Implications of $\mathcal{N} = 2$ Superconformal Symmetry

Sergei Kuzenko, Stefan Theisen

Sektion Physik der Ludwig-Maximilians-Universität
Theresienstraße 37, D-80333 München, Germany
E-mail: sergei, theisen@theorie.physik.uni-muenchen.de

Abstract: We review recent results on the general structure of two- and three- point functions of the supercurrent and the flavor current of $\mathcal{N} = 2$ superconformal field theories.

1. Introduction

This note is a brief review of the results obtained in our recent paper [1] where we analysed the general structure of two- and three- point functions of the supercurrent and the flavor current of $\mathcal{N} = 2$ superconformal field theories. Our research was inspired by (i) similar results obtained by Osborn [2] for $\mathcal{N} = 1$ superconformal field theories; (ii) $\mathcal{N}$–extended superconformal kinematics due to Park [3], in particular the existence of nilpotent superconformal invariants of three points; (iii) the conjecture of Maldacena [4] (see [5] for a review), which relates superconformal gauge theories in four dimensional Minkowski space to extended gauge supergravities in five dimensional anti-de-Sitter space.

In $\mathcal{N} = 1$ superconformal field theory, the conserved currents are contained in two different supermultiplets: (i) the supercurrent $J_{\alpha\dot\alpha}$ [6] containing the energy-momentum tensor Θ_{mn}, the supersymmetry currents $j_{m\hat\alpha}$ ($\hat\alpha = \alpha, \dot\alpha$) and the axial current $j^{(R)}_m$; (ii) the flavor current multiplet $L^{\bar a}$ [7] containing the conserved flavor current $v^{\bar a}_m$ among its components. Both $J_{\alpha\dot\alpha}$ and $L^{\bar a}$ are real $\mathcal{N} = 1$ superfields, and they satisfy the conservation equations

$$\bar D^{\dot\alpha} J_{\alpha\dot\alpha} = D^\alpha J_{\alpha\dot\alpha} = 0 \,, \qquad (1.1)$$
$$\bar D^2 L^{\bar a} = D^2 L^{\bar a} = 0 \,. \qquad (1.2)$$

In $\mathcal{N} = 2$ superconformal field theory, the conserved currents are contained in two different supermultiplets: (i) the supercurrent $\mathcal{J}$ [8, 10] whose components include the energy-momentum tensor Θ_{mn}, the SU(2) R-current $j^{(ij)}_m$ ($i, j = \underline{1}, \underline{2}$), the axial current $j^{(R)}_m$ and the $\mathcal{N} = 2$ supersymmetry currents $j^i_{m\hat\alpha}$; (ii) the flavor current multiplet $\mathcal{L}^{\bar a}_{ij}$ [9, 10] containing the conserved flavor current $v^{\bar a}_m$ among its components. Both $\mathcal{J}$ and $\mathcal{L}^{\bar a}_{ij}$ are real $\mathcal{N} = 2$ superfields ($\overline{\mathcal{L}_{ij}} = \mathcal{L}^{ij}$), and they satisfy the conservation equations

$$D^{ij} \mathcal{J} = \bar D^{ij} \mathcal{J} = 0 \,, \qquad (1.3)$$
$$D^{(i}_\alpha \mathcal{L}^{jk)} = \bar D^{(i}_{\dot\alpha} \mathcal{L}^{jk)} = 0 \,, \qquad (1.4)$$

where $D^{ij} = D^{\alpha(i} D^{j)}_\alpha$, $\bar D^{ij} = \bar D^{(i}_{\dot\alpha} \bar D^{j)\dot\alpha}$.

Any $\mathcal{N} = 2$ superconformal field theory is a special $\mathcal{N} = 1$ superconformal model. Therefore, it is useful to know the decomposition of $\mathcal{J}$ and $\mathcal{L}_{ij}$ into $\mathcal{N} = 1$ multiplets. For that purpose we introduce the $\mathcal{N} = 1$ spinor covariant derivatives $D_\alpha \equiv D^{\underline{1}}_\alpha$, $\bar D^{\dot\alpha} \equiv \bar D^{\dot\alpha}_{\underline{1}}$ and define the $\mathcal{N} = 1$ projection $U| \equiv U(x, \theta^\alpha_i, \bar\theta^j_{\dot\alpha})|_{\theta_{\underline{2}} = \bar\theta^{\underline{2}} = 0}$ of an arbitrary $\mathcal{N} = 2$ superfield U. It follows from (1.3) that $\mathcal{J}$ is composed of three independent $\mathcal{N} = 1$ multiplets

$$J \equiv \mathcal{J}| \,, \qquad J_\alpha \equiv D^{\underline{2}}_\alpha \mathcal{J}| \,, \qquad (1.5)$$
$$J_{\alpha\dot\alpha} \equiv \frac{1}{2} [D^{\underline{2}}_\alpha, \bar D_{\dot\alpha \underline{2}}] \mathcal{J}| - \frac{1}{6} [D^{\underline{1}}_\alpha, \bar D_{\dot\alpha \underline{1}}] \mathcal{J}| \,,$$

while the $\mathcal{N} = 1$ flavor current multiplet is identified as follows

$$L \equiv \mathrm{i}\, \mathcal{L}^{\underline{12}}| \,. \qquad (1.6)$$

Here J and J_α satisfy the conservation equations

$$\bar D^2 J = D^2 J = 0 \,, \qquad (1.7)$$
$$D^\alpha J_\alpha = \bar D^2 J_\alpha = 0 \,. \qquad (1.8)$$

