Axial Vector Amplitudes, Second Order Fermions, and Standard Model Photon – Neutrino Processes

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Abstract: We present an extension of the string-inspired technique suitable to the calculation of amplitudes and effective Lagrangians involving both vector and axial vector gauge fields. The technique is easily adaptable to problems involving constant external fields. We demonstrate the advantages of the formalism by a calculation of the one-loop vector-axialvector amplitude in a general constant electromagnetic field. The relevance of this calculation for photon-neutrino processes is commented upon. We also clarify the properties of the formalism with respect to the chiral symmetry, and its connection to second order fermions.

1. Introduction

The idea of using string theory methods for the calculation of amplitudes in ordinary field theory was originally introduced in the context of QCD [1, 2] and quantum gravity [3]. In particular, it was used for the first complete calculations of the one-loop five gluon amplitude [4] and the one-loop four graviton amplitude [5]. Later it was found that even in QED similar techniques can lead to significant improvements over standard field theory methods [6, 7, 8], particularly for processes involving constant external fields [9, 10, 11, 12, 13, 14, 15].

The present talk is devoted to an extension of the "string-inspired" technique suitable to the calculation of processes involving both vector and axial-vector couplings [16, 17]. In contrast to previous attempts at such a generalization [18, 19] this extension allows one to treat both the real and the imaginary part of these amplitudes in a unified way.

In the introduction we first give some physical motivation for the study of vector – axialvector amplitudes. In section 2 we explain the second-order formalism for spinor QED [20, 21, 22] and its recent extension to the vector-axialvector case [16, 17]. The second-order formalism is then used in section 3 to derive a path-integral representation for the one-loop effective action due to a Dirac fermion loop coupled to abelian vector and axialvector backgrounds. We explain in detail how this path integral can be used for the calculation of one-loop N - point amplitudes. In section 4 we discuss the behaviour of the resulting formalism with respect to the "γ_5 - problem" of dimensional regularisation, including a recalculation of the ABJ anomaly. In Section 5 we apply this technique to calculations in constant background fields. We give explicit results for the Scalar and Spinor QED vacuum polarisation tensors in a general constant field, as well as for the corresponding vector-axialvector amplitude. Our conclusions are given in section 6.

Processes involving photons and neutrinos are presently of importance primarily for astrophysics and cosmology [23]. In the standard model such processes appear at the one-loop level. A typical example would be the diagram shown in fig. 1 which contributes to the process $\gamma\gamma \to \nu\bar{\nu}$.

The $2 \to 2$ processes $\gamma\gamma \to \nu\bar{\nu}$, $\gamma\nu \to \gamma\nu$ and $\nu\bar{\nu} \to \gamma\gamma$ were considered already before the advent of the standard model using the four-Fermi

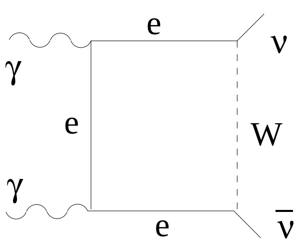


Figure 1: Diagram contributing to $\gamma\gamma \to \nu\bar{\nu}$.

interaction [24]. However in the Fermi limit they vanish due to the Landau-Yang theorem, as was noted by Gell-Mann [25] (for massless neutrinos, and with both photons on-shell). In the standard model this suppression manifests itself by factors of $\frac{\omega}{M_W}$, where ω is the center-of-mass energy and M_W the W boson mass. Nevertheless the $2 \to 2$ processes could be of importance in very high energy reactions [26, 27], and the one-loop helicity amplitudes for $\gamma\gamma \to \nu\bar{\nu}$ were calculated in [27].

There is no such suppression for processes involving two neutrinos and more than two photons, which should therefore be more important at low energies than the four-leg processes. Following early work by Van Hieu and Shabalin [28] recently the two-neutrino three-photon processes such as $\gamma\gamma \to \gamma\nu\bar{\nu}$, $\gamma\nu \to \gamma\gamma\nu$, $\nu\bar{\nu} \to \gamma\gamma\gamma$ have been investigated closely [29, 30, 31]. For center-of-mass energies 2ω between 1 keV and 1 MeV the cross sections for $2 \to 3$ processes turn out to be indeed larger than the corresponding $2 \to 2$ cross sections [29].

Many more photon-neutrino processes become possible if one admits neutrino masses or anomalous magnetic dipole moments [32].

In astrophysical environments it is often not realistic to consider these processes as occuring in vacuum. Plasma effects must be taken into account, as well as the presence of magnetic fields, which around pulsars can have field strengths surpassing the critical value $B_{\rm crit} = \frac{m_e^2}{e} = 4.41 \times 10^{13}$ Gauss. Of particular interest are then processes which do not occur in vacuum but become possible in a medium or B-field. An important example is the plasmon decay $\gamma \to \nu\bar{\nu}$ [33, 34], which is believed to be the dominant source for neutrino production in many types of stars. Similarly the Cherenkov process $\nu \to \nu\gamma$ becomes possible though it turns out to be of much lesser astrophysical relevance [35, 32]. In processes of this type the magnetic field plays a double role. Firstly, it provides an effective photon-neutrino coupling via intermediate charged particles [35, 36, 37]. Secondly, by modifying the photon dispersion relations [38, 39, 40, 41] it opens up phase space for neutrino-photon reactions of the type $1 \to 2 + 3$.

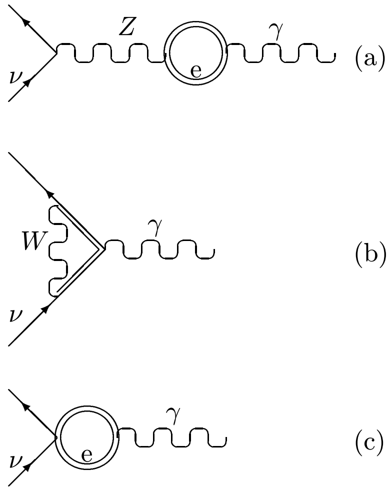


Figure 2: Neutrino-photon coupling in an external magnetic field. (a) Z-A-mixing. (b) Penguin diagram (only for ν_e). (c) Effective coupling in the Fermi limit.

Similarly one would expect the magnetic field to remove the Fermi limit suppression of the above $2 \to 2$ processes. This has recently been verified both for the weak [42] and for the strong field case [43].

In the standard model the effective coupling is provided by the diagrams shown in fig. 2(a) and 2(b) [1]. The double line represents the electron propagator in the presence of the B-field. In the limit of infinite gauge-boson masses both diagrams can be replaced by diagram fig. 2(c). The amplitude then effectively reduces to a purely photonic amplitude with one of the photons replaced by the neutrino current. One is therefore led to the study of mixed vector-axialvector

[1] I thank A.N. Ioannisian for providing this figure.

amplitudes, or alternatively of the corresponding effective action [29, 42, 44].

