Testing the AdS/CFT correspondence beyond large N

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ABSTRACT: According to the AdS/CFT correspondence, the $d=4$, $\mathcal{N}=4$ SU(N) super Yang-Mills theory is dual to the type IIB string theory compactified on $AdS_5 \times S^5$. Most of the tests performed so far are confined to the leading order at large N or equivalently string tree-level. To probe the correspondence beyond this leading order and obtain $\frac{1}{N^2}$ corrections is difficult since string one-loop computations on an $AdS_5 \times S^5$ background generally are beyond feasibility. However, we will show that the chiral SU$(4)_R$ anomaly of the super YM theory provides an ideal testing ground to go beyond leading order in N. In this paper, we review and develop further our previous results [1] that the $1/N^2$ corrections to the chiral anomaly on the super YM side can be exactly accounted for by the supergravity/string effective action induced at one loop.

KEYWORDS: Duality in Gauge Field Theories, Anomaly, D-branes.

1. Introduction

We begin by briefly reviewing some relevant basic aspects of the AdS/CFT correspondence [2, 3, 4], see in particular [5].

Consider type IIB string theory with a number N of D3 branes. There are open strings ending on these D3 branes and closed strings in the bulk. The effective low energy action consists of IIB supergravity describing the bulk closed strings, and $\mathcal{N}=4$ supersymmetric U(N) gauge theory (SYM) describing the open strings ending on the D3 branes. The gauge group of the latter actually is SU(N) since the U(1) decouples. Considering low energies at fixed α' is equivalent to fixed energy and taking the $\alpha' \to 0$ limit. In this limit the gravitational coupling $\kappa \sim g_s\alpha'^2$ vanishes and the interactions between the branes and the bulk can be neglected, as well as all higher derivative terms in the brane action. Only free bulk supergravity and pure $\mathcal{N}=4$ SU(N) SYM in $d=4$ dimensions remain. The latter is a conformal field theory (CFT).

There is a different way to describe the same physics. D3 branes may be viewed as certain solutions of the supergravity field equations, namely

$$\begin{aligned} \mathrm{d}s^2 &= \frac{1}{\sqrt{f}}\left(-\mathrm{d}t^2+\mathrm{d}x_1^2+\mathrm{d}x_2^2+\mathrm{d}x_3^2\right) \\ &\quad +\sqrt{f}\left(\mathrm{d}r^2+r^2\mathrm{d}\Omega_5^2\right), \\ f &= 1+\frac{R^4}{r^4}\ , \quad R^4 = 4\pi g_s\alpha'^2 N\ . \end{aligned} \tag{1.1}$$

Here t, x_1, x_2, x_3 are the longitudinal coordinates (on the D3) while r and Ω_5 describe the transverse space. N is the number of (coincident) D3 branes. There is also a five-form field strength F which is proportional to N. Again, one wants to study the low energy excitations in this description and compare with the first one. Due to the non-trivial function $f(r)$ in the metric there is a red-shift factor between energies measured at r and energies measured at $r=\infty$:

$$E_\infty = f^{-1/4}E_r = \left(1+\frac{R^4}{r^4}\right)^{-1/4} E_r\ . \tag{1.2}$$

One sees that if $r \to 0$, $E_\infty \to 0$ for any finite E_r. So there are two types of low energy excitations: low energy at finite r (bulk) or finite energy near the horizon ($r=0$), and the two types decouple yielding (free) bulk supergravity and near horizon supergravity. Concentrate on the near horizon limit: as $r \to 0$ one has $f \sim \frac{R^4}{r^4}$.

Since also $\alpha' \to 0$, it is convenient to introduce the finite quantity $u = r/\alpha'$ so that the near horizon metric becomes

$$\frac{1}{\alpha'}\mathrm{d}s^2 = \lambda^{-1/2}u^2\left(-\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2\right) + \lambda^{1/2}\frac{\mathrm{d}u^2}{u^2} + \lambda^{1/2}\mathrm{d}\Omega_5^2 \,, \quad (1.3)$$

where we introduced the finite quantity

$$\lambda = 4\pi g_s N = \frac{R^4}{\alpha'^2} \,. \quad (1.4)$$

The metric (1.3) is the metric of $AdS_5 \times S^5$.

Comparing both descriptions of the same physics one then is led to identify the conformally invariant, $\mathcal{N} = 4$ SU(N) SYM theory in $d = 4$ with the supergravity or small α' limit of IIB string theory on $AdS_5 \times S^5$. We will be more precise shortly.