The spinor object J_α contains the second supersymmetry current and two of the three SU(2) currents, namely those which correspond to the symmetries belonging to SU(2)/U(1). Finally, the scalar J contains the current corresponding to the special combination of the $\mathcal{N}=2$ U(1) R-transformation and SU(2) σ_3-rotation which leaves $\theta_{\underline{1}}$ and $\bar\theta^{\underline{1}}$ invariant.

2. Superconformal building blocks

In $\mathcal{N}$–extended global superspace $\mathbf{R}^{4|4\mathcal{N}}$ parametrised by $z^A=(x^a,\theta^\alpha_i,\bar\theta^i_{\dot\alpha})$, an infinitesimal superconformal transformation

$$z^A \longrightarrow z^A + \xi\cdot z^A , \qquad (2.1)$$
$$\xi=\bar\xi=\xi^a(z)\partial_a+\xi^\alpha_i(z)D^i_\alpha+\bar\xi^i_{\dot\alpha}(z)\bar D^{\dot\alpha}_i$$

is generated by a superconformal Killing vector ξ defined to satisfy

$$[\xi\ ,\ D^i_\alpha] \propto D^j_\beta\ . \qquad (2.2)$$

From here it follows

$$\xi^\alpha_i=-\frac{\mathrm{i}}{8}\bar D_{\dot\beta i}\xi^{\dot\beta\alpha}\ ,\qquad \bar D_{\dot\beta j}\xi^\alpha_i=0 \qquad (2.3)$$

while the vector component of ξ is constrained by

$$D^i_{(\alpha}\xi_{\beta)\dot\beta}=\bar D^{(\dot\alpha}_i\xi^{\dot\beta)\beta}=0\ , \qquad (2.4)$$
$$\Longrightarrow\quad \partial_a\xi_b+\partial_b\xi_a=\tfrac{1}{2}\,\eta_{ab}\,\partial_c\xi^c\ .$$

For $\mathcal{N}<4$, the algebra of superconformal Killing vectors is isomorphic to the $\mathcal{N}$–extended superconformal algebra, su$(2,2|\mathcal{N})$.

Let us introduce the parameters of generalized Lorentz $\omega_{(\alpha\beta)}$, scale–chiral σ and SU($\mathcal{N}$) transformations $\Lambda_i{}^j$ ($\Lambda^\dagger-\Lambda=\mathrm{tr}\,\Lambda=0$) generated by ξ

$$\begin{aligned}[\xi\ ,\ D^i_\alpha] &= -(D^i_\alpha\xi^\beta_j)D^j_\beta\\ &=\omega_\alpha{}^\beta D^i_\beta-\mathrm{i}\Lambda_j{}^i\,D^j_\alpha\\ &\quad-\frac{1}{\mathcal{N}}\Big((\mathcal{N}-2)\sigma+2\bar\sigma\Big)D^i_\alpha\ . \end{aligned}\qquad (2.5)$$

A *primary* superfield $\mathcal{O}(z)$, carrying some number of undotted and dotted spinor indices and transforming in some representation of the R–symmetry SU($\mathcal{N}$), satisfies the following infinitesimal transformation law under the superconformal group

$$\begin{aligned}\delta\,\mathcal{O} &= -\xi\,\mathcal{O}+\frac{1}{2}\,\omega^{ab}M_{ab}\,\mathcal{O}+\mathrm{i}\,\Lambda_j{}^i\,R_i{}^j\,\mathcal{O}\\ &\quad -2\,(q\,\sigma+\bar q\,\bar\sigma)\ \mathcal{O}\ .\end{aligned}\qquad (2.6)$$

Here M_{ab} are the Lorentz generators, and $R_i{}^j$ are the generators of SU($\mathcal{N}$). The constant parameters q and $\bar q$ determine the dimension $(q+\bar q)$ and U(1) R–symmetry charge $(q-\bar q)$ of the superfield, respectively.

