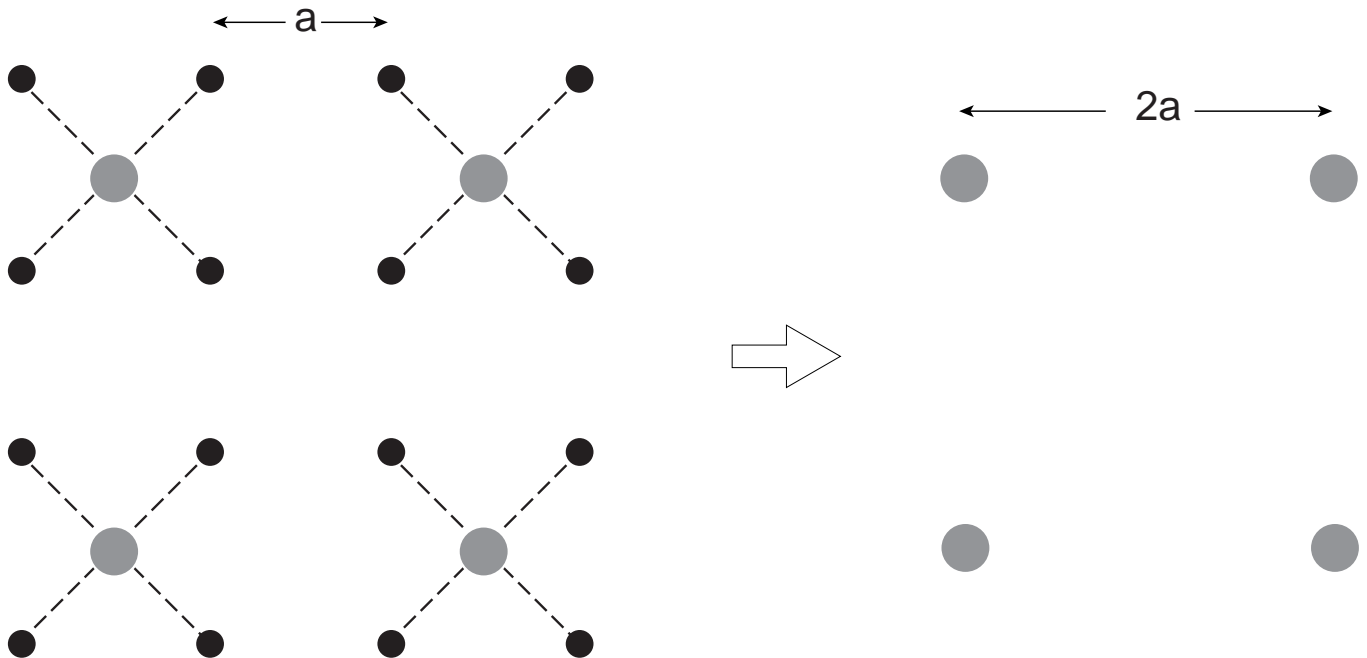


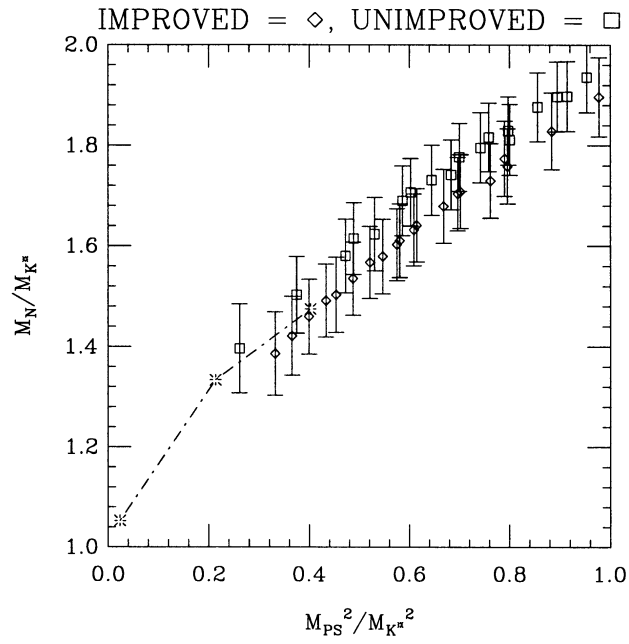
Figure 3: Jet inclusive cross section at Tevatron



