On the perturbative corrections around D-string instantons

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ABSTRACT: We study $\mathcal{F}^4$-threshold corrections in an eight dimensional S-dual pair of string theories, as a prototype of dual string vacua with sixteen supercharges. We show that the orbifold CFT description of D-string instantons gives rise to a perturbative expansion similar to the one appearing on the fundamental string side. By an explicit calculation, using the Nambu-Goto action in the static gauge, we show that the first subleading term agrees precisely on the two sides. We then give a general argument to show that the agreement extends to all orders.

1. Introduction

During the last years, it have been an enormous progress in our understanding of the non perturbative aspects of superstring theories. The key to this development is the discovery of duality symmetries, which relate the strong and weak coupling limits of apparently different string theories. This give us a way to compute certain strong coupling quantities in one string theory by mapping it to a weak coupling result in a dual string theory. The consistency checks of this non-perturbative dualities are strangest for effective couplings called BPS-saturated effective couplings, some times they are linked to anomalies. Duality can be used to calculate their non-perturbative corrections. One can then identify the non-perturbative effects responsible for such corrections. For theories with more than N=2 supersymmetry such non-perturbative effects are due to instantons, which can be associated in string theory to Euclidean branes wrapped around an appropriate compact manifold [2]. Here we will be interested in a particular test of this strong-weak duality in the simplest context of the eight dimensional dual pair with sixteen supercharges obtained via an orbifold/orientifold of type IIB theory considered in [3]. The dual orbifold–orientifold actions are defined by the orbifold $(-)^{F_L}\sigma_V$ ("fundamental side"), where all the R-R states to which the D-brane usually couple are remouved and orientifold $\Omega\sigma_V$ ("type I side"), corresponding to a generalized Ω-projection of the closed oriented type IIB string giving rise to vanishing massless tadpoles thus preventing the introduction of the open-string excitations, with σ_V a shift of order two in the two torus. The complete matching of the BPS states on the two sides of this duality map was computed [3], and it have been shown that N winding modes on the fundamental side corresponds to bound states of ND-strings on the type one side. Moreover, the moduli dependence for O(2,2) $\mathcal{F}^4$ gauge couplings, for the gauge fields coming from the KK reduction of the metric and antisymmetric tensor, were considered. These $\mathcal{F}^4$ couplings have there analogues in the heterotic/type I duality for toroidal compactification to $D \geq 5$ [4]. In that context these $\mathcal{F}^4$ couplings were special since they are related by supersymmetry to CP-odd couplings, for which it is well known that only one-loop corrections are possible [5]. The analysis in the heterotic/type I duality was made possible since for $D \geq 5$, on the heterotic side, the BPS spectrum is completely perturbative. Therefore, allowing for an exact treatment which yields an exact one-loop result even non-perturbatively. A similar

*This work has been done in collaboration with E.Gava,J.F.Morales and K.S.Narain [1] analysis for the type II models under consideration here, has not been done, but we expect similar result to be true. We will assume that this is the case, i.e. that the one-loop formula obtained for the moduli dependence of some $\mathcal{F}^4$ terms in the fundamental side is exact. As it will turn out that the non-perturbative result on the dual type I side will support this assumption.

The one-loop formula can be expressed [6, 7, 3], in the type I variables, as a sum of a finite number of perturbative corrections and an infinite series of D-string instanton contributions. Indeed, for $D \geq 5$, the D-string instanton is the only source of non-perturbative contributions on the type I side. The form of the N-instanton contribution in this sum suggests that it can be read off from the $O(N)$ gauge theory describing N nearby D-strings, whose worldsheets are wrapped on a two-torus [7]. More precisely, the leading behavior in the volume of the compactification torus for the $SO(32)$ $\mathcal{F}^4$ and $\mathcal{R}^4$ gauge and gravitational couplings respectively were shown in agreement [7] with the exact formula found in a perturbative computation in the heterotic side [6]. Similar results were found in [3]. In all the cases, the D-string instanton coupling to the gauge fields are obtained from the classical D-string instanton action, while the contributions from quantum fluctuations around the D-string instanton background are encoded in the elliptic genus of the corresponding $O(N)$ gauge theory. The perturbative computations of [6, 3] show that, on top of the leading D-string instanton corrections, there are a finite number of contributions corresponding to perturbative corrections in the D-string instanton background. One can ask the question whether the CFT describing the infrared limit of the D-string gauge theory captures some information about these subleading terms. The aim of this letter is to show that this is indeed the case.

Let us first start by briefly reviewing the computation of the moduli dependence of $\mathcal{F}^4$ couplings in the the fundamental side and how the result agrees, at the leading order in a large volume expansion, with what one gets from the CFT describing the D-string system. We will subsequently show that the CFT description also gives rise to a perturbative expansion which has the same structure as the one appearing on the fundamental string side. In particular, we will show a perfect agreement for the first subleading correction.

