q-Deformed Heisenberg-Algebra

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ABSTRACT: Quantum group considerations lead to a discrete spacetime structure. This structure can be imbedded into $\mathbf{R}^n$. The simplest model that shows these features is based on the one dimensional q-deformed Heisenberg algebra and will be discussed in this lecture.

The canonical commutation relations are at the basis of a quantum mechanical system:

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i \qquad (1)$$

The elements of this algebra are supposed to be selfadjoint

$$\overline{\hat{x}} = \hat{x} \quad , \quad \overline{\hat{p}} = \hat{p} \qquad (2)$$

A physical system is defined through a representation of this algebra in a Hilbertspace where self-adjoint elements of the algebra have to be represented by (essentially) selfadjoint linear operators.

In quantum mechanics the elements of (1) are represented in the Hilbertspace of square-integrable functions by:

$$\hat{x} = x \quad , \quad \hat{p} = -i\frac{\partial}{\partial x} \qquad (3)$$

Starting from the algebra (1), the spectrum of the linear operator $\hat{x}$ can be interpreted as the manifold on which the physical system lives - i.e. the configuration space.

In quantum mechanics it is $\mathbb{R}_1 (x \in \mathbb{R}_1)$. The element $\hat{p}$ is a differential operator on this manifold.

We shall change the algebra (1) in accord with quantum group considerations. It is natural to assume that $\hat{x}$ is an element of a quantum plane and to relate $\hat{p}$ to a derivative in such a

plane. With an obvious change in notation we study the algebra

$$\partial x = 1 + qx\partial \qquad (4)$$

More precisely, we study the free algebra generated by the elements x and ∂ and divided by the ideal generated by (4).

If we assume x to be selfadjoint

$$\overline{x} = x \qquad (5)$$

we see that this cannot be the case for $i\partial$ because we find from (4) that:

$$\overline{\partial}x = -\frac{1}{q} + \frac{1}{q}x\overline{\partial} \qquad (6)$$

Thus we could study the algebra generated by x, ∂ and $\overline{\partial}$ and divide by (4), (6) and an ideal generated by $\partial\overline{\partial}$ relations. We compute from (4) and (6) $\partial\overline{\partial}x$ and $\overline{\partial}\partial x$ and find that

$$\overline{\partial}\partial = q\partial\overline{\partial} \qquad (7)$$

is consistent with these calculations. If we now try to define an operator $\hat{p}$ by $\hat{p} = -\frac{i}{2}(\partial - \overline{\partial})$ we find that the $x, \hat{p}$ relations do not close. The real part of ∂ has to be introduced as well. Thus our Heisenberg algebra would have one space and two momentum operators - a system that will hardly find a physical interpretation.

It turns out that $\overline{\partial}$ can be related to ∂ and x in a nonlinear way. This relation involves the

scaling operator Λ:

$$\Lambda \equiv q^{\frac{1}{2}} \left(1 + (q-1)x\partial\right)$$
$$\Lambda x = qx\Lambda \qquad (8)$$
$$\Lambda \partial = q^{-1}\partial\Lambda$$

The scaling property follows from (4).

We now define

$$\tilde{\partial} = -q^{-\frac{1}{2}}\Lambda^{-1}\partial \qquad (9)$$

Λ^{-1} is defined by an expansion in $(q-1)$. We find

$$\tilde{\partial}x = -\frac{1}{q} + \frac{1}{q}x\tilde{\partial}$$
$$\tilde{\partial}\partial = q\partial\tilde{\partial} \qquad (10)$$

Comparing this with (6) and (7) it follows from (9) and (10) that conjugation in the x, ∂ algebra can be defined by

$$\overline{x} = x \quad , \quad \overline{\partial} = -q^{-\frac{1}{2}}\Lambda^{-1}\partial \qquad (11)$$

Conjugating Λ and using (11) shows that

$$\overline{\Lambda} = \Lambda^{-1} \qquad (12)$$

Λ is a unitary element of the algebra, this justifies the factor $q^{\frac{1}{2}}$ in the definition of Λ.

The existence of a scaling operator Λ and the definition of the conjugation (11) seems to be very specific for the x, ∂ algebra (4). It is however generic in the sense that a scaling operator and a definition of conjugation based on it can be found for all the quantum planes defined by $SO_q(n)$ and $SO_q(1,n)$.