2. Second-Order Fermions

Let us thus consider the (Euclidean) one-loop effective action for a Dirac fermion coupled to (abelian) vector and axialvector background fields,

$$\Gamma[A, A_5] = \ln \mathrm{Det}[\not p + e\not A + e_5\gamma_5\not A_5 - im] \quad (2.1)$$

It is easily shown that

$$(\not p + e\not A + e_5\gamma_5 \not A_5)^2 = -(\partial_\mu + i\mathcal{A}_\mu)^2 + V \quad (2.2)$$

where

$$\begin{aligned} \mathcal{A}_\mu &\equiv eA_\mu - e_5\gamma_5\sigma_{\mu\nu}A_5^\nu \\ V &\equiv -i\frac{e}{2}\sigma_{\mu\nu}\Bigl(\partial_\mu A_\nu - \partial_\nu A_\mu\Bigr) + ie_5\gamma_5 A^\mu_{5,\mu} \\ &\quad +(D-2)e_5^2A_5^2 \end{aligned} \quad (2.3)$$

($\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$). We have used the four - dimensional Dirac algebra relations, but continued to D dimensions using an anticommuting γ_5. Appealing to the usual argument that

$$\begin{aligned} &\mathrm{Det}\Bigl[(\not p + e\not A + \gamma_5 e_5 \not A_5) - im\Bigr] \\ &= \mathrm{Det}\Bigl[(\not p + e\not A + \gamma_5 e_5 \not A_5) + im\Bigr] \\ &= \mathrm{Det}^{1/2}\Bigl[(\not p + e\not A + \gamma_5 e_5 \not A_5)^2 + m^2\Bigr] \end{aligned} \quad (2.4)$$

we can write the effective action also in the following form,

$$\Gamma = -\frac{1}{2}\,\mathrm{Tr}\int_0^\infty \frac{dT}{T}\,\exp\Bigl\{-T[-(\partial_\mu + i\mathcal{A}_\mu)^2 + V + m^2]\Bigr\} \quad (2.5)$$

Up to the global sign, this is formally identical with the effective action for a scalar loop in a background containing a (Clifford algebra valued) gauge field $\mathcal{A}$ and a potential V. Note that the exponent is not hermitian. This distinguishes the present approach from previous ones [18, 19], and makes it possible to avoid the splitting of the effective action into its real and imaginary parts.

For the pure vector (i.e. QED) case this representation of the effective action is quite well-known. In this case eq.(2.2) expresses the square of the Dirac operator in terms of the Klein-Gordon operator with an additional spin term $\sim \sigma_{\mu\nu}F^{\mu\nu}$. This has been used to construct a non-standard "second-order" version of spinor QED which is equivalent to the ordinary formulation, but has a quite different set of Feynman rules [20, 21, 22]. Those rules are shown in the upper part of fig. 3 (in Euclidean conventions).

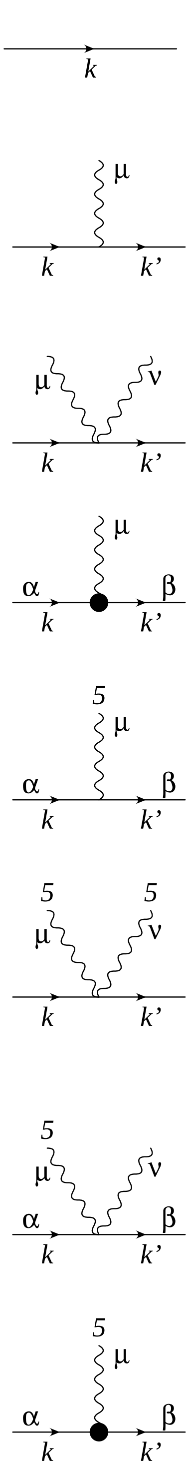
The electron propagator is of scalar form as shown in the first line (the photon propagator is as usual). The first two vertices are the same as in scalar QED, while the third one represents the above spin term. Closed fermion loops require a spinor trace and a factor of $-\frac{1}{2}$. For the treatment of external fermions in the second order formalism see [22].

In the presence of both vectors and axialvectors there are four more vertices shown in the lower part of fig. 3. Those can also be directly read off eq.(2.2). Again these rules are guaranteed to yield the same amplitudes as in the usual first-order formalism, but rewritten in a very different Feynman diagram expansion.

3. Path Integral Representation of Vector – Axialvector Couplings

Rather than applying the second-order formalism directly, we use it for the derivation of a worldline path integral suitable for vector – axialvector calculations [16, 17]. Using standard coherent state methods [45] the functional trace in our representation eq.(2.5) for the effective action can be transformed into the following quantum mechanical double path integral,

$$\begin{aligned} \Gamma &= -\frac{1}{2}\int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}\int \mathcal{D}x\int \mathcal{D}\psi\;\mathrm{e}^{-\int_0^T d\tau\,L(\tau)} \\ L &= \frac{1}{4}\dot x^2 + \frac{1}{2}\psi\cdot\dot\psi + ie\dot x\cdot A - ie\psi\cdot F\cdot\psi \\ &\quad + ie_5\hat\gamma_5\Bigl[-2\dot x\cdot\psi\psi\cdot A_5 + \partial\cdot A_5\Bigr] \\ &\quad +(D-2)e_5^2A_5^2 \end{aligned} \quad (3.1)$$



$$\frac{1}{k^2+m^2}$$

$$e\,(k+k')_\mu$$

$$-2e^2 g_{\mu\nu}$$

$$e(\sigma_{\mu\nu})_{\beta\alpha}\,(k'-k)^\nu$$

$$e_5(\gamma_5\sigma_{\mu\nu})_{\beta\alpha}\,(k+k')^\nu$$

$$2e_5^2 g_{\mu\nu}$$

$$-2ee_5(\gamma_5\sigma_{\mu\nu})_{\beta\alpha}$$

$$e_5(\gamma_5)_{\beta\alpha}\,(k'-k)_\mu$$

Figure 3: Second order Feynman rules.

Here T denotes the usual Schwinger proper-time parameter for the fermion circulating in the loop. The "coordinate" path integral $\int \mathcal{D}x$ has to be performed over the space of closed loops in spacetime $x^\mu(\tau)$ with period T, $x^\mu(T) = x^\mu(0)$. The second path integral $\int \mathcal{D}\psi$, which takes the fermion spin into account, is to be integrated over the space of Grassmann-valued functions $\psi^\mu(\tau)$. The boundary conditions on the Grassmann path integral are, after expansion of the interaction exponential, determined by the power of $\hat{\gamma}_5$ appearing in a given term; they are (anti) periodic with period T if that power is even (odd). After the boundary conditions are determined $\hat{\gamma}_5$ can be replaced by unity. Comparing (3.1) with (2.5) one notes that the coherent state formalism has produced a formal correspondence $[\gamma^\mu,\gamma^\nu] \to 4\psi^\mu\psi^\nu$, $\gamma_5 \to \hat{\gamma}_5$.

