Noncommutative Riemann Surfaces

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Abstract: We compactify M(atrix) theory on Riemann surfaces Σ with genus $g > 1$. Following [1], we construct a projective unitary representation of $\pi_1(\Sigma)$ realized on $L^2(\mathbf{H})$, with $\mathbf{H}$ the upper half–plane. As a first step we introduce a suitably gauged $\mathrm{sl}_2(\mathbf{R})$ algebra. Then a uniquely determined gauge connection provides the central extension which is a 2–cocycle of the 2nd Hochschild cohomology group. Our construction is the double–scaling limit $N \to \infty$, $k \to -\infty$ of the representation considered in the Narasimhan–Seshadri theorem, which represents the higher–genus analog of 't Hooft's clock and shift matrices of QCD. The concept of a noncommutative Riemann surface Σ_θ is introduced as a certain $C^\star$–algebra. Finally we investigate the Morita equivalence.

1. Introduction

The $P_- = N/R$ sector of the discrete light–cone quantization of uncompactified M–theory is given by the supersymmetric quantum mechanics of $U(N)$ matrices. The compactification of M(atrix) theory [2]–[4] as a model for M–theory [5] has been studied in [6]. In [7]–[10] it has been treated using noncommutative geometry [11]. These investigations apply to the d–dimensional torus T^d, and have been further dealt with from various viewpoints in [12]–[18]. These structures are also relevant in noncommutative string and gauge theories [19, 20]. In this paper, following [1], we address the compactification M(atrix) theory on Riemann surfaces with genus $g > 1$.

A Riemann surface Σ of genus $g > 1$ is constructed as the quotient $\mathbf{H}/\Gamma$, where $\mathbf{H}$ is the upper half–plane, and $\Gamma \subset \mathrm{PSL}_2(\mathbf{R})$, $\Gamma \cong \pi_1(\Sigma)$, is a Fuchsian group acting on $\mathbf{H}$ as

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \qquad \gamma z = \frac{az+b}{cz+d}. \tag{1.1}$$

In the absence of elliptic and parabolic generators, the $2g$ Fuchsian generators γ_j satisfy

$$\prod_{j=1}^{g} \left(\gamma_{2j-1}\gamma_{2j}\gamma_{2j-1}^{-1}\gamma_{2j}^{-1}\right) = \mathbf{1}. \tag{1.2}$$

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Inspired by M(atrix) theory, let us promote the complex coordinate $z = x + iy$ to an $N \times N$ complex matrix $Z = X + iY$, with $X = X^\dagger$ and $Y = Y^\dagger$. This suggests defining fractional linear transformations of Z through conjugation with some non–singular matrix $\mathcal{U}$:

$$\mathcal{U} Z \mathcal{U}^{-1} = (aZ + b\mathbf{1})(cZ + d\mathbf{1})^{-1}. \tag{1.3}$$

Accordingly, operators $\mathcal{U}_k$ representing the Fuchsian generators γ_k can be constructed, such that

$$\prod_{k=1}^{g} \left(\mathcal{U}_{2k-1}\mathcal{U}_{2k}\mathcal{U}_{2k-1}^{-1}\mathcal{U}_{2k}^{-1}\right) = e^{2\pi i\theta}\mathbf{1}. \tag{1.4}$$

While we will find the solution to (1.4), we will consider slightly different versions of (1.3). This construction cannot be implemented for finite N, as taking the trace of (1.3) shows. It can be interpreted as defining a sort of *M(atrix) uniformization*, in which the Möbius transformation of the M(atrix) coordinate Z is defined through (1.3).

2. Compactification in $g > 1$

Next we present an explicit Ansatz to compactify 11–dimensional supergravity on a Riemann surface with $g > 1$. The Einstein equations read

$$R_{MN} - \frac{1}{2} G_{MN} R$$

$$= \frac{1}{3}(H_{ML_1L_2L_3}H_{NL'_1L'_2L'_3}G^{L_1L'_1}G^{L_2L'_2}G^{L_3L'_3}$$
$$-\frac{1}{8}G_{MN}H_{L_1L_2L_3L_4}H_{L'_1L'_2L'_3L'_4}$$
$$\times G^{L_1L'_1}G^{L_2L'_2}G^{L_3L'_3}G^{L_4L'_4}), \qquad (2.1)$$

where H_{MNPQ} is the field strength of C_{MNP}. We try an Ansatz by diagonally decomposing G_{MN} into 2–, 4– and 5–dimensional blocks, with H_{MNPQ} taken along the 4–dimensional subspace:

$$G_{MN} = \mathrm{diag}\,(g^{(2)}_{\alpha\beta}, g^{(4)}_{mn}, g^{(5)}_{ab}),$$
$$H_{MPQR} = \epsilon_{mpqr} f. \qquad (2.2)$$

