Diffeomorphisms and Weyl tranformations in AdS_3 gravity

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Abstract: It is shown that the AdS_3 gravity action with boundary terms is non invariant under diffeomorphisms and that its Lie derivative has the form of the Weyl anomaly in two dimensions. This variation is compensated by a Weyl transformation of the boundary metric when the radial derivative of the metric on the boundary is expressed in terms of the stress tensor of a Liouville field. The obtained invariance of the action under the combined transformation of a diffeomorphism and a Weyl transformation allows to interpret the computed Lie derivative as minus the Weyl anomaly of the two-dimensional effective action.

Brown and Henneaux have shown [1] that the asymptotic symmetry group of anti-de Sitter gravity in three dimensions is the conformal group in two dimensions with a central charge $c = 3l/2G$. We are interested in understanding this central charge as the Weyl anomaly of the two-dimensional effective action. This anomaly has been calculated in [2] by means of a regularization procedure and for a constant Weyl parameter[1]. Here we will relate it to the non invariance of the action under diffeomorphisms (see also [4]). We compute the Lie derivative of the AdS_3 on-shell action and show that it has the form of the Weyl anomaly in two dimensions with the value of the Brown-Henneaux central charge. However the variation of the action can be compensated by a Weyl transformation on the boundary, provided that the radial derivative of the metric on the boundary is expressed in terms of the stress tensor of a Liouville field. Moreover the Liouville equation and the Einstein equation are shown to be satisfied at the same time. Therefore the invariance of the action under the combined transformation of the diffeomorphism and the Weyl transformation is established and the Lie derivative of the action can be interpreted as minus the Weyl anomaly of the two-dimensional effective action. In this way, the relation between AdS_3 gravity and Liouville theory [5] is recovered at the level of the Weyl transformations properties and of the equations of motion. The work presented here was done in collaboration with F. Englert, M. Rooman and P. Spindel, and a extended version of these results can be found in [6].

The action considered is the Einstein-Hilbert action for three-dimensional gravity with a negative cosmological constant, improved by the Gibbons-Hawking surface term and a constant surface term:

$$S = \frac{1}{16\pi G}\int_M \sqrt{-^{(3)}g}(^{(3)}R + \frac{2}{l^2})d^3x + \frac{1}{8\pi G}\int_{\partial M} \sqrt{-^{(3)}g}(D_\mu n^\mu + \frac{1}{l})d^2x\,. \quad (1)$$

The constant term makes the Lie derivative of the action finite. The Gibbons-Hawking term gets rid of the second derivatives of the metric and makes the action stationary on the equations of motion for small variations of the metric that keep it fixed on the boundary. Here, because of

[1] In a recent paper, the anomaly has been obtained for arbitrary Weyl parameters by computing the boundary effective action [3].

the cosmological constant in the Einstein equations, the metric diverges at spatial infinity and the boundary metric is ill-defined. Yet a conformal class of boundary metrics can be builded by multiplying the metric by an arbitrary function (which is called the defining function) that vanishes on the boundary and taking the value of this product on the boundary as the boundary metric. As this metric changes by a conformal transformation when the defining function is changed, a conformal equivalence class of boundary metrics has been defined.

According to Fefferman and Graham [7], a suitable choice of coordinates allows to put any solution of the Einstein equations in the form:

$$ {}^{(3)}g_{\mu\nu}dx^\mu dx^\nu = \frac{l^2}{4y^2}dy^2 + \frac{1}{y}g_{ij}(y,x)dx^i dx^j \,, \quad (2) $$

where $y=0$ on the boundary. The metric g_{ij} appearing in (2) can be expanded order by order in the radial coordinate y, leading to the following development:

$$ {}^{(3)}g_{\mu\nu}dx^\mu dx^\nu = \frac{l^2}{4y^2}dy^2 + (\frac{1}{y}g_{(0)ij} + g_{(2)ij})dx^i dx^j + \mathcal{O}(y) \,, \quad (3) $$

where $g_{(0)ij} = g_{ij}(y=0,x)$ is a representative of the conformal class of boundary metrics and $g_{(2)ij}$ is the first radial derivative of g_{ij} on the boundary. The Einstein equations constraint the metric g_{ij} order by order in y. The equations of motion for $g_{(2)}$ are given by:

$$ Tr(g_{(0)}^{-1}g_{(2)}) = -\frac{l^2}{2}R(g_{(0)}) \,, \quad (4) $$

$$ D_i(g_{(0)}^{-1ik}g_{(2)kj}) - D_j(g_{(0)}^{-1ik}g_{(2)ik}) = 0 \,. \quad (5) $$

We see that, unlike in higher dimensions where $g_{(2)}$ is uniquely determined by the Einstein equations, there is an equation of motion for its trace-free part. When $g_{(2)}$ is chosen, the rest of the solution is determined. This shows that the on-shell action depends on the boundary metric $g_{(0)}$ and on the trace-free part of $g_{(2)}$.