We proceed to the perturbative calculation of this path integral following the recipes of the "string-inspired formalism". Before tackling the full vector – axialvector case it will, however, be useful to first study the simpler cases of the Scalar QED and Spinor QED photon scattering amplitudes.

3.1 Scalar Quantum Electrodynamics

For the case of scalar QED, the analogue of (3.1) is well-known and goes, in fact, back to Feynman [46]. The one-loop effective action due to a scalar loop for a Maxwell background can be written as

$$\Gamma[A] = \int_0^\infty \frac{dT}{T}\, e^{-m^2T} \int_{x(T)=x(0)} \mathcal{D}x(\tau) \times \mathrm{e}^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ie\,\dot{x}\cdot A(x(\tau))\right)} \qquad (3.2)$$

If we expand the "interaction exponential",

$$\exp\Bigl[-\int_0^T d\tau\, ieA_\mu\dot{x}^\mu\Bigr] = \sum_{N=0}^\infty \frac{(-ie)^N}{N!}\prod_{i=0}^N \int_0^T d\tau_i \times \Bigl[\dot{x}^\mu(\tau_i)A_\mu(x(\tau_i))\Bigr] \qquad (3.3)$$

the individual terms correspond to Feynman diagrams describing a fixed number of interactions of the scalar loop with the external field (fig. 4). The corresponding N – photon scattering amplitude is then obtained by specializing to a background consisting of a sum of plane waves with definite polarizations,

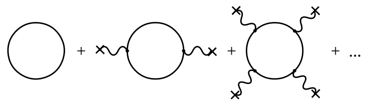


Figure 4: Expanding the path integral in powers of the background field.

$$A_\mu(x) = \sum_{i=1}^{N} \varepsilon_{i\mu} \mathrm{e}^{ik_i\cdot x} \tag{3.4}$$

and picking out the term containing every ε_i once. This yields the following representation for the N - photon amplitude,

$$\Gamma[\{k_i,\varepsilon_i\}] = (-ie)^N \int_0^\infty \frac{dT}{T}\, e^{-m^2T} (4\pi T)^{-\frac{D}{2}} \times \Bigl\langle V_A^0[k_1,\varepsilon_1] \ldots V_A^0[k_N,\varepsilon_N] \Bigr\rangle \tag{3.5}$$

Here V_A^0 denotes the same photon vertex operator as is used in string perturbation theory [47],

$$V_A^0[k,\varepsilon] \equiv \int_0^T d\tau\, \varepsilon\cdot\dot x(\tau)\, \mathrm{e}^{ikx(\tau)} \tag{3.6}$$

At this stage the path integral has become Gaussian, which reduces its evaluation to the task of Wick contracting the expression

$$\Bigl\langle \dot x_1^{\mu_1} \mathrm{e}^{ik_1\cdot x_1} \cdots \dot x_N^{\mu_N} \mathrm{e}^{ik_N\cdot x_N} \Bigr\rangle \tag{3.7}$$

The Green's function to be used is simply the one for the second-derivative operator, acting on periodic functions. To derive this Green's function, first observe that $\int \mathcal{D}x(\tau)$ contains the constant functions, which must be removed to obtain a well-defined inverse. One therefore restricts the integral over the space of all loops by fixing the average position x_0^μ of the loop,

$$x_0^\mu \equiv \frac{1}{T}\int_0^T d\tau\, x^\mu(\tau) \tag{3.8}$$

For effective action calculations this reduces the effective action to the effective Lagrangian. In scattering amplitude calculations, the integral over x_0 just gives momentum conservation. The reduced path integral $\int \mathcal{D}y(\tau)$ over $y(\tau) \equiv x(\tau) - x_0$ has an invertible kinetic operator. This inverse is

$$2\Bigl\langle \tau_1 \mid \Bigl(\frac{d}{d\tau}\Bigr)^{-2} \mid \tau_2 \Bigr\rangle = G_B(\tau_1,\tau_2) \tag{3.9}$$

with the "bosonic" worldline Green's function

$$G_B(\tau_1,\tau_2) = \mid \tau_1 - \tau_2 \mid - \frac{(\tau_1-\tau_2)^2}{T} \tag{3.10}$$

For the performance of the Wick contractions it is convenient to formally exponentiate all the $\dot x_i$'s, writing

$$\varepsilon_i\cdot\dot x_i \mathrm{e}^{ik_i\cdot x_i} = \mathrm{e}^{\varepsilon_i\cdot\dot x_i + ik_i\cdot x_i} \mid_{\mathrm{lin}(\varepsilon_i)} \tag{3.11}$$

This allows one to rewrite the product of N photon vertex operators as an exponential. Then one needs only to "complete the square" to arrive at the following closed expression for the one-loop N - photon amplitude,

$$\begin{aligned}\Gamma[k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N] &= (-ie)^N (2\pi)^D \delta(\textstyle\sum k_i) \\ &\times \int_0^\infty \frac{dT}{T} [4\pi T]^{-\frac{D}{2}} e^{-m^2T} \prod_{i=1}^N \int_0^T d\tau_i \\ &\times \exp\Biggl\{ \sum_{i,j=1}^N \bigl[\frac{1}{2} G_{Bij} k_i\cdot k_j + i\dot G_{Bij} k_i\cdot\varepsilon_j \\ &\quad + \frac{1}{2}\ddot G_{Bij} \varepsilon_i\cdot\varepsilon_j\bigr]\Biggr\} \mid_{\mathrm{multi-linear}} .\end{aligned} \tag{3.12}$$

Here it is understood that only the terms linear in all the $\varepsilon_1,\ldots,\varepsilon_N$ have to be taken. Besides the Green's function G_B also its first and second derivatives appear,

$$\begin{aligned}\dot G_B(\tau_1,\tau_2) &= \mathrm{sign}(\tau_1-\tau_2) - 2\frac{(\tau_1-\tau_2)}{T} \\ \ddot G_B(\tau_1,\tau_2) &= 2\delta(\tau_1-\tau_2) - \frac{2}{T}\end{aligned} \tag{3.13}$$

Dots generally denote a derivative acting on the first variable, $\dot{G}_B(\tau_1,\tau_2) \equiv \frac{\partial}{\partial\tau_1}G_B(\tau_1,\tau_2)$, and we abbreviate $G_{Bij} \equiv G_B(\tau_i,\tau_j)$ etc. The factor $[4\pi T]^{-\frac{D}{2}}$ represents the free Gaussian path integral determinant.