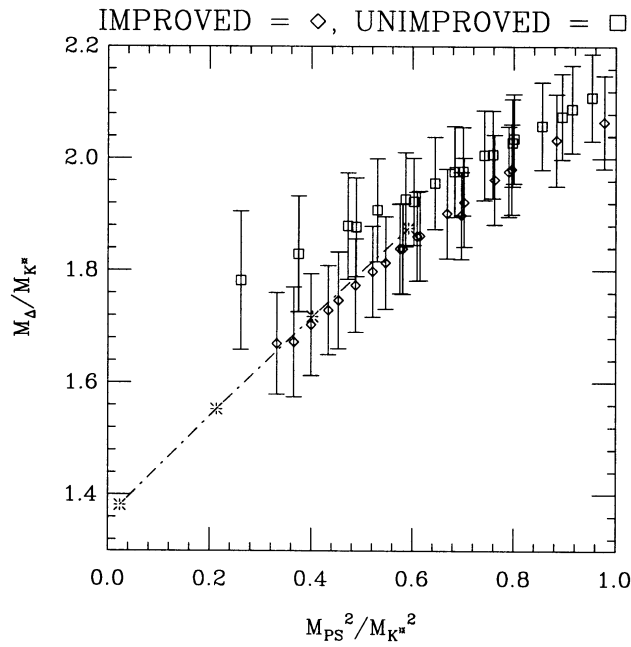
**Figure 4:** Theoretical predictions are compared with observed scaling violations at Hera( DESY-Hamburg)



**Figure 5:** Schematic view of a blocking procedure from local variables ( small circles) to block variables (big circles)



**Figure 6:** The spectrum of the baryon octet in the quenched approximation



**Figure 7:** The spectrum of the baryon decuplet in the quenched approximation

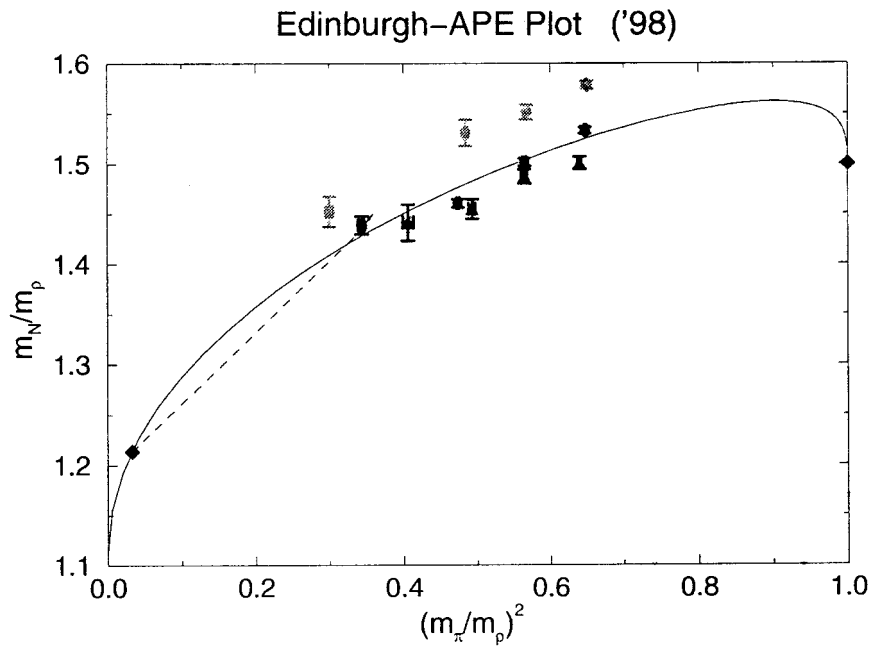


Figure 8: The state of the art for the spectrum in full QCD

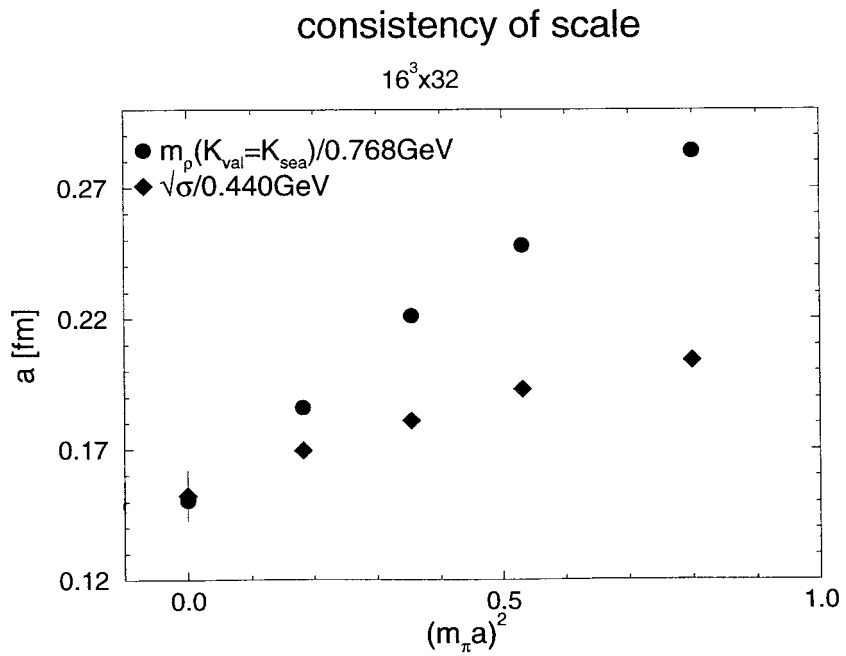


Figure 9: Chiral extrapolation of the lattice spacing in physical units in full QCD