The Einstein equations then decompose as

$$R^{(k)}_{i_kj_k} - \frac{1}{2}g^{(k)}_{i_kj_k}(R^{(2)} + R^{(4)} + R^{(5)})$$
$$= \epsilon_k \mathrm{det}\, g^{(4)}\, f^2\, g^{(k)}_{i_kj_k}, \qquad (2.3)$$

where $k = 2, 4, 5$, $(i_2, j_2) = (\alpha, \beta)$, $(i_4, j_4) = (m, n)$, $(i_5, j_5) = (a, b)$, and $\epsilon_2 = \epsilon_4 = -\epsilon_5 = 1$. Some manipulations lead to

$$R^{(k)} = c_k f^2 \mathrm{det}\, g^{(4)}, \qquad (2.4)$$

with $c_2 = -4/3$, $c_4 = 16/3$ and $c_5 = -10/3$. We observe that $f = 0$ would reproduce the toroidal case. A non–vanishing f is a deformation producing $g > 1$. It suffices that $g^{(4)}$ have positive signature for $R^{(2)}$ to be negative, as required in $g > 1$. Then a choice for the 4– and 5–dimensional manifolds is S^4 and AdS^5.

3. Differential representation of Γ

3.1 The unitary gauged operators

For $n = -1, 0, 1$ and $e_n(z) = z^{n+1}$ we consider the $\mathrm{sl}_2(\mathbf{R})$ operators $\ell_n = e_n(z)\partial_z$. We define

$$L_n = e_n^{-1/2}\ell_n e_n^{1/2} = e_n\left(\partial_z + \frac{1}{2}\frac{e'_n}{e_n}\right). \qquad (3.1)$$

These satisfy

$$[L_m, L_n] = (n-m)L_{m+n}, \quad [\bar{L}_m, L_n] = 0,$$
$$[L_n, f] = z^{n+1}\partial_z f. \qquad (3.2)$$

For $k = 1, 2, \ldots, 2g$, consider the operators

$$T_k = e^{\lambda^{(k)}_{-1}(L_{-1}+\bar{L}_{-1})}e^{\lambda^{(k)}_0(L_0+\bar{L}_0)}e^{\lambda^{(k)}_1(L_1+\bar{L}_1)}, \qquad (3.3)$$

with the $\lambda^{(k)}_n$ picked such that $T_k z T_k^{-1} = \gamma_k z = (a_k z + b_k)/(c_k z + d_k)$ so that by (1.2)

$$\prod_{k=1}^{g}\left(T_{2k-1}T_{2k}T^{-1}_{2k-1}T^{-1}_{2k}\right) = \mathbf{1}. \qquad (3.4)$$

On $L^2(\mathbf{H})$ we have the scalar product

$$\langle\phi|\psi\rangle = \int_{\mathbf{H}} d\nu \bar{\phi}\psi, \qquad (3.5)$$

where $d\nu(z) = i dz \wedge d\bar{z}/2 = dx \wedge dy$. The T_k provide a unitary representation of Γ.

Next consider the *gauged* $\mathrm{sl}_2(\mathbf{R})$ operators [1]

$$\mathcal{L}^{(F)}_n = F(z,\bar{z})L_n F^{-1}(z,\bar{z})$$
$$= e_n\left(\partial_z + \frac{1}{2}\frac{e'_n}{e_n} - \partial_z \ln F(z,\bar{z})\right), \qquad (3.6)$$

where $F(z,\bar{z})$ is an undetermined phase function, to be determined later on. The $\mathcal{L}^{(F)}_n$ also satisfy the algebra (3.2). The adjoint of $\mathcal{L}^{(F)}_n$ is given by

$$\mathcal{L}^{(F)\dagger}_n = -F\overline{e_n^{1/2}}\partial_{\bar{z}}\overline{e_n^{1/2}}F^{-1}, \qquad (3.7)$$

with $\mathcal{L}^{(F)\dagger}_n = -\bar{\mathcal{L}}^{(F^{-1})}_n$. Finally we define

$$\Lambda^{(F)}_n = \mathcal{L}^{(F)}_n - \mathcal{L}^{(F)\dagger}_n = \mathcal{L}^{(F)}_n + \bar{\mathcal{L}}^{(F^{-1})}_n. \qquad (3.8)$$

The $\Lambda^{(F)}_n$ enjoy the fundamental property that both their chiral components are gauged in the same way by the function F, that is

$$\Lambda^{(F)}_n = F(L_n + \bar{L}_n)F^{-1}, \qquad (3.9)$$

while also satisfying the $\mathrm{sl}_2(\mathbf{R})$ algebra:

$$[\Lambda^{(F)}_m, \Lambda^{(F)}_n] = (n-m)\Lambda^{(F)}_{m+n},$$
$$[\Lambda^{(F)}_n, f] = (z^{n+1}\partial_z + \bar{z}^{n+1}\partial_{\bar{z}})f. \qquad (3.10)$$

It holds that

$$e^{\Lambda^{(F)}_n} = F e^{L_n + \bar{L}_n}F^{-1}, \qquad (3.11)$$

which is a unitary operator since $\Lambda^{(F)\dagger}_n = -\Lambda^{(F)}_n$.