The expression (3.12) is identical with the "Bern-Kosower Master Formula" for the special case considered [2]. Let us consider explicitly the vacuum polarisation case, $N = 2$. For $N = 2$ the expansion of the exponential factor yields the following expression,

$$\Bigl(\ddot{G}_{B12}\varepsilon_1\cdot\varepsilon_2+\dot{G}^2_{B12}\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1\Bigr)\mathrm{e}^{G_{B12}k_1\cdot k_2} \tag{3.14}$$

Now one could just plug in the explicit formulas from eqs.(3.10),(3.13) and do the integrals. It is useful, however, to first perform a partial integration on the first term of eq. (3.14) in either τ_1 or τ_2. The integrand then turns into

$$\Bigl(\varepsilon_1\cdot\varepsilon_2 k^2 - \varepsilon_1\cdot k\varepsilon_2\cdot k\Bigr)\dot{G}^2_{B12}\mathrm{e}^{-G_{B12}k^2} \tag{3.15}$$

($k = k_1 = -k_2$). Thus the partial integration leads to the appearance of a transversal projector, making the transversality of the vacuum polarization amplitude manifest at the parameter integral level. We rescale to the unit circle, $\tau_i = Tu_i, i = 1,2$, and use translation invariance in τ to fix the zero to be at the location of the second vertex operator, $u_2 = 0, u_1 = u$. We have then

$$\begin{aligned} G_B(\tau_1,\tau_2) &= Tu(1-u)\\ \dot{G}_B(\tau_1,\tau_2) &= 1-2u \end{aligned} \tag{3.16}$$

After performing the trivial T - integration one arrives at

$$\begin{aligned} \Gamma^{\mu\nu}_{\rm scal}[k] &= \frac{e^2}{(4\pi)^{\frac{D}{2}}}\Bigl(k^\mu k^\nu - g^{\mu\nu}k^2\Bigr)\Gamma\bigl(2-\frac{D}{2}\bigr)\\ &\times\int_0^1 du(1-2u)^2\Bigl[m^2+u(1-u)k^2\Bigr]^{\frac{D}{2}-2} \end{aligned} \tag{3.17}$$

The result of the final integration is, of course, the same as is found in the standard field theory calculation of the corresponding two Feynman diagrams.

3.2 Spinor Quantum Electrodynamics

For spinor QED, the worldline representation of the one-loop effective action is just eq.(3.1) with $A_5 = 0$,

$$\begin{aligned} \Gamma[A] &= -\frac{1}{2}\int_0^\infty\frac{dT}{T}e^{-m^2T}\int\mathcal{D}x\int\mathcal{D}\psi\\ &\times\exp\Bigl[-\int_0^T d\tau\Bigl(\frac{1}{4}\dot{x}^2+\frac{1}{2}\psi\dot{\psi}\\ &+ieA\cdot x - ie\psi\cdot F\cdot\psi\Bigr)\Bigr] \end{aligned} \tag{3.18}$$

The calculation of the x - path integral proceeds as before. Concerning the ψ - path integral, first note that ψ is antiperiodic in the absence of $\hat{\gamma}_5$, so that there is no zero mode. To find the appropriate "fermionic" worldline Green's function G_F, we thus need to invert the first derivative in the space of anti-periodic functions. This yields

$$2\Bigl\langle\tau_1\mid\bigl(\frac{d}{d\tau}\bigr)^{-1}\mid\tau_2\Bigr\rangle = \mathrm{sign}(\tau_1-\tau_2)\equiv G_F(\tau_1,\tau_2) \tag{3.19}$$

Thus we have now the following two Wick contraction rules,

$$\begin{aligned} \langle y^\mu(\tau_1)y^\nu(\tau_2)\rangle &= -g^{\mu\nu}G_B(\tau_1,\tau_2)\\ \langle\psi^\mu(\tau_1)\psi^\nu(\tau_2)\rangle &= \frac{1}{2}g^{\mu\nu}G_F(\tau_1,\tau_2) \end{aligned} \tag{3.20}$$

With our conventions the free ψ - path integral is normalized as

$$N_A \equiv \int\mathcal{D}\psi\exp\Bigl[-\int_0^T d\tau\frac{1}{2}\psi\dot{\psi}\Bigr] = 4 \tag{3.21}$$

The photon vertex operator V_A now has an additional Grassmann piece representing the coupling of the photon to the spin degree of freedom in the loop,

$$V_A^{\frac{1}{2}}[k,\varepsilon]\equiv\int_0^T d\tau\Bigl(\varepsilon\cdot\dot{x}+2i\varepsilon\cdot\psi k\cdot\psi\Bigr)\mathrm{e}^{ikx} \tag{3.22}$$

Let us again look at the vacuum polarization case, $N = 2$. We need now to Wick-contract two copies of the above vertex operator. The calculation of $\int \mathcal{D}x$ is identical with the scalar QED calculation. Only the calculation of $\int \mathcal{D}\psi$ is new, and amounts to a single Wick contraction,

$$(2i)^2\Bigl\langle \psi_1^\mu\psi_1\cdot k_1\psi_2^\nu\psi_2\cdot k_2\Bigr\rangle = -G_{F12}^2\Bigl[g^{\mu\nu}k^2 - k^\mu k^\nu\Bigr] \tag{3.23}$$

Note that, up to the global normalization, the parameter integral for the spinor loop is obtained from the one for the scalar loop simply by substituting, in eq.(3.15),

$$\dot G_{B12}^2 \to \dot G_{B12}^2 - G_{F12}^2 = -\frac{4}{T}G_{B12} \tag{3.24}$$

The complete change thus amounts to supplying eq.(3.17) with the global factor of -2, and replacing $(1-2u)^2$ by $-4u(1-u)$. This leads to

$$\Gamma^{\mu\nu}_{\rm spin}[k] = 8\frac{e^2}{(4\pi)^{\frac{D}{2}}}\Bigl(k^\mu k^\nu - g^{\mu\nu}k^2\Bigr)\Gamma\bigl(2-\frac{D}{2}\bigr) \times\int_0^1 du\, u(1-u)\Bigl[m^2+u(1-u)k^2\Bigr]^{\frac{D}{2}-2} \tag{3.25}$$

again in agreement with field theory. The above "substitution rule" carries over to arbitrary N as follows [2]. Up to the global normalization factor, the integrand for the spin - $\frac{1}{2}$ case can be obtained from the spin - 0 integrand simply by replacing, after the partial integration procedure, every "cycle" $\dot G_{Bi_1i_2}\dot G_{Bi_2i_3}\cdots\dot G_{Bi_ni_1}$ by

$$\dot G_{Bi_1i_2}\dot G_{Bi_2i_3}\cdots\dot G_{Bi_ni_1} \to \text{same} \; - G_{Fi_1i_2}G_{Fi_2i_3}\cdots G_{Fi_ni_1} \tag{3.26}$$

This rule reduces the transition from the scalar loop case to the spinor loop case to a pattern matching problem.