2. Type IIB on $T^2/(-)^{F_L}\sigma_V$

The $\mathcal{F}^4$ couplings we are interested in are obtained from the one-loop string amplitudes:

$$\mathcal{A}_\ell = \langle (V_8^L)^\ell (V_9^L)^{4-\ell} \rangle \tag{2.1}$$

with

$$\begin{aligned} V_i^L &= \int d^2z (G_{\mu i} + iB_{\mu i})(\partial X^\mu - \frac{1}{4} p_\nu S \gamma^{\mu\nu} S) \\ &\quad (\bar\partial X^i - \frac{1}{4} p_\rho \tilde S \gamma^{i\rho} \tilde S) e^{ipX} \end{aligned} \tag{2.2}$$

the vertex operators of the left $O(2,2)$ gauge fields arising from the $F_i \equiv G_{\mu i} + iB_{\mu i}$ components of the metric and antisymmetric tensor. Here and in the following we will denote with capital indices $M = (\mu, i)$ the ten dimensional noncompact $\mu = 0, 1, \dots 7$ and compact $i = 8, 9$ directions. The combinations with the minus sign represent the graviphoton vertex operators, which carry always additional power of momenta and are therefore irrelevant for the computation of $\mathcal{F}^4$ couplings. It is convenient to define a generating function $Z(\nu_i, \tau, \bar\tau)$, in term of which the above amplitudes read:

$$\mathcal{A}_\ell = \tilde t_8 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \tau_2^4 \frac{\partial^\ell}{\partial \nu_8^\ell} \frac{\partial^{4-\ell}}{\partial \nu_9^{4-\ell}} Z(\nu_i, \tau, \bar\tau) \tag{2.3}$$

with $\tilde t_8 = t_8 F_8^\ell F_9^{4-\ell}$, t_8 is the tensor arising from the trace over the right-moving fermions zero-modes and $\mathcal{F}$ the fundamental domain for the modulus of world-sheet torus. $Z(\nu_i, \tau, \bar\tau)$ is the partition function arising from a perturbed Polyakov action, whose bosonic part is:

$$\begin{aligned} S(\nu_i) &= \frac{2\pi}{\alpha'} \int d^2\sigma (\sqrt{g} G_{\mu\nu} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \\ &\quad + iB_{\mu\nu}^{NS} \varepsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu + \sqrt{g} \frac{\alpha'}{2\pi\tau_2} \nu_i \bar\partial X^i), \end{aligned} \tag{2.4}$$

and $\bar\partial = \frac{1}{\tau_2}(\partial_{\sigma_2} - \tau \partial_{\sigma_1})$. The geometry of the target torus are described as usual by the two complex moduli

$$\begin{aligned} T &= T_1 + iT_2 = \frac{1}{\alpha'}(B_{89}^N + i\sqrt{G}) \\ U &= U_1 + iU_2 = (G_{89} + i\sqrt{G})/G_{88} \, , \end{aligned} \tag{2.5}$$

where G_{ij} and B_{ij}^N are the σ-model metric and NS-NS antisymmetric tensor. Since the theory we are considering involves half shift along X^8 direction, it is convenient to replace the radius R^8 by $2R^8$. Indeed this new radius is what appears on the dual $IIB/\Omega\sigma_V$ theory (upto the scaling by the string coupling constant). As a result T, U and ν_8 are replaced by $2T$, $U/2$ and $2\nu_8$ respectively and furthermore all the windings along σ_1 and σ_2 are integer valued.

The partition function $Z(\nu_i, \tau, \bar\tau)$ is given by

$$\int_{\mathcal F} \frac{d^2\tau}{\tau_2^2} Z(\nu_i, \tau\bar\tau) \simeq \int_{\mathcal F} \frac{d^2\tau}{\tau_2^2} \sum_\epsilon \Gamma^\epsilon_{2,2}(\nu_i)\mathcal{A}_\epsilon \quad (2.6)$$

with

$$\Gamma^\epsilon_{2,2}(\nu_i) = \frac{4T_2}{U_2}\sum_{W,\epsilon} e^{2\pi i T \det W} e^{-\frac{\pi T_2}{\tau_2 U_2}\left|(1\ U) W \left(\begin{smallmatrix}\tau\\ -1\end{smallmatrix}\right)\right|^2} e^{-\frac{\pi}{\tau_2}(\nu_8\ \nu_9) W \left(\begin{smallmatrix}\tau\\ -1\end{smallmatrix}\right)} \quad (2.7)$$

and $\mathcal{A}_\epsilon$ the anti-holomorphic BPS partition functions

$$\mathcal{A}_{+-} \equiv \frac{1}{2}\frac{\vartheta_2(\bar q)^4}{\eta(\bar q)^{12}} \quad , \quad \mathcal{A}_{-+} \equiv \frac{1}{2}\frac{\vartheta_4(\bar q)^4}{\eta(\bar q)^{12}} \quad , \quad \mathcal{A}_{--} \equiv \frac{1}{2}\frac{\vartheta_3(\bar q)^4}{\eta(\bar q)^{12}} \quad (2.8)$$