How do the different parameters on the SYM side compare to those of the supergravity/string theory? In the former we have the coupling g_{YM} and the integer N determining the gauge group SU(N). On the supergravity/string side we have α' and R (with only the dimensionless ratio R^2/α' being a relevant parameter) and the coupling g_s. A first relation is already obtained in eq. (1.1), namely $R^4 = 4\pi g_s \alpha'^2 N$ or equivalently eq. (1.4). A second relation is obtained from the D3 brane action from which one reads the YM coupling in terms of the string coupling. Thus

$$\frac{1}{g_{\mathrm{YM}}^2} = \frac{1}{4\pi\, g_s} \quad \text{and} \quad N = \frac{1}{4\pi\, g_s}\frac{R^4}{\alpha'^2} \quad (1.5)$$

expresses the SYM parameters g_{YM} and N in terms of the string/supergravity parameters g_s and R^2/α' and vice versa. In large N gauge theories the relevant loop-counting parameter is the 't Hooft coupling $g_{\mathrm{YM}}^2 N$ rather than g_{YM}^2. Combining both eqs (1.5) yields $g_{\mathrm{YM}}^2 N = \frac{R^4}{\alpha'^2}$ which by eq (1.4) is just the quantity called λ. It is useful to rewrite the relations between the parameters of the two descriptions as

$$\lambda = \frac{R^4}{\alpha'^2} \quad , \quad \frac{N}{\lambda} = \frac{1}{4\pi\, g_s} \quad (1.6)$$

with λ now meaning the 't Hooft coupling.

Let us first comment on the first relation: perturbative SYM theory is a good description if the 't Hooft coupling λ is small. Supergravity, rather than string theory, should be a good description if the radius of curvature of AdS_5 and S^5 is large, meaning $R^2 \gg \alpha'$ or λ large. The two regimes are opposite as is often the case with dualities. This of course avoids the obvious contradiction that both descriptions look so different.

At fixed 't Hooft coupling λ, the second relation (1.6) tells us that $\frac{1}{N^2}$ corresponds to the string loop-counting parameter g_s^2, so that the large N limit of SYM corresponds to classical string theory (or classical supergravity if also $\lambda \gg 1$), and $\frac{1}{N^2}$ corrections should correspond to one-loop effects in string theory.

One now has various possible conjectures: 1.) The weakest one is: SYM is dual to AdS supergravity only for $\lambda \to \infty$, but the full string theory is different. This version would not be very useful. 2.) The SYM theory is dual to string theory on AdS for finite λ but only as $N \to \infty$ or equivalently $g_s \to 0$. This includes α' corrections beyond the supergravity approximation, but no string loops. 3.) This is the strongest version, generally referred to as the Maldacena conjecture: the SYM theory is dual to string theory for all λ and all N, i.e. all R^4/α'^2 and g_s.

While there is now reasonable evidence for version 2.) of the conjecture (see e.g. [5]) the strong version 3.) is hard to test since standard string or supergravity loop computations on an $AdS_5 \times S^5$ background are difficult, to say the least, if not unfeasible, with the present state of the art.

Many successful tests are group theoretic in nature. Some are not restricted to tree-level or even perturbation theory, but on the other hand they do not really provide any real test at one-loop. Examples are global symmetries (disregarding possible anomalies for the moment): AdS_5 space-time has an SO$(4,2)$ symmetry which also is the conformal group of the $\mathcal{N} = 4$ SYM theory. The "internal" symmetry is SO(6) $\simeq$ SU(4): this is the isometry of the S^5 as well as the R-symmetry of the SYM theory. The latter actually is anomalous which will be important for us. Both theories have the same amount of supersymmetry, the full supergroup being SU$(2,2|4) \supset$ SO$(4,2) \times$ SU$(4)_R$. Also the duality symmetry

$SL(2,\mathbf{Z})$ is the same as it acts on

$$\tau = \frac{4\pi i}{g_{\rm YM}^2} + \frac{\theta}{2\pi} = \frac{i}{g_s} + \frac{\chi}{2\pi} \ . \qquad (1.7)$$

A promising arena for performing tests beyond the large N limit or string tree-level is to look at certain anomalies. On the SYM side anomaly coefficients are easily established one-loop effects in λ that are protected against higher order corrections. Typically such an anomaly coefficient will depend on the number of fermion fields in the SYM theory, i.e. on N. The goal then is to reproduce the exact N-dependence from the string theory including subleading terms $\sim \frac{1}{N^2}$ coming from string loops. If we want to have any chance to be able to do this calculation, the relevant quantity to compute in string theory should be of a topological nature, like a Chern-Simons term, so that the actual metric on AdS_5 is irrelevant.