Let b be a real number, and A a Hermitean connection 1–form to be identified presently. Set

$$\mathcal{U}_k = e^{ib\int_z^{\gamma_k z} A}T_k, \qquad (3.12)$$

where the integration contour is taken to be the Poincaré geodesic connecting z and $\gamma_k z$. As the gauging functions introduced in (3.6) we will take the functions $F_k(z,\bar{z})$ that solve the equation

$$F_k T_k F_k^{-1} = e^{ib\int_z^{\gamma_k z} A}T_k, \qquad (3.13)$$

that is

$$F_k(\gamma_k z, \gamma_k \bar{z}) = e^{-ib\int_z^{\gamma_k z} A}F_k(z,\bar{z}). \qquad (3.14)$$

3.2 The gauged algebra

With the choice (3.13) for F_k, (3.9) becomes

$$\begin{aligned}\Lambda^{(F)}_{n,k} &= F_k(L_n+\bar{L}_n)F_k^{-1}\\ &= z^{n+1}\left(\partial_z+\frac{n+1}{2z}-\partial_z\ln F_k\right)\\ &+\bar{z}^{n+1}\left(\partial_{\bar z}+\frac{n+1}{2\bar z}-\partial_{\bar z}\ln F_k\right). \quad (3.15)\end{aligned}$$

The $\Lambda^{(F)}_{n,k}$ satisfy the algebra

$$\begin{aligned}&[\Lambda^{(F)}_{m,j},\Lambda^{(F)}_{n,k}]\\ &=(n-m)\Lambda^{(F)}_{m+n,j}+F_k^{-1}|e_n|\Lambda^{(F)}_{n,k}|e_n|^{-1}F_k\\ &\times F_j^{-1}|e_m|\Lambda^{(F)}_{m,j}|e_m|^{-1}F_j(\ln F_j-\ln F_k),\\ &[\Lambda^{(F)}_{n,k},f]=(z^{n+1}\partial_z+\bar{z}^{n+1}\partial_{\bar z})f. \quad (3.16)\end{aligned}$$

Upon exponentiating $\Lambda^{(F)}_{n,k}$ one finds

$$\mathcal{U}_k=e^{\lambda^{(k)}_{-1}\Lambda^{(F)}_{-1,k}}\,e^{\lambda^{(k)}_{0}\Lambda^{(F)}_{0,k}}\,e^{\lambda^{(k)}_{1}\Lambda^{(F)}_{1,k}}, \quad (3.17)$$

that is, the $\mathcal{U}_k$ are unitary, and

$$\mathcal{U}_k^{-1}=T_k^{-1}e^{-ib\int_z^{\gamma_k z}A}=e^{-ib\int_{\gamma_k^{-1}z}^{z}A}T_k^{-1}. \quad (3.18)$$

3.3 Computing the phase

It is immediate to see that the $\mathcal{U}_k$ defined in (3.12) satisfy (1.4) for a certain value of θ:

$$\begin{aligned}&\prod_{k=1}^{g}\left(\mathcal{U}_{2k-1}\mathcal{U}_{2k}\mathcal{U}^\dagger_{2k-1}\mathcal{U}^\dagger_{2k}\right)=e^{ib\int_z^{\gamma_1 z}A}T_1\\ &\times e^{ib\int_z^{\gamma_2 z}A}T_2e^{-ib\int_{\gamma_1^{-1}z}^{z}A}T_1^{-1}e^{-ib\int_{\gamma_2^{-1}z}^{z}A}T_2^{-1}\ldots\\ &=\exp\left[ib\left(\int_z^{\gamma_1 z}A+\int_{\gamma_1 z}^{\gamma_2\gamma_1 z}A\right)\right]\times\\ &\exp\left[ib\left(\int_{\gamma_2\gamma_1 z}^{\gamma_1^{-1}\gamma_2\gamma_1 z}A+\int_{\gamma_1^{-1}\gamma_2\gamma_1 z}^{\gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1 z}A+\ldots\right)\right]\\ &\times\prod_{k=1}^{g}\left(T_{2k-1}T_{2k}T^{-1}_{2k-1}T^{-1}_{2k}\right)=e^{ib\oint_{\partial\mathcal{F}_z}A}, \quad (3.19)\end{aligned}$$

where $\mathcal{F}_z=\{z,\gamma_1 z,\gamma_2\gamma_1 z,\gamma_1^{-1}\gamma_2\gamma_1 z,\ldots\}$ is a fundamental domain for Γ. The basepoint z, plus the action of the Fuchsian generators on it, determine $\mathcal{F}_z$, as the vertices are joined by geodesics.