The $\Gamma_{2,2}$ lattice has been written in (2.7) as a sum over all possible world-sheet instantons

$$\begin{pmatrix} X^8 \\ X^9 \end{pmatrix} = W \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} \equiv \begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \end{pmatrix}\begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} \quad (2.9)$$

with worldsheet and target space coordinates σ_1, σ_2 and X^8, X^9 respectively, both taking values in the interval (0,1]. The entries m_1, n_1 are even or odd integers depending on the specific orbifold sector, while m_2, n_2 run over all integers. We denote the three relevant sectors: n_1 odd, m_1 odd and both odd by $\epsilon = +-, -+, --$ respectively. The untwisted sector, $\epsilon = ++$, will not contribute, since it has too many zero modes to be soaked by the four vertex insertions at this order in the momenta. Note that, due to the existence of these different sectors, the duality group Γ on U is not the full $SL(2,Z)$ group but the subgroup defined by the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad b \in 2\mathbf{Z} \quad (2.10)$$

The integration over the fundamental domain can be done using the standard trick [9], which reduces the sum over all wrapping mode configurations in a given $SL(2,\mathbf{Z})$ orbit to a single integration over an unfolded domain. The unfolded domains are either the strip or the upper half plane depending on whether the orbit is *degenerate* ($\det W = 0$) or *non degenerate* ($\det W \neq 0$). In [6, 3] these contributions were identified with the perturbative and D-string instanton corrections respectively in the type I side. The result for the non-degenerate contributions were written as [3]

$$\langle (F_8)^\ell (F_9)^{4-\ell}\rangle_{nondeg} = \frac{\mathcal{V}_{10}}{(4\pi^2\alpha')^4} t_8 F_8^\ell F_9^{4-\ell}\frac{1}{T_2} \sum_{n,\epsilon}\sum_{m_1,n_1,n_2} \mathcal{A}^n_\epsilon \frac{\partial^\ell}{\partial\nu_8^\ell}\frac{\partial^{4-\ell}}{\partial\nu_9^{4-\ell}} I_n(\nu_8,\nu_9) \quad (2.11)$$

where $\mathcal{A}^n_\epsilon$ are the coefficient in the $\bar q$ expansion of the modular forms (2.8) and I_n are the result of the modular integrations

$$I_n = \frac{(U_2T_2)^{\frac12}}{m_1} e^{-2\pi i T_1 m_1 n_2} e^{-2\pi i n(\frac{n_1+U_1 n_2}{m_1})} e^{\pi i \nu_8(\frac{nU_2}{m_1 T_2} - m_1)}\sqrt{\frac{\pi}{\beta}} e^{-2\sqrt{\beta\gamma}} \quad (2.12)$$

with

$$\beta = \pi(n_2^2 U_2 T_2 + \nu_9 n_2 - \nu_8 U_1 n_2 - \frac{U_2\nu_8^2}{4T_2})$$
$$\gamma = \frac{\pi T_2}{U_2}(m_1 + \frac{nU_2}{T_2 m_1})^2. \quad (2.13)$$

Here the sum over m_1 and n_1 are over even and odd integers, depending on the ϵ-sectors, and for a fixed m_1 the range of n_1 goes from 0 to $m_1 - 1$.

3. Type IIB on $T^2/\Omega\sigma_V$

In terms of the type IIB on $T^2/\Omega\sigma_V$ variables this result should come from D-instanton contributions since the duality relations ($T_2^F = T_2^I/\lambda_I$) [1]implies that it goes like $e^{-\frac{NT_2^I}{\lambda_I}}$. Indeed it has been shown in [3] that its leading order in the T_2

[1]where the subscripts "F" and "I" are used to distinguish orbifold ($T^2/(-)^{F_L}\sigma_V$) and orientifold ($T^2/\Omega\sigma_V$) compactifications of the type IIB string.

expansion

$$\langle (F_8)^\ell (F_9)^{4-\ell} \rangle_{nondeg} = \frac{\mathcal{V}_{10}}{(4\pi^2\alpha')^4} t_8 F_8^\ell F_9^{4-\ell} \frac{1}{T_2} \sum_\epsilon \sum_{m_1,n_1,n_2} \frac{m_1^4}{m_1|n_2|} \frac{U^\ell}{U_2^4} e^{-2\pi i T m_1 |n_2|} \mathcal{A}_\epsilon(\frac{n_1 + \bar{U} n_2}{m_1}) \quad (3.1)$$

is exactly reproduced by a direct computation of D-instanton contributions in the type I side. In this identification the D-instanton number $N = LM$ is mapped to the determinant $m_1 n_2$ of the representative matrices M, while different wrapping mode configurations labelled by m_1, n_1, n_2 are mapped to L, s, M, describing the different sectors of the orbifold conformal field theory to which the effective $O(N)$ gauge theory flows in the infrared. The coupling of the N D-instanton background to the F_8, F_9 gauge fields is read from the classical D-instanton action

$$S_{D-inst} = 2\pi N T^F - \frac{\pi N}{4\alpha' U_2 \lambda_I} p_\nu (G_{\mu 9} + U G_{\mu 8}) S_0^{cm} \gamma^{\mu\nu} S_0^{cm} + ... \quad (3.2)$$

where $T^F = \frac{1}{\alpha'}(B_{89} + \frac{i}{\lambda_I}\sqrt{G})$ and $S_0^{cm} = \frac{1}{N}\sum_{i=1}^N S^i$ are the fermionic zero modes corresponding to the center of mass of the N copies of D-string worldsheets. Indeed, bringing down four powers of these terms, we soak the eight fermionic zero modes and combining with the quantum D-instanton partition function $\frac{1}{NT_2}\sum_{M,L,s} M^{-4} \mathcal{A}(\frac{s+\bar{U}M}{L})$, the perturbative result (3.1) is reproduced. Note that the center of mass fermions here are defined with unconventional normalization in order to make the comparison with the fundamental string side more transparent.