2. The chiral $SU(4)_R$ anomaly in the $\mathcal{N}=4$ SYM

The anomaly we will consider is the chiral $SU(4)_R$ anomaly. As already mentioned, $SU(4)_R$ is a (classical) global symmetry of the $\mathcal{N}=4$ SYM theory. Due to the presence of chiral fermions transforming in complex conjugate representations of $SU(4)_R$ this symmetry is spoiled at one loop and there is an anomaly: the one-loop effective action in the presence of *external* $SU(4)_R$ gauge fields is no longer invariant under $SU(4)_R$ and the non-invariance is proportional to the number of fermions. Since they are also in the adjoint representation of the "true" gauge group $SU(N)$ there are N^2-1 of them, and the anomaly is proportional to N^2-1. As we recall below, the leading term $\sim N^2$ is accounted for by tree-level supergravity [6]. It is the -1 correction which should originate from a string/supergravity loop effect, and it indeed does as we showed in [1] and explain in the remainder of this paper.

Before explaining the string/supergravity loop correction let us review how the leading N^2 term is obtained in the string/supergravity description. Here $SU(4)\simeq SO(6)$ acts as an isometry on the S^5. As a consequence, the AdS_5 supergravity is actually a gauged supergravity [7, 8, 9] and there is an $SU(4)_R$ gauge group with gauge fields $\tilde{A}^a_\mu(x,z)$, $\mu=0,1,\dots 4$ and $a=1,\dots 15 = \dim SU(4)$. This gauge group is of course not to be confused with the $SU(N)$ of the conformal SYM theory. Note also that in the latter, $SU(4)_R$ is a global symmetry, hence there are $SU(4)_R$ currents $J^a_\mu(x)$, $\mu=0,1,2,3$, but no associated gauge fields. We can nevertheless couple these currents to *external* gauge fields $A^a_\mu(x)$, $\mu=0,1,2,3$ which act as sources for these currents. Then by a standard argument, the non-invariance of the one-loop effective action $\Gamma[A_\mu]$ under gauge transformations of these external gauge fields is equivalent to the covariant non-conservation of the currents: let δ_v be such a gauge transformation with parameter v, then

$$\begin{aligned}\delta_v\Gamma[A_\mu] &= \int \delta_v A^a_\mu \frac{\delta\Gamma}{\delta A^a_\mu} = \int \delta_v A^a_\mu J^{a,\mu} \\ &= \int (D_\mu v)^a J^{a,\mu} = -\int v^a (D_\mu J^\mu)^a \qquad (2.1)\end{aligned}$$

with

$$(D_\mu J^\mu)^a \sim -(N^2-1) d^{abc}\epsilon^{\mu\nu\rho\sigma}\partial_\mu A^b_\nu \partial_\rho A^c_\sigma + \dots \qquad (2.2)$$

the precise numerical coefficient being given below.

There is a standard prescription [4] in the AdS/CFT correspondence how to compute correlation functions: we will give this prescription for the case of present interest. For any ($SU(N)$ gauge-invariant) operator $\mathcal{O}(x)$ like the currents $J^a_\mu(x)$ of the SYM theory, introduce a source $\phi_0(x)$ like the $A^a_\mu(x)$. Then the generating functional for correlators of J^a_μ is just

$$e^{-\Gamma[A]} \equiv \langle e^{-\int d^4x A^a_\mu(x) J^{a,\mu}(x)}\rangle_{\rm SYM} \ . \qquad (2.3)$$

In AdS_5 string theory there is a field $\phi(x,z)$ such that at the boundary $z=0$ of AdS_5 (note that $z=1/u$) which is just the four-dimensional space of the SYM theory one has $\phi(x,z=0)=\phi_0(x)$. In our case this is just $\tilde{A}^a_\mu(x,z=0)=A^a_\mu(x)$ (for $\mu=0,1,2,3$ only) where the $\tilde{A}^a_\mu$ are the gauge fields of the gauged supergravity. The prescription [4] then is

$$e^{-\Gamma[A]} = Z_{\rm string}\Big|_{\tilde{A}^a_\mu(x,z=0)=A^a_\mu(x)} \ , \qquad (2.4)$$

meaning that the string partition function should be evaluated subject to the boundary condition