3.4 Uniqueness of the gauge connection

For (3.19) to provide a projective unitary representation of Γ, $\int_{\mathcal{F}_z}dA$ should be z–independent. Changing z to z' can be expressed as $z\to z'=\mu z$ for some $\mu\in\mathrm{PSL}_2(\mathbf{R})$. Then $\mathcal{F}_z\to\mathcal{F}_{\mu z}=\{\mu z,\gamma_1\mu z,\gamma_2\gamma_1\mu z,\gamma_1^{-1}\gamma_2\gamma_1\mu z,\ldots\}$. Now consider $\mathcal{F}_z\to\mu\mathcal{F}_z=\{\mu z,\mu\gamma_1 z,\mu\gamma_2\gamma_1 z,\mu\gamma_1^{-1}\gamma_2\gamma_1 z,\ldots\}$. The congruence $\mu\mathcal{F}_z\cong\mathcal{F}_{\mu z}$ follows from two facts: that the vertices are joined by geodesics, and that $\mathrm{PSL}_2(\mathbf{R})$ maps geodesics into geodesics. Since Γ is defined up to conjugation, $\Gamma\to\mu\Gamma\mu^{-1}$, if $\mu\mathcal{F}_z$ is a fundamental domain, so is $\mathcal{F}_{\mu z}$. Thus, to have z–independence we need $\forall\mu\in\mathrm{PSL}_2(\mathbf{R})$

$$\int_{\mathcal{F}_z}dA=\int_{\mathcal{F}_{\mu z}}dA=\int_{\mu\mathcal{F}_z}dA=\int_{\mathcal{F}}dA. \quad (3.20)$$

This fixes the (1,1)–form dA to be $\mathrm{PSL}_2(\mathbf{R})$–invariant. It is well known that the Poincaré form is the unique $\mathrm{PSL}_2(\mathbf{R})$–invariant (1,1)–form, up to an overall constant factor. This is a particular case of a more general fact [21]. The Poincaré metric $ds^2=y^{-2}|dz|^2=2g_{z\bar z}|dz|^2=e^{\varphi}|dz|^2$ has curvature $R=-g^{z\bar z}\partial_z\partial_{\bar z}\ln g_{z\bar z}=-1$, so that $\int_{\mathcal{F}}d\nu e^{\varphi}=-2\pi\chi(\Sigma)$, where $\chi(\Sigma)=2-2g$ is the Euler characteristic. As the Poincaré (1,1)–form is $dA=e^{\varphi}d\nu$, this uniquely determines the gauge field to be

$$A=A_z dz+A_{\bar z}d\bar z=dx/y, \quad (3.21)$$

up to gauge transformations. Using $\oint_{\partial\mathcal{F}}A=\int_{\mathcal{F}}dA$ we finally have that (3.19) becomes

$$\prod_{k=1}^{g}\left(\mathcal{U}_{2k-1}\mathcal{U}_{2k}\mathcal{U}^\dagger_{2k-1}\mathcal{U}^\dagger_{2k}\right)=e^{2\pi ib\chi(\Sigma)}. \quad (3.22)$$

3.5 Non–Abelian extension

Up to now we considered the case in which the connection is Abelian. However, it is easy to extend our construction to the non–Abelian case in which the gauge group $U(1)$ is replaced by $U(N)$. The operators $\mathcal{U}_k$ now become

$$\mathcal{U}_k=Pe^{ib\int_z^{\gamma_k z}A}T_k, \quad (3.23)$$

where the T_k are the same as before, times the $N\times N$ identity matrix. Eq.(3.19) is replaced by

$$\prod_{k=1}^{g}\left(\mathcal{U}_{2k-1}\mathcal{U}_{2k}\mathcal{U}^\dagger_{2k-1}\mathcal{U}^\dagger_{2k}\right)=Pe^{ib\oint_{\partial\mathcal{F}_z}A}. \quad (3.24)$$

Given an integral along a closed contour σ_z with basepoint z, the path–ordered exponentials for a connection A and its gauge transform $A^U = U^{-1}AU + U^{-1}dU$ are related by [22]

$$Pe^{i\oint_{\sigma_z} A} = U(z)Pe^{i\oint_{\sigma_z} A^U} U^{-1}(z) =$$
$$U(z)Pe^{i\oint_{\sigma_z} d\sigma^\mu \int_0^1 ds s\sigma^\nu U^{-1}(s\sigma)F_{\nu\mu}(s\sigma)U(s\sigma)} U^{-1}(z). \tag{3.25}$$

Applying this to (3.24), we see that the only possibility to get a coordinate–independent phase is for the curvature (1,1)–form $F = dA + [A, A]/2$ to be the identity matrix in the gauge indices times a (1,1)–form η, that is $F = \eta\mathbf{1}$. It follows that

$$Pe^{ib\oint_{\partial\mathcal{F}} A} = e^{ib\int_{\mathcal{F}} F}. \tag{3.26}$$

However, the above is only a necessary condition for coordinate–independence. Nevertheless, we can apply the same reasoning as in the Abelian case to see that η should be proportional to the Poincaré (1,1)–form. Denoting by E the vector bundle on which A is defined, we have $k = \deg(E) = \frac{1}{2\pi}\mathrm{tr}\int_{\mathcal{F}} F$. Set $\mu(E) = k/N$ so that $\int_{\mathcal{F}} F = 2\pi\mu(E)\mathbf{1}$ and $\eta = -\frac{\mu(E)}{\chi(\Sigma)}e^\varphi d\nu$, *i.e.*

$$F = 2\pi\mu(E)\omega\mathbf{1}, \tag{3.27}$$

where $\omega = \left(e^\varphi / \int_{\mathcal{F}} d\nu e^\varphi\right) d\nu$. Thus, by (3.26) we have that Eq.(3.24) becomes

$$\prod_{k=1}^{g}\left(\mathcal{U}_{2k-1}\mathcal{U}_{2k}\mathcal{U}_{2k-1}^\dagger\mathcal{U}_{2k}^\dagger\right) = e^{2\pi ib\mu(E)}\mathbf{1}, \tag{3.28}$$

which provides a projective unitary representation of $\pi_1(\Sigma)$ on $L^2(\mathbf{H}, \mathbf{C}^N)$.