4. Perturbative corrections around D-string instantons

The exact formula (2.11) shows the presence of T_2-subleading terms which should correspond to a finite number of perturbative corrections around the N D-instanton background. In order to make more transparent the translation in terms of the type I variables it is convenient to use the complex source ν, and its complex conjugate $\bar{\nu}$, defined as:

$$\nu \equiv \frac{1}{2U_2}(\nu_9 + U\nu_8), \quad (4.1)$$

In terms of these new variables, after a long but straightforward algebra, one can express, to the order of interest, the $\bar{\nu}$-expansion of the generating function $\mathcal{I}(\nu,\bar{\nu}) = \sum_n \mathcal{A}_\epsilon^n I_n$ of (2.11) formally as:

$$\mathcal{I}(\nu,\bar{\nu}) = \sum_{r=0}^4 \frac{\bar{\nu}^r}{r!} D_{\bar{U}}^r \left[\frac{1}{(T_2 - \frac{\nu}{2\pi})^{r+1}} \mathcal{I}_0(\nu) \right] + O(\bar{\nu}^5), \quad (4.2)$$

where

$$\mathcal{I}_0(\nu) = \frac{\mathcal{V}_{10}}{(4\pi^2\alpha\prime)^4} \sum_\epsilon \sum_{m_1,n_1,n_2} \frac{e^{-2\pi i \bar{T} m_1 n_2}}{m_1 n_2^5} \mathcal{A}_\epsilon(\frac{n_1 + \bar{U} n_2}{m_1}) e^{2 m_1 n_2 \nu} \quad (4.3)$$

and $D_{\bar{U}}$ is the $\bar{U}$-covariant derivative, which acting on a modular form of weight $-2r$ gives a the weight $-2r+2$ form

$$D_{\bar{U}} \Phi_r = (\frac{i}{\pi}\partial_{\bar{U}} - \frac{r}{\pi U_2})\Phi_r \quad (4.4)$$

In (4.2) the first 4 terms in the series have been explicitly indicated, being the only relevant ones to the present discussion, The function appearing there has no definite modular transformation properties, and the covariant derivative $D_{\bar{U}}^r$ should be understood as acting on the coefficient of ν^{4-r} in the expansion of the function inside the brackets in (4.2). This gives the $\bar{F}^r F^{4-r}$ gauge couplings we are interested in, with $F = \frac{1}{2U_2}(F_9 + UF_8)$ and $\bar{F} = \frac{1}{2U_2}(F_9 + \bar{U}F_8)$:

$$\langle \bar{F}^r F^{4-r} \rangle = \frac{\mathcal{V}_{10}}{(4\pi^2\alpha\prime)^4} \sum_N \frac{e^{-2\pi i N\bar{T}}}{N T_2^{r+1}} \sum_{k=0}^{4-r} \frac{1}{(NT_2)^k} \frac{(r+k)!}{k!(r!)^2(4-r-k)!} H_N(D_{\bar{U}}^r A_{-+}) \quad (4.5)$$

where H_N is the Hecke operator for the subgroup Γ :

$$H_N(D_{\bar{U}}^r A_{-+}) = \sum_\epsilon \sum_{N|L} \sum_{s=0}^{L-1} L^{4-2r} D_{\bar{\mathcal{U}}}^r A_\epsilon(\bar{\mathcal{U}})|_{\bar{\mathcal{U}} = \frac{M\bar{U}+s}{L}} \quad (4.6)$$

where (L,s) are (even,odd), (odd,even) and (odd,odd) in the three ϵ sectors $+-$, $-+$ and $--$ respectively. Note that $D_{\bar{U}}^r A_{-+}(\bar{U})$ is a modular form of weight $2r-4$ with respect to the subgroup Γ. The above definition of the Hecke operator coincides with the standard one appearing for example in [10] upto an overall N dependent factor.

We would like now to show how these corrections can be reproduced by the effective D-string instanton action. To this aim, we start with the D-string instanton action in the presence of constant field strength background for the gauge fields $G_{\mu i}, B_{\mu i}$ under consideration. Each field strength appears in the action with two fermion zero modes and therefore the amplitude is obtained by extracting the 8 fermion zero modes in the partition function. One immediately encounters two problems:

1) One needs to keep in the action higher order terms in the fermion zero modes. Unfortunately in curved backgrounds the covariant Green-Schwarz actions that exist in the literature include only terms upto quartic in fermions, with higher order terms in principle being calculable using space-time supersymmetry and kappa symmetry. On dimensional grounds, as will become clear in the following, each pair of fermion zero modes appears with $1/T_2$ and therefore, such higher order terms can contribute to order $1/T_2^2$ correction (compared to the leading order term). In the absence of information on such higher order terms we will be restricted below to calculate only the first subleading correction. However later we will argue that the complete action should correctly reproduce the full perturbative expansion.