3.3 The General Case

We are now ready for the general vector – axialvector case. The only real novelty is in the case of an odd number of axial vectors, since here the Grassmann path integral appears with periodic boundary conditions. It acquires therefore a zero mode which must be separated out in the same way as for the coordinate path integral, splitting

$$\psi^\mu(\tau) = \psi_0^\mu + \xi^\mu(\tau) \tag{3.27}$$

$$\int_0^T d\tau\, \xi^\mu(\tau) = 0 \tag{3.28}$$

The zero mode integration then produces the ε - tensor expected for a spinor loop with an odd number of axial insertions,

$$\int d^4\psi_0\psi_0^{\mu_1}\psi_0^{\mu_2}\psi_0^{\mu_3}\psi_0^{\mu_4} = \varepsilon^{\mu_1\cdots\mu_4} \tag{3.29}$$

The remaining ξ - path integral is again performed using the appropriate Wick contraction rule, which is

$$\langle\xi^\mu(\tau_1)\,\xi^\nu(\tau_2)\rangle = g^{\mu\nu}\frac{1}{2}\dot G_B(\tau_1,\tau_2) \tag{3.30}$$

The free Grassmann path integral with periodic boundary conditions is normalized to $N_P = 1$.

To define the analogue of the photon vertex operator (3.22) for the axial coupling, it is convenient to linearize the term quadratic in A_5. This can be achieved through the introduction of an auxiliary path integration, writing

$$\exp\Bigl[-(D-2)e_5^2\int_0^T d\tau A_5^2\Bigr] = \int\mathcal{D}z \times\exp\Bigl[-\int_0^T d\tau\Bigl(\frac{z^2}{4}+ie_5\sqrt{D-2}z\cdot A_5\Bigr)\Bigr] \tag{3.31}$$

This allows us to define an axial-vector vertex operator as follows,

$$V_{A_5}[k,\varepsilon] \equiv \hat\gamma_5\int_0^T d\tau\Bigl(i\varepsilon\cdot k + 2\varepsilon\cdot\psi\dot x\cdot\psi +\sqrt{D-2}\,\varepsilon\cdot z\Bigr)\,{\rm e}^{ik\cdot x} \tag{3.32}$$

With this definition we obtain the following representation for the one-loop amplitude with M vectors and N axialvectors,

$$
\begin{aligned}
\Gamma[\{k_i,\varepsilon_i\},\{k_{5j},\varepsilon_{5j}\}] &= -\frac{1}{2}N_{A,P}(-i)^{M+N}e^M e_5^N \\
&\times\int_0^\infty \frac{dT}{T}\, e^{-m^2T}(4\pi T)^{-\frac{D}{2}}\Bigl\langle V_A^{\frac{1}{2}}[k_1,\varepsilon_1]\dots \\
&\dots V_A^{\frac{1}{2}}[k_M,\varepsilon_M]V_{A_5}[k_{51},\varepsilon_{51}]\dots V_{A_5}[k_{5N},\varepsilon_{5N}]\Bigr\rangle
\end{aligned}
\tag{3.33}
$$

The Wick-contraction of this expression can still be done in closed form [17], though the result is too lengthy to be given here.

4. Relation to the "γ_5 - Problem"

As is well-known, theories involving both vector and axialvector couplings can, in the presence of UV divergences, generally not be regularised in a way which would respect both the vector and the axialvector gauge invariance. The most elementary manifestation of this fact is the ABJ anomaly [48, 49]. Since the worldline formalism is normally used together with dimensional regularization, we should investigate how our formalism behaves with respect to the chiral symmetry in the dimensional continuation.

It must be remembered that, in field theory, one has essentially a choice between two evils. If one preserves the anticommutation relations between γ_5 and the other Dirac matrices [50] then the chiral symmetry is preserved for parity-even fermion loops, but Dirac traces with an odd number of γ_5's are not unambiguously defined in general, requiring additional prescriptions. The main alternative is to use the 't Hooft-Veltman-Breitenlohner-Maison prescription [51, 52]. In this case there are no ambiguities, but the chiral symmetry is explicitly broken, so that in chiral gauge theories finite renormalizations generally become necessary to avoid violations of the gauge Ward identities [53, 54].

Since our path integral representation was derived using an anticommuting γ_5, we have not broken the chiral symmetry. In particular, in the massless case the amplitude with an even number of axialvectors should coincide with the corresponding vector amplitude. It is easy to verify this fact for the two-point case. The Wick contraction of two axialvector vertex operators (3.32) leads, after rescaling $\tau_{1,2} = Tu_{1,2}$ and putting $u_2 = 0$ as above, to the following expression for the (massless) two-point amplitude,

$$
\begin{aligned}
\Gamma^{\mu\nu}[k] &= 2\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}} \\
&\times\Biggl\{2(D-2)Tg^{\mu\nu} - 2(D-1)Tg^{\mu\nu} \\
&+\int_0^1 du\, \mathrm{e}^{-Tu(1-u)k^2}\Bigl[2(D-1)Tg^{\mu\nu} \\
&+(1-2u)^2T^2(g^{\mu\nu}k^2-k^\mu k^\nu)+T^2k^\mu k^\nu\Bigr]\Biggr\}
\end{aligned}
\tag{4.1}
$$

The result of the integrations is, using Γ - function identities, easily identified with the one for the massless QED vacuum polarisation, eq.(3.25). This confirms that the chiral symmetry is unbroken for parity-even loops.

Next let us see whether we correctly get the AVV anomaly, i.e. the anomalous divergence for the $\langle A_5AA\rangle$ amplitude. In field theory this amplitude would be given by the two triangle diagrams shown in fig. 5.

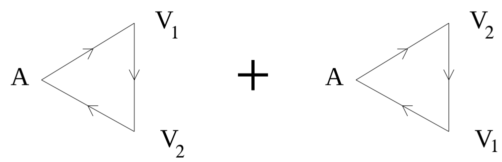


Figure 5: Sum of AVV triangle diagrams.