$\tilde{A}^a_\mu(x, z = 0) = A^a_\mu(x)$ for $\mu = 0, 1, 2, 3$. Writing

$$Z_{\rm string} = {\rm e}^{-S^{\rm cl}_{\rm string} - S^{\rm 1-loop}_{\rm string} - \cdots} \equiv {\rm e}^{-S^{\rm eff}_{\rm string}} \tag{2.5}$$

eq (2.4) together with eq (2.1) implies that, if the AdS/CFT correspondence is correct, we should have

$$\delta_v S^{\rm eff}_{\rm string}\Big|_{\tilde{A}^a_\mu(x,z=0)=A^a_\mu(x)} = \delta_v \Gamma[A] = -\int v^a (D_\mu J^\mu)^a \tag{2.6}$$

which is non-vanishing according to (2.2). Thus the $SU(4)_R$ gauge variation of $S^{\rm eff}_{\rm string}$ should directly reproduce the SYM chiral $SU(4)_R$ anomaly. Actually for the purpose of reproducing the leading N^2 part of the anomaly it is enough to consider the classical supergravity action [6].

Let us now determine the exact anomaly coefficient of the $\mathcal{N} = 4$ SYM theory in 4 dimensions. This theory has four complex Weyl fermions λ in the fundamental representation of $SU(4)_R$ with the chirality part (0,1/2) in **4** and (1/2,0) in $\mathbf{4}^*$ (see for example [10]. Our conventions here are equivalent to those of [10].) Moreover, all fields are also in the adjoint representation of the "true" gauge group $SU(N)$ which acts as a "flavour" group with respect to the $SU(4)_R$. Thus there are actually $N^2 - 1$ complex Weyl fermions λ in the **4**, resp. $\mathbf{4}^*$. The correctly normalised R-symmetry anomaly is given by

$$\delta_v \Gamma[A] = (N^2 - 1) \int_{S^4} \omega^1_4(v, A). \tag{2.7}$$

The differential forms

$$\omega^1_4(v, A) = \frac{1}{24\pi^2} {\rm Tr}\,[vd(AdA + \frac{1}{2}A^3)],$$
$$\omega_5(A) = \frac{1}{24\pi^2} {\rm Tr}\,[A(dA)^2 + \frac{3}{2}A^3 dA + \frac{3}{5}A^5] \tag{2.8}$$

satisfy the descent equations $d\omega_5 = \frac{1}{24\pi^2} {\rm Tr}\, F^3$, and $\delta_v \omega_5 = d\omega^1_4$ with $F = dA + A^2$, $A = A^a T^a$ and $v = v^a T^a$ as usual, the T^a being the generators of SU(4) in the fundamental **4** representation. For later use we note that for T^a in a general representation **R** of SU(4), the corresponding quantities with the trace taken in **R** are

$$\omega^{1\,\mathbf{R}}_{2n} = A(\mathbf{R})\ \omega^1_{2n}\ , \qquad \omega^{\mathbf{R}}_{2n+1} = A(\mathbf{R})\ \omega_{2n+1}, \tag{2.9}$$

where $A(\mathbf{R})$ is the anomaly coefficient defined by the ratio of the d-symbols taken in the representation **R** and in the fundamental representation. In general $2n$ or $2n + 1$ dimensions, since the d-symbol is given by a symmetrized trace of $n + 1$ Lie algebra generators, it is easy to show that the complex conjugate representation $\mathbf{R}^*$ has an anomaly coefficient

$$A(\mathbf{R}^*) = (-1)^{n+1} A(\mathbf{R}). \tag{2.10}$$

Due to the connection of anomalies and Chern-Simons actions in one higher dimension, it is natural to expect that the four-dimensional field theory anomaly is dual to a Chern-Simons action in the gauged AdS_5 supergravity. This is indeed the case as was first pointed out in [4]. The tree level supergravity action on AdS_5 contains the following terms [4, 6, 7, 9]

$$S_{\rm cl}[A] = \frac{1}{4g^2_{SG}} \int d^5x \sqrt{g}\, F^a_{\mu\nu} F^{\mu\nu a} + k \int_{AdS_5} \omega_5. \tag{2.11}$$

Note that here F is the field strength associated with the five-dimensional gauge field $\tilde{A}_\mu$. The exact values of the coefficients $\frac{1}{4g^2_{SG}}$ and k will be important for us. Their ratio is fixed by supersymmetry [7, 9]. They may be obtained by dimensional reduction of the ten-dimensional IIB supergravity on S^5 using the fact that the radius of S^5 is given by eq. (1.4) as $R^4/\alpha'^2 = 4\pi g_s N$. Then it is easy to determine the normalization of the gauge kinetic energy term and one finds

$$g^2_{SG} = \frac{16\pi^2}{N^2}, \quad k = N^2. \tag{2.12}$$

Note that the action (2.11) with the values (2.12) has been used to compute the 2-point and 3-point correlators of the currents J^a_μ in the SYM theory [6]. To leading order in N this gives the correct result.