2) Even upto the terms quartic in fermion fields, the Green-Schwarz action presupposes that the background fields satisfy the supergravity equations of motion. Of course constant field strengths satisfy the gauge field equations, however they can contribute to the stress energy tensor and therefore can modify the classical equations for dilaton and the 8-dimensional metric. However this problem will not appear if we restrict ourselves to the first subleading correction in $1/T_2$. Indeed as seen from equation (4.5), the first subleading term appears in two ways: i) for $\bar{\nu}=0$ (i.e. $F_{\mu\nu z}=0$) there is a correction coming from the first order expansion of $1/(T_2-\frac{\nu}{2\pi})$. This can get contribution from terms in the action that are quadratic in $F_{\mu\nu\bar{z}}$. For $F_{\mu\nu z}=0$ arbitrary constant values of $F_{\mu\nu\bar{z}}$ will solve the equations of motion since the resulting stress energy tensor is zero. ii) the first order in $\bar{\nu}$ already comes with $1/T_2^2$ and therefore does not require an expansion of the term $1/(T_2-\frac{\nu}{2\pi})^2$. This means that in the action we need to include only the first order term in both the F's which certainly solves the linearized equations of motion. All the expressions in the following should be understood to make sense only in the first subleading term in $1/T_2$.

The Green-Schwarz, covariant world-sheet action for a single string coupled to the $\mathcal{N}=1$, 10-D supergravity multiplet, is given by [11] ($\alpha'=\frac{1}{2}$):

$$\begin{aligned} S_{D-inst} &= 2\pi\int d^2\sigma[\sqrt{\frac{|det\hat{G}_{ab}|}{\lambda_I}}+\frac{1}{2}\epsilon^{ab}R_{ab}] \\ &\quad +\dots \end{aligned} \tag{4.7}$$

The ... above indicate terms involving 6 and higher powers of fermion fields. Here we have directly rewritten the action of [11] in the Nambu-Goto form, by using the equations of motion for the world-sheet metric. The resulting induced metric $\hat{G}_{ab}$ and the antisymmetric coupling R_{ab} are given by:

$$\begin{aligned} \hat{G}_{ab} &= \partial_{(a}X^M\partial_{b)}X^N G_{MN}-2i\partial_{(a}X^M\bar{\Theta}\gamma_M D_{b)}\Theta \\ &\quad -\bar{\Theta}\gamma_M D_a\Theta\bar{\Theta}\gamma^M D_b\Theta \\ R_{ab} &= i\partial_{[a}X^M\partial_{b]}X^N B_{MN}+\frac{2i}{\lambda_I}\partial_{[a}X^M\bar{\Theta}\gamma_M\sigma_3 D_{b]}\Theta \\ &\quad +\frac{1}{\lambda_I}\bar{\Theta}\gamma_M D_{[a}\Theta\bar{\Theta}\gamma^M\sigma_3 D_{b]}\Theta. \end{aligned} \tag{4.8}$$

Here the covariant derivative D_a is defined to be:

$$D_a=\partial_a-\frac{1}{4}\gamma^{\hat{K}\hat{L}}\tilde{\omega}_N^{\hat{K}\hat{L}}\partial_a X^N, \tag{4.9}$$

where $\tilde{\omega}_N^{\hat{K}\hat{L}}=\omega_N^{\hat{K}\hat{L}}+\frac{\lambda_I}{2}H_N^{\hat{K}\hat{L}}\sigma_3$, with ω the spin connection and H the 3-form field strength of the RR antisymmetric tensor B. The capital, Latin indices are 10-dimensional curved indices, whereas hatted ones are $SO(1,9)$ flat indices. $\Theta^T=(\theta,\tilde{\theta})$ denotes a doublet of $SO(1,9)$ Majorana-Weyl spinors on which σ_3 acts. θ and $\tilde{\theta}$ have the same $SO(1,9)$ chirality and are world-sheet scalars. Finally, $\bar{\Theta}\equiv(\theta^T\gamma^0,\tilde{\theta}^T\gamma^0)$. Note that the fields appearing above are supposed to satisfy the 10-dimensional supergravity equations of motion.