3.6 The gauge length

A basic object is the *gauge length* function

$$d_A(z, w) = \int_z^w A, \tag{3.29}$$

where the contour integral is along the Poincaré geodesic connecting z and w. In the Abelian case

$$d_A(z, w) = \int_{\mathrm{Re}\, z}^{\mathrm{Re}\, w} \frac{dx}{y} = -i\ln\left(\frac{z - \bar{w}}{w - \bar{z}}\right), \tag{3.30}$$

which is equal to the angle α_{zw} spanned by the arc of geodesic connecting z and w. Observe that the gauge length of the geodesic connecting two punctures, *i.e.* two points on the real line, is π. This is to be compared with the usual divergence of the Poincaré distance. Under a $\mathrm{PSL}_2(\mathbf{R})$–transformation μ, we have ($\mu_x \equiv \partial_x\mu x$)

$$d_A(\mu z, \mu w) = d_A(z, w) - \frac{i}{2}\ln\left(\frac{\mu_z\bar{\mu}_w}{\bar{\mu}_z\mu_w}\right). \tag{3.31}$$

Therefore, the gauge length of an n–gon

$$d_A^{(n)}(\{z_k\}) = \sum_{k=1}^{n} d_A(z_k, z_{k+1}) = \pi(n-2) - \sum_{k=1}^{n}\alpha_k, \tag{3.32}$$

where $z_{n+1} \equiv z_1$, $n \geq 3$, and α_k are the internal angles, is $\mathrm{PSL}_2(\mathbf{R})$–invariant. One can check that the $\mathrm{PSL}_2(\mathbf{R})$–transformation (3.31) corresponds to a gauge transformation of A. Furthermore, as we will see, the triangle length, that by Stokes' theorem corresponds to the Poincaré area, is proportional to the Hochschild 2–cocycle.

3.7 Pre–automorphic forms

A related reason for the relevance of the gauge length function is that it also appears in the definition of the F_k. The latter, which apparently never appeared in the literature before, are of particular interest. Let us recast (3.13) as

$$F_k(\gamma_k z, \gamma_k\bar{z}) = \left(\frac{\gamma_k z - \bar{z}}{z - \gamma_k\bar{z}}\right)^b F_k(z, \bar{z}). \tag{3.33}$$

Since $(\gamma_k z - \bar{z})/(z - \gamma_k\bar{z})$ transforms as an automorphic form under Γ, we call the F_k *pre–automorphic forms.* Eq.(3.14) indicates that finding the most general solution to (3.33) is a problem in geodesic analysis. In the case of the inversion $\gamma_k z = -1/z$ and b an even integer, a solution to (3.33) is $F_k = (z/\bar{z})^{\frac{b}{2}}$. By (3.30) $F_k = (z/\bar{z})^{\frac{b}{2}}$ is related to the *A–length* of the geodesic connecting z and 0:

$$e^{\frac{i}{2}b\int_z^0 A} = F_k(z, \bar{z}) = (z/\bar{z})^{\frac{b}{2}}. \tag{3.34}$$

An interesting formal solution to (3.33) is

$$F_k(z, \bar{z}) = \prod_{j=0}^{\infty}\left(\frac{\gamma_k^{-j}z - \gamma_k^{-j-1}\bar{z}}{\gamma_k^{-j-1}z - \gamma_k^{-j}\bar{z}}\right)^b. \tag{3.35}$$

To construct other solutions, we consider the uniformizing map $J_{\mathbf{H}} : \mathbf{H} \longrightarrow \Sigma$, which enjoys the property $J_{\mathbf{H}}(\gamma z) = J_{\mathbf{H}}(z)$, $\forall\gamma \in \Gamma$. Then, if F_k satisfies (3.33), this equation is invariant under $F_k \to G(J_{\mathbf{H}}, \bar{J}_{\mathbf{H}})F_k$. Since $|F_k| = 1$, we should require $|G| = 1$, otherwise G is arbitrary.