In usual considerations of supergravity on AdS, one considers gauge configurations $\tilde{A}_\mu$ that vanish at the boundary and so the Chern-Simons term is gauge invariant since $\delta\omega_5 = d\omega^1_4$ and the integral vanishes. For the considerations of the AdS/CFT correspondence however, we precisely want nonvanishing boundary values for $\tilde{A}_\mu$ as explained above, cf eq (2.4). Then under a gauge variation $\delta_v \tilde{A}$, the variation of the Chern-Simons term is a boundary term

$$\delta_v S_{cl} = k \int_{S^4} \omega_4^1(v, A) \ . \tag{2.13}$$

(We take the SYM theory to be defined on compactified Euclidean space, i.e. on S^4.) Now by eq (2.6), approximating $S^{\rm eff}_{\rm string} \to S_{\rm cl}$ and using (2.13) one can read off the $SU(4)_R$ anomaly obtained from the supergravity action (2.11). It is

$$\delta_v \Gamma[A] = \delta_v S_{\rm cl} = N^2 \int_{S^4} \omega_4^1(v, A) \ , \tag{2.14}$$

which agrees with the gauge theory computation (2.7) to leading order in N.

We thus see that the IIB supergravity action contains a Chern-Simons term at tree level which can account for the chiral anomaly of the gauge theory to leading order in N. But there is also a mismatch of "-1" which is of order $1/N^2$ relative to the leading term. As discussed above, this should correspond to a 1-loop effect in IIB string theory. Thus we are lead to examine the string one-loop effective action.

3. One-loop induced Chern-Simons action

Loop effects in AdS_5 supergravity are technically very diffciult to compute due to the complicated propagators in AdS geometry. Here however, this is possible due to the topological character of the Chern-Simons action.

Fermionic contributions

Consider a Dirac fermion ψ in odd dimensions (flat) minimally coupled to vector bosons A_μ of a group G. At the quantum level, a regularization needs to be introduced to make sense of the theory and one cannot preserve both the gauge symmetry (small and large) and the parity at the same time [11, 12]. If one chooses to preserve the gauge symmetry by doing a Pauli-Villars regularization, then there will be an induced Chern-Simons term generated at one loop. The result is independent of the fermion mass. In our notation, the induced Chern-Simons term is

$$\Delta\Gamma = \pm\frac{1}{2}\int \omega^{\mathbf{R}}_{2n+1} = \pm\frac{1}{2}A(\mathbf{R})\int \omega_{2n+1}, \tag{3.1}$$

where $\mathbf{R}$ is the representation of the Dirac fermion. The $\pm$ sign depends on the regularization and can often be fixed within a specific context.

This result was originally [11, 12] obtained for fermions coupled to gauge fields in a flat spacetime and has been extended to full generality for arbitrary curved backgrounds and any odd dimension $d = 2n+1$ [13]. The induced parity violating terms are given (up to a normalization factor) by the secondary characteristic class $Q(A, \omega)$ satisfying

$$dQ(A, \omega) = \hat{A}(R) ch(F)|_{2n+2}, \tag{3.2}$$

where ω is the gravitational connection. Since $\hat{A}(R) = 1 + \mathcal{O}(R^2)$ and $\mathrm{Tr}\, F = 0$ for SU-groups, it is clear that for $n = 2$ ($d = 5$) there are no mixed gauge/gravitational terms. Also, there can be no purely gravitational term since it would correspond to a gravitational anomaly in four dimensions which is not possible. Hence for the present case of SU(4) with $n = 2$, (3.2) simply reduces to $dQ = ch(F)|_6$ giving rise to the Chern-Simons action upon descent, which does not depend on the geometry of AdS_5 at all! Hence the result of (3.1) for a Dirac fermion in flat space(-time) remains valid on AdS_5.

Now we need the particle spectrum of the type IIB string theory on $AdS_5 \times S^5$. The only explicitly known states are the KK states coming from the compactification of the 10 dimensional IIB supergravity multiplet [14]. So we will examine them first. We will argue in the discussion section that the other string states are not likely to modify the result.

Particles in AdS_5 are classified by unitary irreducible representations of SO(2,4). SO(2,4) has the maximal compact subgroup SO(2)×SU(2) ×SU(2) and so its irreducible representations are labelled by the quantum numbers (E_0, J_1, J_2). The complete KK spectrum of the IIB supergravity on $AdS_5 \times S^5$ was obtained in [14, 8] together with information on the representation content under $SU(4)_R$. We reproduce these results in the table below. Actually, all fermions are symplectic Majorana, giving half the anomaly of a Dirac fermion. But there also is a mirror table with conjugate $SU(4)_R$ representations and SU(2)×SU(2) quantum numbers exchanged (opposite chiralities). So these "mirror" fermions give the same anomaly as those in the table and the net effect is that we may restrict ourselves to the fermions of the table treating them as if they were Dirac fermions.