We then proceed by choosing the static gauge, which amounts to identifying the Euclidean world-sheet complex coordinates $z \equiv \frac{1}{\sqrt{2\tau_2}}(\sigma^1 + \tau\sigma^2)$, $\bar{z}$, with the spacetime torus coordinates $X^z \equiv \frac{1}{\sqrt{2\tau_2}}(X^8 + UX^9)$, $X^{\bar{z}}$ respectively and the complex structure of the world-sheet torus, τ, with that of the spacetime torus, U. In addition, local kappa-symmetry allows to reduce the fermionic degrees of freedom by imposing [2]

$$\begin{aligned}\gamma^z\theta &\equiv \frac{1}{\sqrt{2U_2}}(\gamma^8 + U\gamma^9)\theta = 0\\ \gamma^{\bar{z}}\tilde{\theta} &\equiv \frac{1}{\sqrt{2U_2}}(\gamma^8 + \bar{U}\gamma^9)\theta = 0\end{aligned} \tag{4.10}$$

The resulting gauge fixed fermions θ ($\tilde{\theta}$) become left-moving (right-moving) world-sheet fermions in the $\mathbf{8_s}$ ($\mathbf{8_c}$) of $SO(8)$.

We will be interested in a constant background for the field strength of $A_{\mu i} \equiv \frac{G_{\mu i}}{\lambda_I} + B_{\mu i}$[3]. To extract the relevant couplings it is convenient to choose for the vielbeins the representation

$$e_M^{\hat{L}} = \begin{pmatrix} e_i^j & G_{\mu i} \\ 0 & \eta_{\mu\nu} \end{pmatrix} + O(G_{\mu i}^2) \tag{4.11}$$

with e_i^j the square root of the torus metric (in the complex basis $e_z^z = e_{\bar{z}}^{\bar{z}} = \sqrt{T_2^I}$, $e_{\bar{z}}^z = e_z^{\bar{z}} = 0$). Since we are interested in extracting four powers of the field strengths $A_{\mu i}$ from the expansion of (4.7) and then set $G_{\mu i}$ to zero, we can set always $G_{\mu i}$ to zero unless a derivative hits on it. This is so because such terms would anyway disappear in the final result due to gauge invariance. Furthermore since the connections involve at most one derivative of the metric, the terms containing two or more derivatives of $G_{\mu I}$ are irrelevant for our present discussion. Here and in the following we will always set these components to zero.

Using the ansatz (4.11), and recalling from [3] that, after orientifolding, θ is periodic (and therefore has 8 zero modes, denoted by θ_0), whereas $\tilde{\theta}$ is anti-periodic, it can be seen that the contributions of interest in (4.8) come from two kind of terms, $\bar{\theta}_0\gamma^{\bar{z}}\gamma^{\hat{K}\hat{L}}\tilde{\omega}_i^{\hat{K}\hat{L}}\theta_0$ for $i = z, \bar{z}$ and $\bar{\theta}_0\gamma^{\mu}\gamma^{\hat{K}\hat{L}}$ $\tilde{\omega}_\rho^{\hat{K}\hat{L}}\theta_0$. The first type of terms gives the following fermionic bilinears

$$\begin{aligned}\eta &= \frac{i}{4}\partial_{[\mu}A_{\nu]\bar{z}}\bar{\theta}_0\gamma^{\mu\nu\bar{z}}\theta_0\\ \bar{\eta} &= \frac{i}{4}\partial_{[\mu}A_{\nu]z}\bar{\theta}_0\gamma^{\mu\nu\bar{z}}\theta_0,\end{aligned} \tag{4.12}$$

whereas the second type gives

$$\epsilon_{\rho\mu} = \frac{i}{4}\partial_{[\nu}A_{\rho]\bar{z}}\bar{\theta}_0\gamma^{\mu\nu\bar{z}}\theta_0. \tag{4.13}$$

In the above formulae and in the following γ_z and $\gamma_{\bar{z}}$ denote $SO(2)$ gamma matrices and the T_2 dependence will be explicitly displayed. Contributions to the $\mathcal{F}^4$ couplings will come from the D-instanton partition function once four powers of these combinations are brought down to soak the 8 fermionic zero modes. This is why we replaced the fermions in (4.12,4.13) by their zero mode part θ_0. $\mathcal{F}^4$ couplings will be defined then by the fourth order terms in the η, $\bar{\eta}$ expansion of the resulting effective action. The action (4.7) can then be written as [4]

$$\begin{aligned}S_{D-inst} &= 2\pi i T^F - 4\pi\eta\\ &+2\pi\int d^2z\Big[\partial_z X^\mu\partial_{\bar{z}}X_\mu - 2i\bar{\tilde{\theta}}\gamma_{\bar{z}}\partial_z\tilde{\theta} - 2i\bar{\theta}\gamma_z\partial_{\bar{z}}\theta\\ &+\frac{\bar{\eta}}{T_2-\eta}(\partial_{\bar{z}}X^\mu\partial_{\bar{z}}X_\mu - 2i\bar{\tilde{\theta}}\gamma_{\bar{z}}\partial_{\bar{z}}\tilde{\theta}) + ...\Big]\end{aligned} \tag{4.14}$$

where ... represent higher quantum fluctuations, and we have rescaled X^μ, $\tilde{\theta}$ as

$$\begin{aligned}X^\mu &\to (1 - \frac{2\epsilon}{T_2^F})_{\mu\nu}X^\nu \qquad \mu = 0,\dots 7\\ \tilde{\theta}_\alpha &\to (1 - \frac{\eta}{T_2^F})^{\frac{1}{2}}\tilde{\theta}_\alpha \qquad \alpha = 1,\dots 8.\end{aligned} \tag{4.15}$$