According to eq.(3.33) the sum of these diagrams is given by

$$
\begin{aligned}
\langle A^\mu[k_1]A^\nu[k_2]A_5^\rho[k_3]\rangle &= -\frac{i}{2}\int_0^\infty \frac{dT}{T}\, e^{-m^2T} \\
&\times\int\mathcal{D}x\int\mathcal{D}\psi\ \exp\Bigl\{-\int_0^T d\tau\,\Bigl(\frac{1}{4}\dot{x}^2+\frac{1}{2}\psi\cdot\dot{\psi}\Bigr)\Bigr\} \\
&\times\int_0^T d\tau_1\Bigl(\dot{x}_1^\mu+2i\psi_1^\mu k_1\cdot\psi_1\Bigr)\mathrm{e}^{ik_1\cdot x_1} \\
&\times\int_0^T d\tau_2\Bigl(\dot{x}_2^\nu+2i\psi_2^\nu k_2\cdot\psi_2\Bigr)\mathrm{e}^{ik_2\cdot x_2} \\
&\times\int_0^T d\tau_3\Bigl(ik_3^\rho+2\psi_3^\rho\dot{x}_3\cdot\psi_3\Bigr)\mathrm{e}^{ik_3\cdot x_3}
\end{aligned}
\tag{4.2}
$$

From this expression it is obvious, even before doing any integrations, that the amplitude will be transversal in the vector current indices. If one multiplies the right hand side by, say, k_1^μ, then the Grassmann part of the τ_1 - integrand will vanish by the anticommuation rules, and the remaining bosonic part of the photon vertex operator becomes a total derivative in τ_1, which vanishes upon integration due to periodicity. This mechanism is, of course, well-known from string theory. Nothing analogous holds true for the axial-vector vertex operator.

Thus the structure of the path integral eq.(3.1) already forces the vector currents to be divergence-free, and we clearly have to look at the axial vector current to find the anomalous divergence. We are interested in this divergence only, rather than in a calculation of the complete amplitude, so that we can simplify by contracting eq.(4.2) with k_3^ρ. Also we can put legs 1 and 2 on-shell, $k_1^2 = k_2^2 = 0$, and it suffices to consider the massless case.

Since for this amplitude the Grassmann path integral has periodic boundary conditions, according to eq.(3.27) we have to rewrite

$$\psi_i^\alpha(\tau) = \psi_{0i}^\alpha + \xi_i^\alpha(\tau)$$

and then to keep only those terms which contain precisely four factors of the zero mode piece ψ_0. Using the zero mode integration rule eq.(3.29) as well as eq.(3.30) and the usual correlator for the coordinate field, we obtain (deleting the energy-momentum conservation factor)

$$\begin{aligned}
k_3^\rho \langle A^\mu A^\nu A_5^\rho \rangle \;&=\; 2\varepsilon^{\mu\nu\kappa\lambda} k_1^\kappa k_2^\lambda \int_0^\infty \frac{dT}{T} (4\pi T)^{-2} \\
&\times \prod_{i=1}^{3} \int_0^T d\tau_i \exp\Bigl[(G_{B12} - G_{B13} - G_{B23}) k_1 \cdot k_2\Bigr] \\
&\times \biggl\{ \Bigl[2 + (\dot G_{B12} + \dot G_{B23} + \dot G_{B31})(\dot G_{B13} - \dot G_{B23})\Bigr] \\
&\qquad \times k_1 \cdot k_2 - (\ddot G_{B13} + \ddot G_{B23}) \biggr\}
\end{aligned} \tag{4.3}$$

Momentum conservation has been used to eliminate k_3. Removing the second derivatives $\ddot G_{B13}$ ($\ddot G_{B23}$) by a partial integration in τ_1 (τ_2), and using the identities

$$\begin{aligned}
\dot G_{Bij}^2 \;&=\; 1 - 4\frac{G_{Bij}}{T} \\
\dot G_{Bij} + \dot G_{Bjk} + \dot G_{Bki} \;&=\; -\mathrm{sign}_{ij}\mathrm{sign}_{jk}\mathrm{sign}_{ki}
\end{aligned} \tag{4.4}$$

the expression in braces is transformed into

$$4\frac{k_1 \cdot k_2}{T}(G_{B13} + G_{B23} - G_{B12}) \tag{4.5}$$

This is precisely the same expression which appears also in the exponential factor in (4.3). After the usual rescaling to the unit circle, and performance of the trivial T - integral, we find therefore a complete cancellation between the Feynman numerator and denominator polynomials [2]. Thus without further integration we obtain already the desired result for the anomalous divergence,

$$k_3^\rho \langle A^\mu A^\nu A_5^\rho \rangle = \frac{8}{(4\pi)^2} \varepsilon^{\mu\nu\kappa\lambda} k_1^\kappa k_2^\lambda \tag{4.6}$$

5. Constant External Fields

5.1 QED in a Constant External Field

Let us extend this formalism to the practically important case of QED in a constant external field. In field theory the Dirac equation in a constant field can be solved exactly, so that one can absorb the field already at the level of the Feynman rules by a redefinition of the electron propagator [56, 57, 58]. Quite analogously in the worldline formalism we can absorb a constant field into the worldline Green's functions [9, 10, 11, 12].

Let us thus assume that we have, in addition to the background field $A^\mu(x)$ which serves as a generator of the external photons, an additional field $\bar A^\mu(x)$ with constant field strength tensor $\bar F_{\mu\nu}$. Using Fock–Schwinger gauge centered at x_0 we may take $\bar A^\mu(x)$ to be of the form

$$\bar A_\mu(x) = \frac{1}{2} y_\nu \bar F_{\nu\mu} \tag{5.1}$$

[2] The on-shell conditions are not necessary for this cancellation to occur [17]. Moreover the same mechanism was found to work also for the calculation of the chiral anomaly in $D = 8$ [55].

The constant field contribution to the worldline Lagrangian may then be written as

$$\Delta\mathcal{L} = \frac{1}{2} iey_\mu \bar{F}_{\mu\nu}\dot{y}_\nu - ie\psi_\mu \bar{F}_{\mu\nu}\psi_\nu \qquad (5.2)$$

Since it is quadratic in the worldline fields it need not be considered as part of the interaction Lagrangian; we can absorb it into the free worldline propagators. The details are given in [12]; the result is that the worldline Green's functions G_B, G_F should, in a constant external electromagnetic field, be replaced by (deleting the "bar" and abbreviating $z = eFT$)

$$\begin{aligned}\mathcal{G}_B(\tau_1,\tau_2) &= \frac{T}{2z^2}\biggl(\frac{z}{\sin(z)}\,\mathrm{e}^{-iz\dot{G}_{B12}} + iz\dot{G}_{B12} - 1\biggr)\\ \mathcal{G}_F(\tau_1,\tau_2) &= G_{F12}\frac{\mathrm{e}^{-iz\dot{G}_{B12}}}{\cos(z)}\end{aligned} \qquad (5.3)$$

These expressions should be understood as power series in the field strength matrix F. They are, in general, nontrivial Lorentz matrices, so that the Wick contraction rules eq.(3.20) have to be replaced by

$$\begin{aligned}\langle y^\mu(\tau_1)y^\nu(\tau_2)\rangle &= -\mathcal{G}_B^{\mu\nu}(\tau_1,\tau_2)\\ \langle \psi^\mu(\tau_1)\psi^\nu(\tau_2)\rangle &= \frac{1}{2}\mathcal{G}_F^{\mu\nu}(\tau_1,\tau_2)\end{aligned} \qquad (5.4)$$