4. Hochschild cohomology of Γ

The Fuchsian generators $\gamma_k \in \Gamma$ are projectively represented by means of unitary operators $\mathcal{U}_k$ acting on $L^2(\mathbf{H})$. The product $\gamma_k\gamma_j$ is represented by[1] $\mathcal{U}_{jk}$, which equals $\mathcal{U}_j\mathcal{U}_k$ up to a phase:

$$\mathcal{U}_j\mathcal{U}_k = e^{2\pi i\theta(j,k)}\mathcal{U}_{jk}. \tag{4.1}$$

Associativity implies

$$\theta(j,k) + \theta(jk,l) = \theta(j,kl) + \theta(k,l). \tag{4.2}$$

We can easily determine $\theta(j,k)$:

$$\mathcal{U}_j\mathcal{U}_k = \exp\left(ib\int_z^{\gamma_j z} A + ib\int_{\gamma_j z}^{\gamma_k\gamma_j z} A\right)\times$$
$$\exp\left(-ib\int_z^{\gamma_k\gamma_j z} A\right)\mathcal{U}_{jk} = \exp\left(ib\int_{\tau_{jk}} A\right)\mathcal{U}_{jk}, \tag{4.3}$$

where τ_{jk} denotes the geodesic triangle with vertices z, $\gamma_j z$ and $\gamma_k\gamma_j z$. This identifies $\theta(j,k)$ as the gauge length of the perimeter of the geodesic triangle τ_{jk}. By Stokes' theorem this is the Poincaré area of the triangle. A similar phase, introduced independently of any gauge connection, has been considered in [23] in the context of Berezin's quantization of $\mathbf{H}$ and Von Neumann algebras.

The information on the compactification of M(atrix) theory is encoded in the action of Γ on $\mathbf{H}$, plus a projective representation of Γ. The latter amounts to the choice of a phase. Physically inequivalent choices of $\theta(j,k)$ turn out to be in one–to–one correspondence with elements in the 2nd Hochschild cohomology group $H^2(\Gamma, U(1))$ of Γ. This cohomology group is defined as follows. A k–cochain is an angular–valued function $f(\gamma_1,\ldots,\gamma_k)$ with k arguments in Γ. The coboundary operator δ maps the k–cochain f into the $(k+1)$–cochain δf defined as

$$(\delta f)(\gamma_0,\ldots,\gamma_k) = f(\gamma_1,\ldots,\gamma_k)$$
$$+\sum_{l=1}^{k}(-1)^l f(\gamma_0,\ldots,\gamma_{l-1}\gamma_l,\ldots,\gamma_k)$$
$$+(-1)^{k+1}f(\gamma_0,\ldots,\gamma_{k-1}). \tag{4.4}$$

[1] The differential representation of $\mathrm{PSL}_2(\mathbf{R})$ acts in reverse order with respect to the one by matrices.

Clearly $\delta^2 = 0$. A k–cochain annihilated by δ is called a k–cocycle. $H^k(\Gamma, U(1))$ is the group of equivalence classes of k–cocycles modulo the coboundary of $(k-1)$–cochains. The associativity condition (4.2) is just $\delta\theta(j,k) = 0$. Thus θ is a 2–cocycle of the Hochschild cohomology. Projective representations of Γ are classified by $H^2(\Gamma, U(1)) = U(1)$. Hence $\theta = b\chi(\Sigma)$ is the unique parameter for this compactification ($\theta = b\mu(E)$ in the general case).

5. Stable bundles and double scaling limit

We now present some facts about projective, unitary representations of Γ and the theory of holomorphic vector bundles [24, 25]. Let $E \to \Sigma$ be a holomorphic vector bundle over Σ of rank N and degree k. The bundle E is called *stable* if the inequality $\mu(E') < \mu(E)$ holds for every proper holomorphic subbundle $E' \subset E$. We may take $-N < k \leq 0$. We will further assume that Γ contains a unique primitive elliptic element γ_0 of order N (*i.e.*, $\gamma_0^N = \mathbf{1}$), with fixed point $z_0 \in \mathbf{H}$ that projects to $x_0 \in \Sigma$.

Given the branching order N of γ_0, let $\rho: \Gamma \to U(N)$ be an irreducible unitary representation. It is said *admissible* if $\rho(\gamma_0) = e^{-2\pi ik/N}\mathbf{1}$. Putting the elliptic element on the right–hand side, and setting $\rho_k \equiv \rho(\gamma_k)$, (1.2) becomes

$$\prod_{j=1}^{g}\left(\rho_{2j-1}\rho_{2j}\rho_{2j-1}^{-1}\rho_{2j}^{-1}\right) = e^{2\pi ik/N}\mathbf{1}. \tag{5.1}$$

On the trivial bundle $\mathbf{H}\times\mathbf{C}^N \to \mathbf{H}$ there is an action of Γ: $(z,v) \to (\gamma z, \rho(\gamma)v)$. This defines the quotient bundle

$$\mathbf{H}\times\mathbf{C}^N/\Gamma \to \mathbf{H}/\Gamma \cong \Sigma. \tag{5.2}$$

Any admissible representation determines a holomorphic vector bundle $E_\rho \to \Sigma$ of rank N and degree k. When $k = 0$, E_ρ is simply the quotient bundle (5.2) of $\mathbf{H}\times\mathbf{C}^N \to \mathbf{H}$. The Narasimhan–Seshadri (NS) theorem [26] now states that a holomorphic vector bundle E over Σ of rank N and degree k is stable if and only if it is isomorphic to a bundle E_ρ, where ρ is an admissible representation of Γ. Moreover, the bundles E_{ρ_1}

and E_{ρ_2} are isomorphic if and only if the representations ρ_1 and ρ_2 are equivalent.