The definition of the q-deformed Heisenberg algebra will now be based on the definition of the momentum:

$$p = -\frac{i}{2}(\partial - \overline{\partial}) \qquad (13)$$

It is selfadjoint. From the x, ∂ algebra and the definition of $\overline{\partial}$ follows

$$q^{\frac{1}{2}}xp - q^{-\frac{1}{2}}px = i\Lambda^{-1}$$
$$\Lambda x = qx\Lambda \qquad \Lambda p = q^{-1}p\Lambda \qquad (14)$$

and

$$\overline{p} = p \quad , \quad \overline{x} = x \quad , \quad \overline{\Lambda} = \Lambda^{-1} \qquad (15)$$

These algebraic relations can be verified in the x, ∂ representation where the ordered x, ∂ monomials form a basis. We shall take (14) and

(15) as the defining relations for the q-deformed Heisenbergalgebra without making further reference to its x, ∂ representation.

Let us discuss the representations of (14). We assume p to be represented by an essentially selfadjoint operator and therefore diagonalyzable. We assume p to be diagonal. Let p_0 be an eigenvalue and $| 0\rangle$ its eigenstate.

$$p \mid 0\rangle = p_0 \mid 0\rangle \qquad (16)$$

From (14) follows that $\Lambda^n \mid 0\rangle$ will have the eigenvalue $q^n p_0$

$$p \mid n\rangle = q^n p_0 \mid n\rangle$$
$$\Lambda \mid n\rangle = \mid n+1\rangle \qquad (17)$$

As Λ is unitary and p selfadjoint we find:

$$\langle m \mid n\rangle = \delta_{mn} \qquad (18)$$

As Hilbertspace H_{p_0} we take the completion of the linear span D_{p_0}

$$D_{p_0} = \left\{ \sum_{finite} c_n \mid n\rangle \right\} \qquad (19)$$

$$H_{p_0} = \left\{ \sum_{n=-\infty}^{\infty} c_n \mid n\rangle \quad , \quad \sum_{n=-\infty}^{\infty} \mid c_n \mid^2 < \infty \right\}$$

On this Hilbertspace x can be represented as follows:

$$x \mid n\rangle \qquad (20)$$
$$= \frac{i}{p_0} \frac{q^{-n}}{q - \frac{1}{q}} \left(q^{\frac{1}{2}} \mid n-1\rangle - q^{-\frac{1}{2}} \mid n+1\rangle \right)$$

The eqns (17) and (20) define a representation of the algebra (14). In this representation p is essentially selfadjoint and Λ is unitary. The operator x is symmetric:

$$\overline{\langle m \mid x \mid n\rangle} = \langle n \mid x \mid m\rangle \qquad (21)$$

It follows from (14) that another representation of the algebra can be otained by defining

$$p^s = \pm sp \quad , \quad x^s = \pm s^{-1}x \quad , \quad \Lambda^s = \Lambda \qquad (22)$$

where s is a real number. For $1 \leq s < q$ (we now consider the case $q \geq 1$) the representations obtained that way will be inequivalent.

We now show that x cannot be essentially selfadjoint in these representations. We assume that it is and thus can be diagonalized. Let $\mid x_0 \rangle$ be an eigenvector with eigenvalue x_0. The operator Λ lowers and Λ^{-1} raises this eigenvalue. The states obtained that way are orthogonal (x essentially selfadjoint).

$$x \mid x_0 \rangle = x_0 \mid x_0 \rangle \qquad (23)$$

$$\mid x_0 \rangle = \sum_{n=-\infty}^{\infty} c_n \mid n \rangle \quad , \quad \sum_{n=-\infty}^{\infty} \mid c_n \mid^2 = 1$$

From the algebra follows

$$\langle x_0 \mid p \mid x_0 \rangle x_0 (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) = 0 \qquad (24)$$

$$s p_0 \sum_{n=-\infty}^{\infty} q^n \mid c_n \mid^2 = 0$$

This contradicts (23).