We also need the generalization of $\dot{G}_B$, which is

$$\dot{\mathcal{G}}_B(\tau_1,\tau_2) = \frac{i}{z}\biggl(\frac{z}{\sin(z)}\mathrm{e}^{-iz\dot{G}_{B12}} - 1\biggr) \qquad (5.5)$$

The substitution rule eq.(3.26) continues to hold. However, in contrast to the vacuum worldline correlators $\dot{G}_B, G_F$ their generalizations $\dot{\mathcal{G}}_B, \mathcal{G}_F$ have non-vanishing coincidence limits,

$$\dot{\mathcal{G}}_B(\tau,\tau) = i\cot(z) - \frac{i}{z} \qquad (5.6)$$

$$\mathcal{G}_F(\tau,\tau) = -i\tan(z) \qquad (5.7)$$

As a consequence the substitution rule must be extended to include one-cycles,

$$\dot{\mathcal{G}}_B(\tau_i,\tau_i) \to \dot{\mathcal{G}}_B(\tau_i,\tau_i) - \mathcal{G}_F(\tau_i,\tau_i) \qquad (5.8)$$

This is almost all one needs to know for computing one-loop photon scattering amplitudes, or the corresponding effective action, in a constant overall background field. The only further information required at the one–loop level is the change in the free path integral determinants due to the external field. This change is [18]

$$(4\pi T)^{-\frac{D}{2}} \to (4\pi T)^{-\frac{D}{2}}\det{}^{-\frac{1}{2}}\biggl[\frac{\sin(z)}{z}\biggr] \qquad (5.9)$$

$$(4\pi T)^{-\frac{D}{2}} \to (4\pi T)^{-\frac{D}{2}}\det{}^{-\frac{1}{2}}\biggl[\frac{\tan(z)}{z}\biggr] \qquad (5.10)$$

for Scalar and Spinor QED, respectively. Those determinants describe the vacuum amplitude in a constant field, and therefore are just the proper-time integrands of the well-known Euler-Heisenberg-Schwinger formulas [59, 60].

With this machinery set up it is then easy to derive the following generalization of the master formula for N - photon scattering, eq.(3.12), to the constant field case,

$$\begin{aligned}\Gamma[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] &= (-ie)^N\int_0^\infty\frac{dT}{T}[4\pi T]^{-\frac{D}{2}}\\ &\times \mathrm{e}^{-m^2T}\det{}^{-\frac{1}{2}}\biggl[\frac{\sin(z)}{z}\biggr] \times \prod_{i=1}^N\int_0^T d\tau_i\\ &\times\exp\biggl\{\sum_{i,j=1}^N\bigl[\frac{1}{2}k_i\cdot\mathcal{G}_{Bij}\cdot k_j - i\varepsilon_i\cdot\dot{\mathcal{G}}_{Bij}\cdot k_j\\ &\quad+\frac{1}{2}\varepsilon_i\cdot\ddot{\mathcal{G}}_{Bij}\cdot\varepsilon_j\bigr]\biggr\}\Big|_{\mathrm{multi-linear}}\end{aligned} \qquad (5.11)$$

Expanding out this formula for $N = 2$, and performing the partial integration procedure, we obtain the following representation for the (dimensionally regularised) Scalar QED vacuum polarisation tensor in a constant field,

$$\begin{aligned}\Pi^{\mu\nu}_{\mathrm{scal}}[k] &= -\frac{e^2}{[4\pi]^{\frac{D}{2}}}\int_0^\infty\frac{dT}{T}T^{2-\frac{D}{2}}e^{-m^2T}\\ &\quad\times\det{}^{-\frac{1}{2}}\biggl[\frac{\sin(z)}{z}\biggr]\int_0^1 du_1\, I^{\mu\nu}_{\mathrm{scal}}\,\mathrm{e}^{-Tk\cdot\Phi_{12}\cdot k}\end{aligned}$$

$$I^{\mu\nu}_{\rm scal} = \Bigl\{\Bigl[\dot{\mathcal{S}}^{\mu\nu}_{B12}\dot{\mathcal{S}}^{\kappa\lambda}_{B12} - \dot{\mathcal{S}}^{\mu\lambda}_{B12}\dot{\mathcal{S}}^{\nu\kappa}_{B12}\Bigr] + \Bigl(\dot{\mathcal{A}}_{B12} - \dot{\mathcal{A}}_{B11}\Bigr)^{\mu\lambda}\Bigl(\dot{\mathcal{A}}_{B12} - \dot{\mathcal{A}}_{B22}\Bigr)^{\nu\kappa}\Bigr\}k^\kappa k^\lambda \tag{5.12}$$

We have decomposed $\mathcal{G}_B$ and $\mathcal{G}_F$ into their parts symmetric ("S") and antisymmetric ("A") in the Lorentz indices, $\mathcal{G}_{B,F} = \mathcal{S}_{B,F} + \mathcal{A}_{B,F}$.

The application of the substitution rule to this representation gives us the corresponding quantity for Spinor QED,

$$\Pi^{\mu\nu}_{\rm spin}[k] = 2\frac{e^2}{[4\pi]^{\frac{D}{2}}}\int_0^\infty \frac{dT}{T}T^{2-\frac{D}{2}}e^{-m^2T} \times{\rm det}^{-\frac{1}{2}}\left[\frac{\tan(z)}{z}\right]\int_0^1 du_1\; I^{\mu\nu}_{\rm spin}\,{\rm e}^{-Tk\cdot\Phi_{12}\cdot k}$$

$$\begin{aligned} I^{\mu\nu}_{\rm spin} &= \Bigl\{\Bigl[\dot{\mathcal{S}}^{\mu\nu}_{B12}\dot{\mathcal{S}}^{\kappa\lambda}_{B12} - \dot{\mathcal{S}}^{\mu\lambda}_{B12}\dot{\mathcal{S}}^{\nu\kappa}_{B12}\Bigr] \\ &\quad -\Bigl[\mathcal{S}^{\mu\nu}_{F12}\mathcal{S}^{\kappa\lambda}_{F12} - \mathcal{S}^{\mu\lambda}_{F12}\mathcal{S}^{\nu\kappa}_{F12}\Bigr] \\ &\quad +\Bigl(\dot{\mathcal{A}}_{B12} - \dot{\mathcal{A}}_{B11} + \mathcal{A}_{F11}\Bigr)^{\mu\lambda} \\ &\quad \times\Bigl(\dot{\mathcal{A}}_{B12} - \dot{\mathcal{A}}_{B22} + \mathcal{A}_{F22}\Bigr)^{\nu\kappa} \\ &\quad -\mathcal{A}^{\mu\lambda}_{F12}\mathcal{A}^{\nu\kappa}_{F12}\Bigr\}k^\kappa k^\lambda \end{aligned} \tag{5.13}$$