	SU(2) × SU(2)	$SU(4)_R$	
ψ_μ	$(1, 1/2)$	$\mathbf{4}, \mathbf{20}, \cdots$	←
	$(1, 1/2)$	$\mathbf{4}^*, \mathbf{20}^*, \cdots$	
λ	$(1/2, 0)$	$\mathbf{20}^*, \cdots$	←
	$(1/2, 0)$	$\mathbf{4}, \mathbf{20}, \cdots$	
λ'	$(1/2, 0)$	$\mathbf{4}^*, \mathbf{20}^*, \cdots$	←
	$(1/2, 0)$	$\mathbf{4}, \mathbf{20}, \cdots$	
λ''	$(1/2, 0)$	$\mathbf{36}, \mathbf{140}, \cdots$	
	$(1/2, 0)$	$\mathbf{36}^*, \mathbf{140}^*, \cdots$	

(3.3)

Notice that the fermion towers always come in pairs with conjugate representation content, except for a missing $\mathbf{4}^*$ state in the first tower of λ. As a result [1], all contributions cancel two by two except for the contribution from the unpaired $\mathbf{4}$ of the λ tower. The net resulting induced Chern Simons action is

$$\Delta\Gamma = -\frac{1}{2}\int_{AdS_5} \omega_5. \qquad (3.4)$$

While this is almost what we want, it is only half of the desired result. However this is not the whole story.

Doubleton multiplet

There are similar "missing states" in the bosonic towers. Together they are identified with the doubleton multiplet of $SU(2,2|4)$ which consists of a gauge potential, six scalars and four complex spinors. These are nonpropagating modes in the bulk of AdS_5 and can be gauged away completely [14, 15], which is the reason why they don't appear in the physical spectrum. These modes are exactly dual to the U(1) factor of the U(N) SYM living on the boundary [5]. We will now show that the other half of the induced Chern-Simons action is due to the corresponding Faddeev-Popov ghosts.

Let us recall that the doubleton multiplet is absent because it has been gauged away [14] by imposing the gravitino gauge fixing condition (see also [15] for the gauging in the case of $AdS_7\times S^4$ case). The basic idea is that upon compactifying on S^5, the original supersymmetyries in 10 dimensions decompose into an infinite tower of (unwanted) supersymmetries according to the Fourier expansion. This can be fixed however by imposing a certain condition on the variation

$$\delta\psi_\alpha = D_\alpha\epsilon + \frac{i}{2R}\gamma_\alpha\epsilon \qquad (3.5)$$

of the gravitino. Denote the local coordinates of $AdS\times S^5$ by (x^μ, y^α). A general spinor ϵ has the decomposition

$$\epsilon = \sum \epsilon^{I,\pm}(x)\;\Xi^{I,\pm}(y) \qquad (3.6)$$

where $\Xi^{I,\pm}(y)$ are the spinor spherical harmonics on S^5 and satisfy ($\not{D}_y = \gamma^\alpha D_\alpha, \alpha = 5,\ldots 9$)

$$\not{D}_y\Xi^{I,\pm} = \mp i(k+\frac{5}{2})\frac{1}{R}\Xi^{I,\pm} \qquad (3.7)$$

where $k = I \geq 0$ and $\Xi^{I,\pm}$ can be written in terms of the killing spinors $\eta^{I,\pm}$ on S^5. Subsituting (3.6) into (3.5) and using (3.7) we get

$$\delta(\gamma^\alpha\psi_\alpha) = \frac{i}{R}\sum\left[\mp(k+\frac{5}{2})+\frac{5}{2}\right]\epsilon^{I,\pm}(x)\;\Xi^{I,\pm}(y)\,. \qquad (3.8)$$