The story is that x is symmetric but not essentially selfadjoint. It can be shown that x has exactly one eigenvector for each of the eigenvalues $\pm i$. Thus it has a one-parameter family of selfadjoint extensions. But none of these extensions will represent the algebra. This also follows from the argument given above. It is however possible to find reducible representations of the algebra (14) where the selfadjoint extension of x will satisfy the algebra (14). It is obvious from (24)that this is only possible if p_0 is allowed to have both signs. This will be discussed in the next section.

We now show that the algebra (14) can also be represented in the "quantum mechanical" Hilbertspace defined by (1). With the notation

$$\hat{z} = -\frac{i}{2}(\hat{p}\hat{x} + \hat{x}\hat{p}) = -i\hat{p}\hat{x} + \frac{1}{2} = -i\hat{x}\hat{p} - \frac{1}{2} \quad (25)$$

and, for any quantity A:

$$[A] = \frac{q^A - q^{-A}}{q - q^{-1}} \qquad (26)$$

we define

$$p = \hat{p} \qquad (27)$$

$$x = \frac{[\hat{z} + \frac{1}{2}]}{(\hat{z} + \frac{1}{2})}\hat{x}$$

$$\Lambda = q^{-\hat{z}}$$

and show that p, x satisfy the algebraic relations (14) if $\hat{p}$ and $\hat{x}$ satisfy (1). We start from:

$$\hat{x}\hat{z} = (\hat{z} + 1)\hat{x} \quad , \quad \hat{p}\hat{z} = (\hat{z} - 1)\hat{p} \qquad (28)$$

and conclude:

$$\hat{x}f(\hat{z}) = f(\hat{z} + 1)\hat{x} \quad , \quad \hat{p}f(\hat{z}) = f(\hat{z} - 1)\hat{p} \quad (29)$$

Let us now verify the first relation of (14).

$$q^{\frac{1}{2}}\frac{[\hat{z} + \frac{1}{2}]}{(\hat{z} + \frac{1}{2})}\hat{x}\hat{p} - q^{-\frac{1}{2}}\hat{p}\frac{[\hat{z} + \frac{1}{2}]}{(\hat{z} + \frac{1}{2})}\hat{x} = \qquad (30)$$

$$q^{\frac{1}{2}}\frac{[\hat{z} + \frac{1}{2}]}{(\hat{z} + \frac{1}{2})}\hat{x}\hat{p} - q^{-\frac{1}{2}}\frac{[\hat{z} - \frac{1}{2}]}{(\hat{z} - \frac{1}{2})}\hat{p}\hat{x}$$

$$= iq^{\frac{1}{2}}[\hat{z} + \frac{1}{2}] - q^{-\frac{1}{2}}[\hat{z} - \frac{1}{2}] = iq^{\hat{z}} = i\Lambda^{-1}$$

The other relations are verified in the same way. That p is selfadjoint and Λ unitary if $\hat{p}$ and $\hat{x}$ are selfadjoint is obvios. Let us show that x is selfadjoint as well:

$$\hat{x} = \hat{x}\frac{[-\hat{z} + \frac{1}{2}]}{(-\hat{z} + \frac{1}{2})} = \hat{x}\frac{[\hat{z} - \frac{1}{2}]}{(\hat{z} - \frac{1}{2})} \qquad (31)$$

$$= \frac{[\hat{z} + \frac{1}{2}]}{(\hat{z} + \frac{1}{2})}\hat{x} = x$$

The x, p, Λ algebra does not change if $\hat{x}\hat{p}$ are changed by a canonical transformation. A class of such canonical transformations is:

$$\tilde{p} = f(\hat{z})\hat{p} \quad , \quad \tilde{x} = \tilde{x}f^{-1}(\hat{x}) \qquad (32)$$

We immediately see that

$$\tilde{x}\tilde{p} - \tilde{p}\tilde{x} = i$$

The elements $\tilde{p}, \tilde{x}$ will be selfadjoint if

$$\overline{f(\overline{\hat{z}})} = f(\hat{z} + 1) \qquad (33)$$

An example of such an f is:

$$f^{-1}(\hat{z}) = \frac{[\hat{z} - \frac{1}{2}]}{\hat{z} - \frac{1}{2}} \qquad (34)$$

With such a canonical transformation we find:

$$p = \frac{[\tilde{z} - \frac{1}{2}]}{(\tilde{z} - \frac{1}{2})}\tilde{p} \qquad (35)$$

$$x = \tilde{x}$$

$$\Lambda = q^{-\tilde{z}}$$

This is again a representation of the $x\,p\,\Lambda$ algebra.