The standard Hermitean metric on $\mathbf{C}^N$ gives a metric on $\mathbf{H}\times\mathbf{C}^N\to\mathbf{H}$. This metric and the corresponding connection are invariant with respect to the action $(z,v)\to(\gamma z,\rho(\gamma)v)$, when ρ is admissible. Hence they determine a (degenerate) metric g_{NS} and a connection A_{NS} on the bundle $E=E_\rho$. The connection A_{NS} is compatible with the metric g_{NS} and with the holomorphic structure on E, but it has a singularity at the branching point $x_0\in\Sigma$ of the covering $\mathbf{H}\to\Sigma$. The curvature F_{NS} of A_{NS} is a $(1,1)$–current with values in the bundle $\mathrm{End}\,E$, characterized by the property[2]

$$\int_\Sigma f\wedge F_{NS}=-2\pi i\mu(E)\mathrm{tr}\,f(x_0), \tag{5.3}$$

for every smooth section f of the bundle $\mathrm{End}\,E$. The connection A_{NS} is uniquely determined by the curvature condition (5.3) and by the fact that it corresponds to the degenerate metric g_{NS}. The connection A_{NS} on the stable bundle $E=E_\rho$ is called the *NS connection*.

A differential–geometric approach to stability has been given by Donaldson [27]. Fix a Hermitean metric on Σ, for example the Poincaré metric, normalized so that the area of Σ equals 1. Let us denote by ω its associated (1,1)–form. A holomorphic bundle E is stable if and only if there exists on E a metric connection A_D with central curvature $F_D=-2\pi i\mu(E)\omega\mathbf{1}$; such a connection A_D is unique.

The unitary projective representations of Γ we constructed above have a uniquely defined gauge field whose curvature is proportional to the volume form on Σ. With respect to the representation considered by NS, we note that NS introduced an elliptic point to produce the phase, while in our case the latter arises from the gauge length. Our construction is directly connected with Donaldson's approach as $F=iF_D$, where F is the curvature (3.27). However, the main difference is that our operators are unitary differential operators on $L^2(\mathbf{H},\mathbf{C}^N)$ instead of unitary matrices on $\mathbf{C}^N$. This allowed us to obtain a non–trivial phase also in the Abelian case.

[2]Note that our convention for A differs from the one in the mathematical literature by a factor i.

It is however possible to understand the formal relation between our operators and those of NS. To see this we consider the adjoint representation of Γ on $\mathrm{End}\,\mathbf{C}^N$,

$$\mathrm{Ad}\,\rho(\gamma)Z=\rho(\gamma)Z\rho^{-1}(\gamma), \tag{5.4}$$

where $Z\in\mathrm{End}\,\mathbf{C}^N$ is understood as an $N\times N$ matrix. Let us also consider the trivial bundle $\mathbf{H}\times\mathrm{End}\,\mathbf{C}^N\to\mathbf{H}$. There is an action of Γ: $(z,Z)\mapsto(\gamma z,\mathrm{Ad}\,\rho(\gamma)Z)$ that defines the quotient bundle

$$\mathbf{H}\times\mathrm{End}\,\mathbf{C}^N/\Gamma\to\mathbf{H}/\Gamma\cong\Sigma. \tag{5.5}$$

Then, the idea is to consider a vector bundle E' in the double scaling limit $N'\to\infty$, $k'\to-\infty$, with $\mu(E')=k'/N'$ fixed, that is

$$\mu(E')=b\mu(E). \tag{5.6}$$

In this limit, fixing a basis in $L^2(\mathbf{H},\mathbf{C}^N)$, the matrix elements of our operators can be identified with those of $\rho(\gamma)$.

6. Noncommutative Riemann surfaces

Let us now introduce two copies of the upper half–plane, one with coordinates z and $\bar z$, the other with coordinates w and $\bar w$. While the coordinates z and $\bar z$ are reserved to the operators $\mathcal{U}_k$ we introduced previously, we reserve w and $\bar w$ to construct a new set of operators. We now introduce noncommutative coordinates expressed in terms of the covariant derivatives

$$W=\partial_w+iA_w,\qquad \bar W=\partial_{\bar w}+iA_{\bar w}, \tag{6.1}$$

with $A_w=A_{\bar w}=1/(2\,\mathrm{Im}\,w)$, so that

$$[W,\bar W]=iF_{w\bar w}, \tag{6.2}$$

where $F_{w\bar w}=i/[2(\mathrm{Im}\,w)^2]$. Let us consider the following realization of the $\mathrm{sl}_2(\mathbf{R})$ algebra:

$$\hat L_{-1}=-w,\qquad \hat L_0=-\frac{1}{2}(w\partial_w+\partial_w w),$$
$$\hat L_1=-\partial_w w\partial_w. \tag{6.3}$$