So one finds that only the component corresponding to $\Xi^{0,+}$ is gauge invariant while all other components of $\gamma^a\psi_\alpha$ can be gauged away. Thus we arrive at the gravitino gauge fixing condition

$$\gamma^\alpha\psi_\alpha(x,y) \sim \chi^{I_0}(x)\eta^{I_0,+}(y) \qquad (3.9)$$

where $\chi^{I_0}(x)$ are some arbitrary spacetime spinor fields. We refer the reader to [14] for more details. Therefore we see that (3.9) is the closest one can get to the gauge condition $\gamma^\alpha\psi_\alpha = 0$. One can also rewrite this condition as

$$\psi_\alpha = \psi_{(\alpha)} + \chi^{I_0}(x)\gamma^\alpha\eta^{I_0,+}(y) \qquad (3.10)$$

where the part $\psi_{(\alpha)}$ satisfies $\gamma^\alpha\psi_{(\alpha)} = 0$. The other Killing spinors η^- have been gauged away. The coefficient of η^- would be a field in the 4^* of SU(4) and is precisely the doubleton spinors we are after. Now the constraint can be taken care of in the functional approach by introducing in the path integral the factor

$$\int db d\bar{b}\frac{1}{\det M}e^{\int d^5x\bar{b}Mb} \qquad (3.11)$$
$$\delta(\gamma^\alpha\psi_\alpha - \sum\chi^I\eta^{I,+} - b^I(x)\eta^{I,-}(y))\cdot\delta(\text{h.c.})$$

where $b(x)$ is a complex fermionic field in the 4^* of SU(4) and $M = \not{D}_x$. Integrating over $b, \bar{b}$

results in the gauged fixed lagrangian. The factor $(\det M)^{-1}$ can be handled by intoducing ghost fields $c, \bar{c}$, which are bosonic spinor fields on AdS_5 and are in the 4^* of SU(4). Thus

$$\frac{1}{\det M} = \int dc d\bar{c}\, e^{-\int d^5x \bar{c}Mc}. \tag{3.12}$$

and so give rises to another -1/2 contribution to the induced Chern-Simons action. So altogether we get a total induced Chern-Simons term of -1,

$$\Delta\Gamma = -\int_{AdS_5} \omega_5, \tag{3.13}$$

which is exactly the desired result. Notice that the induced Chern-Simons action (coming with a constant integer coefficient) is independent of the radius R and this is consistent with the AdS/CFT proposal since the anomaly and its corrections are independent of λ.

Bosonic contributions

There is another interesting effect related to the Chern-Simons action. It is known that in three dimensions, the gluons at one loop can modify the coefficient of the Chern-Simons action by an integer shift. It has been argued that [4, 1] there is no such shift for the present case. Therefore only spinor loops contribute to the induced Chern-Simons action and we find that at finite N, the coefficient k is shifted by

$$k \to k-1 \quad \text{or} \quad N^2 \to N^2 - 1 \tag{3.14}$$

due to the quantum effects of the full set of Kaluza-Klein states.

A few comments about the absence of a shift due to gluon loops are in order. The bosonic shift in pure Chern-Simons theory was first computed in [16] using a saddle point approximation. Later calculations trying to reproduce this results from the perturbative point of view revealed that the precise shift depends on the choice of regularization scheme [1]. In the present case of 5-dimensions, one may try to employ a regularization scheme and do a 1-loop perturbative calculation to determine the possible shift. However, like in the 3-dimensional case, it can be expected that the result will depend on the choice of regularization and a better way to determine the shift is called for. One possibility might be to do a string theory calculation by embedding the Chern-Simons action in a string setting and to determine the quantum loop effects from the string loop effects. Since string theory is free from divergences, no regularization related ambuigities should occur and a definite answer can be expected.

4. Discussion

We have reproduced the correct shift $N^2 \to N^2-1$ of the anomaly coefficient as a one-loop effect in IIB supergravity/string theory on $AdS_5 \times S^5$. This shift is entirely due to the towers of Kaluza-Klein states. No massive string states need to be invoked. It is indeed likely that the latter play no role at all since anomalies are usually due to massless fields only. Note however that we need the full towers of Kaluza-Klein states to get the correct result. A truncation to five-dimensional *AdS* supergravity alone would not give the desired result. Also the AdS_5 supergravity Chern-Simons term originates from compactifying the full IIB supergravity. This is another indication that string states beyond the Kaluza-Klein towers are unlikely to modify our result.

We have been able to obtain a non-trivial one-loop result within a particularly favourable case. In general, one-loop calculations in AdS_5 are very difficult - already tree computations are quite non-trivial! Of course, the anomaly coefficient $\sim N^2 - 1$ should be exact and there cannot be any further higher-loop corrections $\sim N^2 \frac{1}{N^4}$. Again, since the induced Chern-Simons term in 5 dimensions is closely related to anomalies in 4 dimensions, we expect some sort of non-renormalisation theorem to be at work, although it would be nice to have a proof of this statement.