Let us see how the representation (27) is related to the representations (17), (20), (22). We start from a representation of (1) where $\hat{p}$ is diagonal:

$$\hat{p} \mid k_0\rangle = k_0 \mid k_0\rangle \quad \langle k_0' \mid k_0\rangle = \delta(k' - k_0) \quad (36)$$

Eqn.(22) suggests the notation:

$$p^s \mid n,\sigma\rangle \mid s\rangle = \sigma s q^n p_0 \mid n,\sigma\rangle \mid s\rangle \qquad (37)$$

$$x^s \mid n,\sigma\rangle \mid s\rangle = \sigma s^{-1} p_0^{-1} q^{-n} \frac{i}{q - \frac{1}{q}}$$

$$(q^{\frac{1}{2}} \mid n-1,\sigma\rangle \mid s\rangle - q^{-\frac{1}{2}} \mid n+1,\sigma\rangle \mid s\rangle)$$

$$\langle s \mid s'\rangle = \delta(s - s')$$

$$\sigma = \pm 1 \quad , \quad 1 \le s < q \quad .$$

The representation (27) can be reduced:

$$\mid n,\sigma\rangle \mid s\rangle = \int dk_0 q^{\frac{n}{2}} \delta(k_0 - \sigma s q^n) \mid k_0\rangle \quad (38)$$

Normalization:

$$\langle n',\sigma' \mid \langle s' \mid\mid n,\sigma\rangle \mid s\rangle \qquad (39)$$

$$= \int dk_0 dk_0' q^{\frac{1}{2}(n+n')}$$

$$\delta(k_0 - k_0')\delta(k_0 - \sigma s q^n)\delta(k_0' - \sigma' s' q^{n'})$$

$$= \int dk_0 q^{\frac{1}{2}(n+n')} \delta(k_0 - \sigma s q^n)\delta(k_0 - \sigma' s' q^{n'})$$

$$= q^{\frac{1}{2}(n+n')}\delta(\sigma' s' q^{n'} - \sigma s q^n)$$

$$= q^{\frac{1}{2}(n+n')} \frac{1}{q^n}\delta_{n,n'}\delta_{\sigma,\sigma'}\delta(s - s')$$

$$= \delta_{n,n'}\delta_{\sigma,\sigma'}\delta(s - s') \quad .$$

The transformation (38) can be inverted:

$$\mid k_0\rangle = \qquad\qquad\qquad\qquad\qquad (40)$$

$$\int_1^s ds \sum_{n=-\infty}^{\infty} \sum_{\sigma=+,-} q^{\frac{n}{2}} \delta(k_0 - \sigma s q^n) \mid n,\sigma\rangle \mid s\rangle$$

Let us now act with Λ as defined by (27) on (38). On a wavefunction in the k_0 representation (36) $\hat{x}$ acts as $i\frac{\partial}{\partial k_0}$. The action of Λ then is

$$\Lambda f(k_0) = q^{-\frac{1}{2}} q^{-k_0 \frac{\partial}{\partial k_0}} f(k_0) \qquad (41)$$

$$= q^{-\frac{1}{2}} f(q^{-1} k_0)$$

$$\Lambda q^{\frac{n}{2}} \delta(k_0 - \sigma s q^n) = q^{-\frac{1}{2}} q^{\frac{n}{2}} \delta(q^{-1} k_0 - \sigma s q^n)$$

$$= q^{\frac{1}{2}(n+1)} \delta(k_0 - \sigma s q^{(n+1)})$$

$$\Lambda \mid n\sigma\rangle \mid s\rangle = \mid n+1,\sigma\rangle \mid s\rangle$$

The action of x can be otained from (41) by

$$x = \frac{i}{q - \frac{1}{q}}(q^{\frac{1}{2}}\Lambda^{-1} - q^{-\frac{1}{2}}\Lambda)p^{-1} \qquad (42)$$

Its action on $\mid n,\sigma\rangle \mid s\rangle$ is identical with (37). This shows that the representation (27) decomposes into a direct integral of representations of the type (20) and (22).