We have considered so far the Nambu-Goto action for a single D-string. The results (4.14) can however easily be generalized to the N D-string case. The low energy effective action describing the excitations of the N D-string system is described by $O(N)$ gauge theory studied in [3]. As argued in that reference, after integrating out the very massive degrees of freedom

[2] Strictly speaking, because of the Majorana-Weyl nature of fermions, the gauge fixing condition makes sense only in the Minkowski space-time (and world-sheet). We assume that X^9 is time like in which case z and $\bar{z}$ are real light-like coordinates and τ_2 and U_2 are imaginary. At the very end we will do the analytic continuation back to complex τ and U

[3] Notice that precisely this combination appears in the connections in (4.9)

[4] We omit in this expression a trivial λ_I-rescaling of all bosonic X_μ and fermionic fields $\theta, \tilde{\theta}$

in the infrared limit, we are left with an effective conformal field theory in terms of the diagonal multiplets X^i_μ, θ^i and $\tilde{\theta}^i$, $i = 1, \ldots, N$, on which the orbifold permutation group S_N is acting. Correspondingly, in (4.14) the fermionic bilinears η and $\bar{\eta}$ will involve $\sum_{i=1}^N \bar{\theta}^i \gamma^{\mu\nu\bar{z}} \theta^i = N \bar{\theta}_0^{cm} \gamma^{\mu\nu\bar{z}} \theta_0^{cm} + ...$, θ_0^{cm} being the fermionic zero mode corresponding to the center of mass. Also, the volume factor $\sum_{i=1}^N G_{z\bar{z}} \partial_z X_i^z \partial_{\bar{z}} X_i^{\bar{z}} = N T_2^I$ pick up a factor of N. The effective action is then given by simply replacing $T_2^I, \eta, \bar{\eta}$ and ϵ by $N T_2^I, N\eta, N\bar{\eta}$ and $N\epsilon$. The operators expressed in terms of the non-zero mode fields will be replaced by the sum of the N copies of them, involving X^i_μ θ^i and $\tilde{\theta}^i$.

The moduli dependence of $\mathcal{F}^4$ couplings in the N D-instanton background are then defined by the $\eta^{4-r}\bar{\eta}^r$ terms in the expansion of the exponential of (4.14) written for ND-instantons. Identifying the fermionic bilinears $2\pi\eta, 2\pi\bar{\eta}$ with our previous sources $\nu, \bar{\nu}$, one can show that the perturbative results (4.2), (4.3) are reproduced.

Let us start by considering the $\bar{\eta} = 0$ case. Were it not for the rescaling of the $X^\mu, \tilde{\theta}$ fields (4.15), we would have a free theory (upto Weyl permutations) defining the orbifold partition function $\frac{1}{NT_2} \sum_{M,L,s} M^{-4} \mathcal{A}(\frac{s+\bar{U}M}{L}) e^{2N\eta}$ [3, 12]. Identifying as before L, M, s with m_1, n_2, n_1 respectively we can see by a simple inspection of (4.2), (4.3) that we reproduce the $\bar{\eta} = 0$ term except for the fact that we get $\frac{1}{T_2^F}$ instead of $\frac{1}{T_2^F - \eta}$ that appears in (4.2).

We want now to show that, at least to first order in $\frac{1}{T_2^F}$, the rescaling of the fields (4.15) results in the desired renormalization of the inverse area prefactor $\frac{1}{T_2^F} \to \frac{1}{T_2^F - \eta}$. Indeed, to the first order, the Jacobian of the transformation (4.15) is given by the following correlation function:

$$\int d^2 z \ \left[4\epsilon_{\mu\nu} \langle \partial_z X^\mu \partial_{\bar{z}} X^\nu \rangle - 2i\eta \langle \bar{\tilde{\theta}} \gamma^z \partial_z \tilde{\theta} \rangle \right] \quad (4.16)$$

This expression, which involves expectation values of the kinetic energies of X^μ and $\tilde{\theta}$, needs to be regularized. If we adopt a point-splitting regularization, the term involving $\tilde{\theta}$ is zero because $\tilde{\theta}$ has no zero modes. We are thus left with the bosonic contribution, which can be evaluated using:

$$\langle \partial_z X^\mu(z) \partial_{\bar{z}} X^\nu(w) \rangle = -\frac{\delta^{\mu\nu}}{8\pi}. \quad (4.17)$$

The claimed result follows after performing the z-integration and using $\delta^{\mu\nu}\epsilon_{\mu\nu} = -\eta$. We expect that higher order terms reproduce the expansion of $\frac{1}{T_2^F - \eta}$. We can then write finally

$$\begin{aligned} &\mathcal{I}_0^{Dinst}(\eta) \equiv \langle e^{-S_{ND-inst}(\eta,0)} \rangle \\ &= \frac{\mathcal{V}_{10}}{(2\pi^2)^4} \sum_\epsilon \sum_{L,M,s} \frac{1}{LM^5 (T_2^F - \eta)} e^{-2\pi i \bar{T}_F LM} \\ &\mathcal{A}_\epsilon(\frac{s + \bar{U}M}{L}) e^{4\pi LM\eta} \end{aligned} \quad (4.18)$$

that reproduces the $r = 0$ term in (4.2) after the previous identifications.