From (5.3),(5.5) we have

$$\begin{aligned} \dot{\mathcal{A}}_{Bij} &= i\Bigl(\frac{\cos(z\dot{G}_{Bij})}{\sin(z)} - \frac{1}{z}\Bigr) \\ \dot{\mathcal{A}}_{Bii} &= i\Bigl(\cot(z) - \frac{1}{z}\Bigr) \\ \dot{\mathcal{S}}_{Bij} &= \frac{\sin(z\dot{G}_{Bij})}{\sin(z)} \\ \mathcal{A}_{Fij} &= -iG_{Fij}\frac{\sin(z\dot{G}_{Bij})}{\cos(z)} \\ \mathcal{A}_{Fii} &= -i\tan(z) \\ \mathcal{S}_{Fij} &= G_{Fij}\frac{\cos(z\dot{G}_{Bij})}{\cos(z)} \\ \Phi_{ij} &= \frac{1}{2iz}\Bigl(\dot{\mathcal{A}}_{Bij} - \dot{\mathcal{A}}_{Bii}\Bigr) \\ &= \frac{\cos(z\dot{G}_{Bij}) - \cos(z)}{2z\sin(z)} \end{aligned} \tag{5.14}$$

Both $\Pi^{\mu\nu}_{\rm scal}$ [61] and $\Pi^{\mu\nu}_{\rm spin}$ [62, 63] have been obtained before in field theory, though with considerably more computational effort. A similar gain in efficiency was found for the $N = 3$ case [13]. The three-point calculation is of relevance for the process of photon splitting in strong magnetic fields [38, 64, 65], another process of interest to astrophysicists [66].

5.2 The Vector-Axialvector Polarisation Tensor in a Constant Field

The above treatment of the constant external field carries over to the full vector-axialvector case without difficulty. We have used it for obtaining the following representation for the vector – axialvector two-point amplitude in a constant electromagnetic field [67],

$$\begin{aligned} \langle A^\mu(k_1)A_5^\nu(k_2)\rangle_F &= (2\pi)^4\delta(k_1+k_2)\Pi_5^{\mu\nu}(k) \\ \Pi_5^{\mu\nu}(k) &= \frac{1}{2}\int_0^\infty\frac{dT}{T}\,e^{-m^2T}(4\pi T)^{-2} \\ &\quad\times\int_0^T d\tau_1\int_0^T d\tau_2\; J^{\mu\nu}(\tau_1,\tau_2)\,{\rm e}^{-Tk\Phi_{12}k} \end{aligned}$$

$$\begin{aligned} J^{\mu\nu} &= 4iT\Bigl\{\tilde{F}^{\mu\nu}k\mathcal{U}_{12}k - (\dot{\mathcal{S}}_{12}\tilde{F})^{\mu\nu}k\dot{\mathcal{S}}_{12}k \\ &\quad +\Bigl[(\tilde{F}k)^\mu(\mathcal{U}_{12}k)^\nu + (\mu\leftrightarrow\nu)\Bigr] \\ &\quad -\Bigl[(\tilde{F}\dot{\mathcal{S}}_{12}k)^\nu(\dot{\mathcal{S}}_{12}k)^\mu + (\mu\leftrightarrow\nu)\Bigr]\Bigr\} \\ &\quad +T^2F\cdot\tilde{F}\Bigl\{-\dot{\mathcal{A}}^{\mu\nu}_{12}k\mathcal{U}_{12}k + (\dot{\mathcal{S}}_{12}\dot{\mathcal{A}}_{22})^{\mu\nu}k\dot{\mathcal{S}}_{12}k \\ &\quad -\Bigl[(\dot{\mathcal{A}}_{12}k)^\mu(\mathcal{U}_{12}k)^\nu + (\mu\leftrightarrow\nu)\Bigr] \\ &\quad +\Bigl[(\dot{\mathcal{S}}_{12}k)^\mu(\dot{\mathcal{A}}_{22}\dot{\mathcal{S}}_{12}k)^\nu + (\mu\leftrightarrow\nu)\Bigr]\Bigr\} \end{aligned}$$

Here we introduced one more coefficient function,

$$\begin{aligned} \mathcal{U}_{ij} &= \dot{\mathcal{S}}^2_{ij} - (\dot{\mathcal{A}}_{ij} - \dot{\mathcal{A}}_{ii})\bigl(\dot{\mathcal{A}}_{ij} + \frac{i}{z}\bigr) \\ &= \frac{1-\cos(z\dot{G}_{Bij})\cos(z)}{\sin^2(z)} \end{aligned} \tag{5.15}$$

($\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$). As far as is known to the author this calculation was done in field theory only for the special case of a pure magnetic field

[37, 32]. In addition to being more general the calculation presented here has also the advantage of being manifestly (vector) gauge invariant. In [37, 32] gauge invariance was reached only after performing a certain integration by parts, and appealing to the absence of the chiral anomaly to be able to drop the boundary term.

As was explained in the introduction, in the Fermi limit the calculation of the $\gamma \to \nu\bar{\nu}$ and $\nu \to \nu\gamma$ processes in a constant magnetic field essentially reduces to a determination of $\Pi_{\rm spin}$ and Π_5. The further analysis shows that for the plasmon decay Π dominates over Π_5, while for the Cherenkov process it is the other way round [32].

6. Conclusions

We have presented a recently developed approach to calculations involving a fermion loop and both vector and axialvector couplings [16, 17]. This formalism extends the second order formulation of QED to the mixed vector-axialvector case. It also provides the corresponding generalization of the QED worldline formalism, leading to a generalization of the QED Bern-Kosower master formula [17]. As in the QED case this formalism allows one to incorporate constant external electromagnetic fields in a very economic way.

We discussed the properties of the formalism with regard to the treatment of γ_5 in the dimensional continuation. There is no breaking of the chiral symmetry for parity-even loops. For parity-odd loops the usual expression is obtained for the chiral anomaly, however the anomalous divergences turn out to be unambiguously confined to the axialvector legs.

We demonstrated that this technique provides an easy and elegant way for calculating, in a general constant electromagnetic field, both the ordinary photon polarisation tensor and the vector-axialvector polarisation tensor. The relevance of these quantities to standard model photon-neutrino processes has been discussed in some detail.

While our representation here concentrated on the calculcation of amplitudes, the formalism applies as well to the calculation of the effective action eq.(3.1) itself. The first few terms in the heat-kernel expansion of this effective action were already presented in [16]. Work in this direction is in progress.

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