We consider now diffeomorphisms that keep the metric in the form (2), i.e. such that $\delta_{diff}{}^{(3)}g_{yy} = \delta_{diff}{}^{(3)}g_{yi} = 0$. They are given by [8]:

$$ \delta_{diff}y = 2\delta\sigma(x)y \,, \quad (6) $$

$$ \delta_{diff}x^i = -\frac{l^2}{2}\int_0^y g^{ij}(x,y')\delta\sigma_{,j}dy' \,, \quad (7) $$

and induce on $g_{(0)}$ and $g_{(2)}$ the following transformations:

$$ \delta_{diff}g_{(0)ij} = -2\delta\sigma g_{(0)ij} \,, \quad (8) $$

$$ \delta_{diff}g_{(2)ij} = -l^2 D_i\partial_j\delta\sigma \,. \quad (9) $$

This diffeomorphism keeps $g_{(0)}$ in its conformal class (which is unique in two dimensions). Yet the Lie derivative of the action on the equations of motion gives:

$$ \delta_{diff}S = \frac{l}{16\pi G}\int_{\partial M}\sqrt{-g_{(0)}}R(g_{(0)})\delta\sigma d^2x \,, \quad (10) $$

showing that the action is not invariant under the diffeomorphism.

To restore the invariance, we would like to compensate the above diffeomorphism by a Weyl transformation on the boundary:

$$ \delta_W g_{(0)} = 2\delta\sigma g_{(0)} \,, \quad (11) $$

which clearly cancels the tranformation (8) induced by the diffeomorphism on $g_{(0)}$. We recall that the on-shell action depends not only on the boundary metric $g_{(0)}$ but also on the trace-free part of $g_{(2)}$ because there correspond different $g_{(2)}$'s solutions for the same $g_{(0)}$. During the Weyl transformation, this further degree of freedom needs to be controlled to ensure that we are back to the initial solution, i.e. the transformation properties of $g_{(2)}$ must be specified in order to compensate exactly its variation (9) under the diffeomorphism. To induce this transformation, we note that the dependence of the action on the trace-free part of $g_{(2)}$ can be expressed in terms of a field ϕ living on the boundary. We take for ϕ the following conformal weight:

$$ \delta_W\phi = -\delta\sigma \,, \quad (12) $$

and look for an expression for $g_{(2)}$ as a functional of $g_{(0)}$ and ϕ.

We can check that the relation for $g_{(2)}$ in terms of $g_{(0)}$ and ϕ that implies the correct transformation for $g_{(2)}$ under the Weyl transformation is given by the following expression:

$$ g_{(2)ij} = l^2[-D_i\partial_j\phi + \partial_i\phi\partial_j\phi + g_{(0)ij}(\lambda e^{2\phi} - \frac{1}{2}\partial^k\phi\partial_k\phi)] \,. \quad (13) $$

It can be written more compactly in terms of the on-shell Liouville stress tensor T_{ij}:

$$\frac{1}{l^2}g_{(2)ij} = \frac{8\pi G}{l}T_{ij} - \frac{1}{2}g_{(0)ij}R\,, \qquad (14)$$

where T_{ij} is derived from the Liouville action:

$$S = \frac{l}{8\pi G}\int(\frac{1}{2}\sqrt{-g_{(0)}}g^{ij}_{(0)}\partial_i\phi\partial_j\phi + \frac{1}{2}\sqrt{-g_{(0)}}R\phi + \lambda\sqrt{-g_{(0)}}e^{2\phi})d^2x\,. \qquad (15)$$

The constants in front of the action and T_{ij} have been adjusted to match the value of the Weyl anomaly that will be computed below.

Under the Weyl transformations of $g_{(0)}$ and ϕ,

$$\delta_W g_{(0)} = 2\delta\sigma g_{(0)}\,, \qquad (16)$$

$$\delta_W\phi = -\delta\sigma\,, \qquad (17)$$

equation (13) implies the searched-for transformation for $g_{(2)}$:

$$\delta_W g_{(2)ij} = l^2 D_i\partial_j\delta\sigma\,. \qquad (18)$$

Moreover, we see that the field ϕ satisfies the Liouville equation when the Einstein equation (4) on the trace of $g_{(2)}$ is satisfied:

$$Tr(g^{-1}_{(0)}g_{(2)}) + \frac{l^2}{2}R = l^2(-\Box\phi + \frac{1}{2}R + 2\lambda e^{2\phi}) = 0\,. \qquad (19)$$

The Einstein equation (5) on the trace-free part of $g_{(2)}$ is then automatically verified:

$$D_i(g^{-1ik}_{(0)}g_{(2)kj}) - D_j(g^{-1ik}_{(0)}g_{(2)ik}) = l^2\partial_j\phi(\Box\phi - \frac{1}{2}R - 2\lambda e^{2\phi}) = 0\,, \qquad (20)$$

and expresses the conservation of the Liouville stress tensor. Indeed it can be written, with the use of (19), as:

$$l^2\frac{8\pi G}{l}D_iT^i{}_j = 0\,. \qquad (21)$$

After doing the diffeomorphism (8,9) and the Weyl transformation (16,17), we are back to the initial solution and the action is invariant under this combined transformation:

$$(\delta_{diff} + \delta_W)S = 0\,. \qquad (22)$$

The value of the Weyl anomaly of the two-dimensional effective action can then be deduced from expression (10) (see also [4]):

$$\delta_W S = -\delta_{diff}S = -\int_{\partial M}\sqrt{-g_{(0)}}A\delta\sigma\,, \qquad (23)$$

with

$$A = \frac{3l}{2G}\frac{R}{24\pi}\,. \qquad (24)$$

We recover in this way the central charge of the Brown-Henneaux asymptotic algebra [1].

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