Finally, we would like to make some comments on more or less related situations. There are other dualities like those involving $AdS_7 \times S^4$ where one can expect similar Chern-Simons terms and doubleton multiplets. The issue of the trace-anomaly in AdS_5 should be closely related to the present study. Already the leading N behaviour of this conformal anomaly is non-trivial to establish [17] and to explicitly obtain the sub-

[1] We thank R. Stora for a useful discussion about the issues of regularization.

leading terms might well turn out to be more difficult than for the chiral anomaly studied here. Effects that are of lower order than N^2 have also been considered in [18] which essentially studies situations where the leading effect corresponds to open strings at tree level and hence comes with just one power of N. A somewhat related discussion is [19].

It will also be interesting to investigate these anomaly issues within the non-commutative version of the AdS/CFT correspondence [20] to see the origin of the higher derivative Chern-Simons terms on the supergravity side.

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References

[1] A. Bilal, C.-S. Chu, *A Note on the Chiral Anomaly in the AdS/CFT Correspondence and $1/N^2$ Correction*, Nucl. Phys. **B562** (1999) 181, hep-th/9907106.

[2] J. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231, hep-th/9711200.

[3] S.S. Gubser, I.R. Klebanov, A.M. Polyakov, *Gauge Theory Correlators from Non-Critical String Theory*, Phys. Lett. **B428** (1998) 105, hep-th/9802109.

[4] E. Witten, *Anti de Sitter Space and Holography*, Adv. Theor. Math. Phys. **2** (1998) 253, hep-th/9802150.

[5] For a review, see O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, Y. Oz, *Large N Field Theories, String Theory and Gravity*, hep-th/9905111.

[6] D. Z. Freedman, S. D. Mathur, A. Matusis, L. Rastelli, *Correlation Functions in the CFT(d)/AdS(d+1) Correpondence*, hep-th/9804058.

[7] M. Gunaydin, L.J. Romans, N.P. Warner, *Gauged $N = 8$ Supergravity in Five Dimensions*, Phys. Lett. **B154** (1985) 268.

[8] M. Gunaydin, L.J. Romans, N.P. Warner, *Compact and Non-compact Gauged Supergravity Theories in Five Dimensions*, Nucl. Phys. **B272** (1986) 598.

[9] M. Pernici, K. Pilch, P. van Nieuwenhuizen, *Gauged $N = 8$, $d = 5$ Supergarvity*, Nucl. Phys. **B259** (1985) 460.

[10] S. Ferrara, C. Fronsdal, A. Zaffaroni, *On N=8 Supergravity on AdS_5 and N=4 Superconformal Yang-Mills theory*, Nucl. Phys. **B532** (1998) 153, hep-th/9802203.

[11] A.N. Redlich, *Gauge Noninvariance and Parity Nonconservation of Three-dimensional Fermions*, Phys. Rev. Lett. **52** (1984) 18; *Parity Violation and Gauge Noninvariance of the Effective Gauge Field Action in Three Dimensions*, Phys. Rev. **D29** (1984) 2366.

[12] A.J. Niemi, G.W. Semenoff, *Axial-Anomaly-Induced Fermion Fractionization and Effective Gauge Theory Actions in Odd-Dimensional Spacetimes*, Phys. Rev. Lett. **51** (1983) 2077.

[13] L. Alvarez-Gaume, S. Della Pietra, G. Moore, *Anomalies and Odd Dimensions*, Ann. Phys. **163** (1985) 288.

[14] H.J. Kim, L.J. Romans, P. van Nieuwenhuizen, *Mass Spectrum of Chiral Ten-dimensional $N = 2$ Supergravity on S^5*, Phys. Rev. **D32** (1985) 389.

[15] M. Gunaydin, P. van Nieuwenhuizen, N.P. Warner, Nucl. Phys. **B255** (1985) 63.

[16] E. Witten, *Quantum Field Theory and Jones Polynomial*, Commun. Math. Phys. 121 (1989) 351 .

[17] M. Henningson, K. Skenderis, *The Holographic Weyl Anomaly*, JHEP 9807 (1998) 023, hep-th/9806087.

[18] O. Aharony, J. Pawelczyk, S. Theisen, S. Yankielowicz, *A Note on Anomalies in the AdS/CFT Correspondence*, hep-th/9901134.

[19] M. Blau, E. Gava, K.S. Narain, *On Subleading Contributions to the AdS/CFT Trace Anomaly*, hep-th/9904179.
S. Nojiri, S. D. Odintsov, *On the conformal anomaly from higher derivative gravity in AdS/CFT correspondence*, hep-th/9903033.

[20] J. M. Maldacena and J. G. Russo, *Large N Limit of Non-Commutative Gauge Theories*, hep-th/9908134. C.-S. Chu, *Induced Chern-Simons and WZW action in noncommutative spacetime*, hep-th/0003007.