We then define the unitary operators

$$\hat T_k=e^{\lambda_{-1}^{(k)}(\hat L_{-1}+\bar{\hat L}_{-1})}e^{\lambda_0^{(k)}(\hat L_0+\bar{\hat L}_0)}e^{\lambda_1^{(k)}(\hat L_1+\bar{\hat L}_1)}, \tag{6.4}$$

where the $\lambda_n^{(k)}$ are as in (3.3). Set $\mathcal{V}_k = \hat{T}_k \mathcal{U}_k$. Since the $\hat{T}_k$ satisfy (3.4), it follows that the $\mathcal{V}_k$ satisfy (3.28) and

$$\mathcal{V}_k \partial_w \mathcal{V}_k^{-1} = \hat{T}_k \partial_w \hat{T}_k^{-1} = \frac{a_k \partial_w + b_k}{c_k \partial_w + d_k}. \qquad (6.5)$$

Setting $W = G\partial_w G^{-1}$, *i.e.* $G = (w - \bar{w})^2$, and using $Af(B)A^{-1} = f(ABA^{-1})$, we see that

$$\mathcal{V}_k W \mathcal{V}_k^{-1} = \hat{T}_k W \hat{T}_k^{-1} = G(\tilde{w}) \hat{T}_k \partial_w \hat{T}_k^{-1} G^{-1}(\tilde{w}), \qquad (6.6)$$

where

$$\tilde{w} = \hat{T}_k w \hat{T}_k^{-1} = -e^{-\lambda_0^{(k)}} + 2\lambda_1^{(k)}(\hat{L}_0 - \lambda_{-1}^{(k)} w)$$
$$-\lambda_1^{(k)2} e^{\lambda_0^{(k)}} (\hat{L}_1 + 2\lambda_{-1}^{(k)} \hat{L}_0 - \lambda_{-1}^{(k)2} w), \qquad (6.7)$$

and by (6.5)

$$\mathcal{V}_k W \mathcal{V}_k^{-1} = \hat{T}_k W \hat{T}_k^{-1} = \frac{a_k \tilde{W} + b_k}{c_k \tilde{W} + d_k}, \qquad (6.8)$$

where $\tilde{W}$ differs from W by the connection

$$\tilde{W} = \partial_w + G(\tilde{w})[\partial_w G^{-1}(\tilde{w})]. \qquad (6.9)$$

6.1 Morita equivalence and large N limit

By a natural generalization of the n–dimensional noncommutative torus, one defines a noncommutative Riemann surface Σ_θ in $g > 1$ to be an associative algebra with involution having unitary generators $\mathcal{U}_k$ obeying the relation (3.22). Such an algebra is a $C^\star$–algebra, as it admits a faithful unitary representation on $L^2(\mathbf{H}, \mathbf{C}^N)$ whose image is norm–closed. Relation (3.22) is also satisfied by the $\mathcal{V}_k$. However, while the $\mathcal{U}_k$ act on the commuting coordinates $z, \bar{z}$, the $\mathcal{V}_k$ act on the operators W and $\bar{W}$ of (6.1). The latter, factorized by the action of the $\mathcal{V}_k$ in (6.8), can be pictorially identified with a sort of noncommutative coordinates on Σ_θ.

Each $\gamma \neq \mathbf{1}$ in Γ can be uniquely expressed as a positive power of a primitive element $p \in \Gamma$, *primitive* meaning that p is not a positive power of any other $p' \in \Gamma$ [28]. Let $\mathcal{V}_p$ be the representative of p. Any $\mathcal{V} \in C^\star$ can be written as

$$\mathcal{V} = \sum_{p \in \{prim\}} \sum_{n=0}^{\infty} c_n^{(p)} \mathcal{V}_p^n + c_0 \mathbf{1}, \qquad (6.10)$$

for certain coefficients $c_n^{(p)}$, c_0. A trace can be defined as $\mathrm{tr}\, \mathcal{V} = c_0$.

In the case of the torus one can connect the $C^\star$–algebras of $U(1)$ and $U(N)$. To see this one can use 't Hooft's clock and shift matrices

$$V_1 V_2 = e^{2\pi i \frac{M}{N}} V_2 V_1. \qquad (6.11)$$

The $U(N)$ $C^\star$–algebra is constructed in terms of the V_k and of the unitary operators representing the $U(1)$ $C^\star$–algebra. Morita equivalence is an isomorphism between the two. In higher genus, the analog of the V_k is the $U(N)$ representation $\rho(\gamma)$ considered above. One can obtain a $U(N)$ projective unitary differential representation of Γ by taking $\mathcal{V}_k \rho(\gamma_k)$, with $\mathcal{V}_k$ Abelian. This non–Abelian representation should be compared with the one obtained by the non–Abelian $\mathcal{V}_k$ constructed above. In this framework it should be possible to understand a possible higher–genus analog of the Morita equivalence.

The isomorphism of the $C^\star$–algebras is a direct consequence of an underlying equivalence between the $U(1)$ and $U(N)$ connections. The z–independence of the phase requires F to be the identity matrix in the gauge indices. This in turn is deeply related to the uniqueness of the connection we found. The latter is related to the uniqueness of the NS connection. We conclude that Morita equivalence in higher genus is intimately related to the NS theorem.

Finally let us observe that, as our operators correspond to the $N \to \infty$ limit of projective unitary representations of Γ, these play a role in the $N \to \infty$ limit of QCD as considered in [29].

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