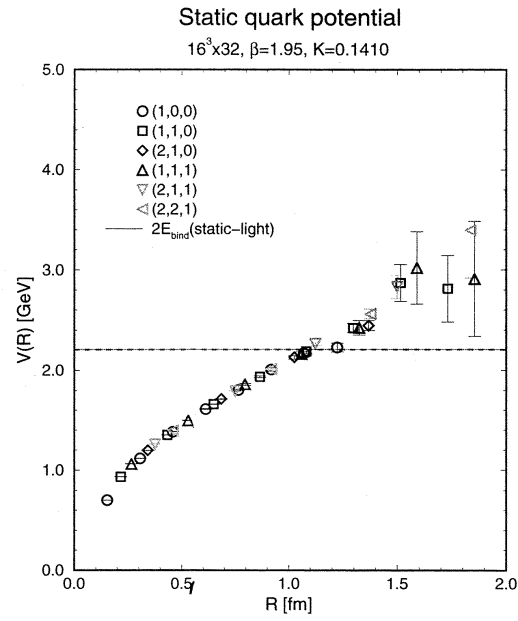


Figure 10: The static quark potential in full QCD

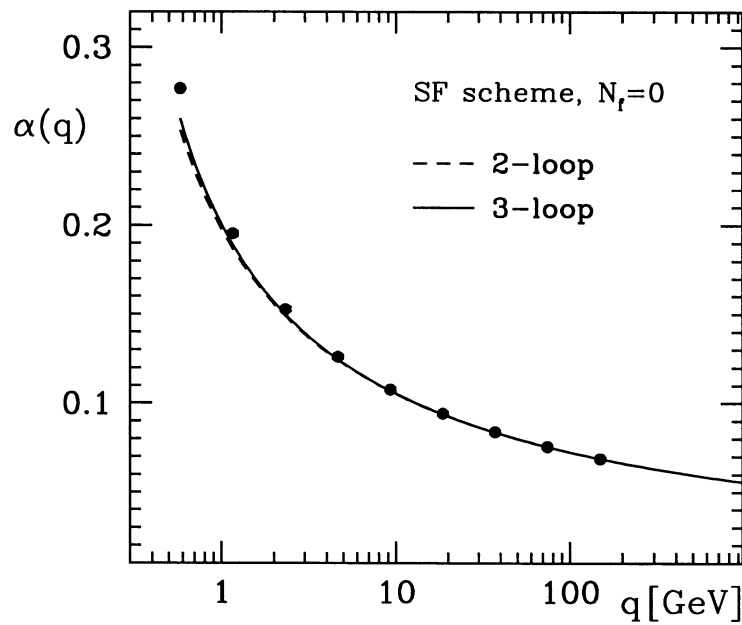
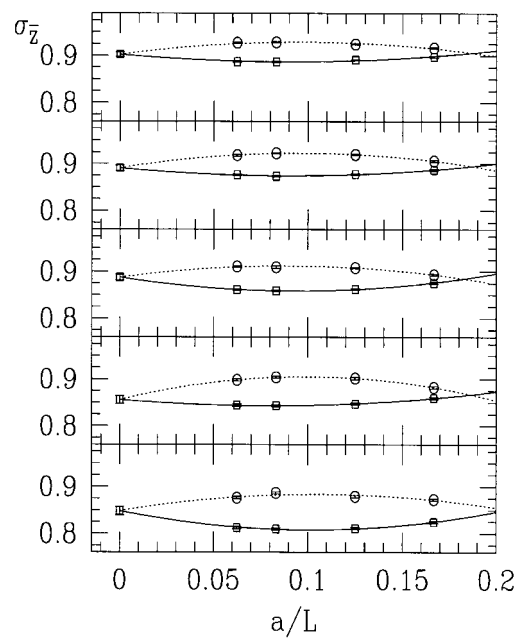


Figure 11: Non-perturbative running of  $\alpha_S$  in quenched QCD



**Figure 12:** Continuum limit of step scaling functions for the running of the the average valence momentum in quenched QCD