We can now go on and consider r $\bar{\eta}$ insertions in (4.18). From the $\bar{\eta}$ coupling in (**??**) one can see that each of these correspond to the insertion of a normalized stress-energy tensor $\frac{1}{T_2^F - \eta} \int d^2 z \overline{T}(\bar{z})$ with $\overline{T}(\bar{z}) \equiv \sum_{i=1}^N \left[\partial_{\bar{z}} X^{\mu i} \partial_{\bar{z}} X^i_\mu(\bar{z}) - 2i \bar{\tilde{\theta}}^i \gamma_{\bar{z}} \partial_{\bar{z}} \tilde{\theta}^i(\bar{z}) \right]$. But this is precisely the Virasoro generator of an infinitesimal shift in the $\bar{z}$ worldsheet coordinate, $\bar{L}_0$, which couples to $\bar{U}$. Therefore its insertions translate into covariant derivatives $D_{\bar{U}}$ acting on (4.18). Note that although the expressions above were to make sense only to the first subleading term in $1/T_2$ the fact that the $\bar{\eta}$ coupling appears with the same normalization factor $\frac{1}{T_2^F - \eta}$ as in the perturbative result (4.2) is quite remarkable.

The calculation above shows the precise matching of the first subleading correction around D-string instantons with the exact result on the fundamental string side. How can we extend this to all orders in $1/T_2^F$? The steps involved are two fold. First of all, we will need terms upto 8th order in fermion zero modes in the covariant Green-Schwarz action. This can in principle be obtained by repeatedly using space-time supersymmetry and kappa symmetry. Secondly, we need to solve the linearized equation for $\bar{\eta}$ in the presence of arbitrary η. This will involve turning on 8-dimensional gravitational and dilaton fields to the first order in $\bar{\eta}$. Although straightforward, the computation involved is rather cumbersome. We will give here an alternative argument to show that the complete action should correctly reproduce the full perturbative expansion. For $N = 1$ the covariant D-string action is the same as the fundamental string action. The static gauge computation presented here should be the same as the light cone gauge computation in the fundamental string side provided one restricts to $detW = 1$ in equation (4.5). This shows that at least for single instanton the two results should agree to all orders in $1/T_2$. For $N > 1$, the orbifold CFT description implies that the result is obtained from the $N = 1$ result by applying the Hecke operator H_N, upto N dependent factors. Indeed, by the zero mode argument of [12, 3], one knows that only the twisted sectors of the form $(L)^M$ contribute and the resulting expression involves the sum over L, M, s. Modular covariance in $\bar{U}$ under the subgroup Γ then fixes the powers of L in the sum over the sectors, giving rise to Hecke operator H_N. The only quantity which is not fixed in the amplitude $\langle \bar{F}^r F^{4-r} \rangle$ by this argument is the N dependence of the leading as well as the subleading terms in $1/T_2^F$ for each r. To determine this N dependence we note that the general form of the N D-string action in the orbifold limit is schematically of the form

$$S_{ND-inst} = NT_2^F \sum_{k=0}^{4} \frac{a_k}{(T_2^F)^k} (F_{\bar{z}}\theta_0^2)^k + \sum_{i=1}^{N} \sum_{k=0}^{4} \frac{1}{(T_2^F)^k} (F_{\bar{z}}\theta_0^2)^k A_k^i + \sum_{i=1}^{N} \frac{1}{T_2^F} (F_z\theta_0^2) \sum_{k=0}^{3} \frac{1}{(T_2^F)^k} (F_{\bar{z}}\theta_0^2)^k B_k^i \quad (4.19)$$

where a_k are numerical coefficients independent of N and A_k^i and B_k^i are dimension (1,1) and (0,2) operators (depending on X, $\tilde{\theta}$ and the non-zero modes of θ) for each of the N copies of the variables. Here we have suppressed all 8-dimensional gamma matrices and Lorentz indices. Note that (4.14) obtained by including only upto quartic terms in fermion fields in the covariant Green-Schwarz action fixes a_0, a_1, a_2, A_0, A_1 and B_0 and these were the objects that entered in the computation of the first subleading corrections. In writing (4.19) we have assumed that the higher dimensional operators will decouple in the infrared limit. The powers of T_2^F in the above are easily seen by scaling arguments while the N dependence follows from the definition of the fermion zero mode $\theta^{cm} = \frac{1}{N}\sum_i \theta^i$.

One can see from this general form of the action, that the leading term for each r comes with $\frac{1}{N} H_N(D_{\bar{U}}^r A_{-+})$ while in the subleading correction in $1/T_2^F$, the volume T_2^F is replaced by NT_2^F. This exactly reproduces the exact formula (4.5) of the fundamental string.

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