In $\mathcal{N}=1$ superconformal theory, the supercurrent $J_{\alpha\dot\alpha}$ and the flavor current L are primary superfields with the superconformal transformations

$$\begin{aligned}\delta J_{\alpha\dot\alpha} &= -\xi\,J_{\alpha\dot\alpha}-3\,(\sigma+\bar\sigma)\,J_{\alpha\dot\alpha}\\ &\quad+(\omega_\alpha{}^\beta\delta_{\dot\alpha}{}^{\dot\beta}+\bar\omega_{\dot\alpha}{}^{\dot\beta}\delta_\alpha{}^\beta)J_{\beta\dot\beta}\ ,\end{aligned}\qquad (2.7)$$
$$\delta L = -\xi\,L-2\,(\sigma+\bar\sigma)\,L\ . \qquad (2.8)$$

In $\mathcal{N}=2$ superconformal theory, the supercurrent $\mathcal{J}$ and the flavor current $\mathcal{L}_{ij}$ are primary superfields with the superconformal transformations

$$\delta\mathcal{J} = -\xi\,\mathcal{J}-2\,(\sigma+\bar\sigma)\,\mathcal{J}\ , \qquad (2.9)$$
$$\begin{aligned}\delta\mathcal{L}_{ij} &= -\xi\,\mathcal{L}_{ij}-2\,(\sigma+\bar\sigma)\,\mathcal{L}_{ij}\\ &\quad+2\mathrm{i}\,\Lambda_{(i}{}^k\,\mathcal{L}_{j)k}\ .\end{aligned}\qquad (2.10)$$

Correlation functions of primary superfields, $\langle\,\mathcal{O}_1(z_1)\,\mathcal{O}_2(z_2)\ldots\mathcal{O}_n(z_n)\,\rangle$, involve some universal building blocks which we are going to describe briefly. Associated with any two points z_1 and z_2 in superspace are (anti-)chiral combinations $x_{\bar12}$, θ_{12} and $\bar\theta_{12}$:

$$\begin{aligned}x^a_{\bar12} &= -x^a_{2\bar1}=x^a_{1-}-x^a_{2+}+2\mathrm{i}\,\theta_{2i}\,\sigma^a\,\bar\theta^i_1\ ,\\ x^a_\pm &\equiv x^a\pm\mathrm{i}\,\theta_i\sigma^a\bar\theta^i\ ;\\ \theta_{12} &= \theta_1-\theta_2\ ,\qquad \bar\theta_{12}=\bar\theta_1-\bar\theta_2\ ,\end{aligned}\qquad (2.11)$$

which are invariant under Poincaré supersymmetry transformations (the notation '$x_{\bar12}$' indicates that $x_{\bar12}$ is antichiral with respect to z_1 and chiral with respect to z_2) but transforms semi-covariantly with respect to the superconformal group (see, e.g. [1]). In extended supersymmetry, there exist primary superfields with isoindices, and their correlation functions generically involve a conformally covariant $\mathcal{N}\times\mathcal{N}$ unimodular matrix[1] [3]

$$\hat{u}_i{}^j(z_{12}) = \left(\frac{x_{\bar{2}1}{}^2}{x_{\bar{1}2}{}^2}\right)^{1/\mathcal{N}} u_i{}^j(z_{12}) ,$$
$$u_i{}^j(z_{12}) = \delta_i{}^j - 4\mathrm{i}\,\frac{\theta_{12\,i}x_{\bar{1}2}\bar{\theta}^j_{12}}{x_{\bar{1}2}{}^2} , \qquad (2.12)$$

with the basic properties

$$\hat{u}^\dagger(z_{12})\ \hat{u}(z_{12}) = \mathbf{1} ,$$
$$\hat{u}^{-1}(z_{12}) = \hat{u}(z_{21}) ,$$
$$\det\ \hat{u}(z_{12}) = 1 , \qquad (2.13)$$

and the transformation rule

$$\delta\hat{u}_i{}^j(z_{12}) = \mathrm{i}\,\Lambda_i{}^k(z_1)\hat{u}_k{}^j(z_{12}) - \mathrm{i}\,\hat{u}_i{}^k(z_{12})\Lambda_k{}^j(z_2) . \qquad (2.14)$$

Given three superspace points z_1, z_2 and z_3, one can define superconformally covariant bosonic and fermionic variables $\mathbf{Z}_1$, $\mathbf{Z}_2$ and $\mathbf{Z}_3$, where $\mathbf{Z}_1 = (\mathbf{X}_1,\ \Theta^i_1,\ \bar{\Theta}_{1\,i})$ are [2, 3]

$$\mathbf{X}_1 = \tilde{x}_{1\bar{2}}{}^{-1}\tilde{x}_{\bar{2}3}\tilde{x}_{3\bar{1}}{}^{-1} ,$$
$$\tilde{\Theta}^i_1 = \mathrm{i}\left(\tilde{x}_{\bar{2}1}{}^{-1}\bar{\theta}^i_{12} - \tilde{x}_{\bar{3}1}{}^{-1}\bar{\theta}^i_{13}\right) ,$$
$$\bar{\mathbf{X}}_1 = \mathbf{X}_1^\dagger = \mathbf{X}_1 - 4\mathrm{i}\,\tilde{\Theta}^i_1\tilde{\bar{\Theta}}_{1\,i} ,$$
$$\tilde{\bar{\Theta}}_{1\,i} = (\tilde{\Theta}^i_1)^\dagger \qquad (2.15)$$

and $\mathbf{Z}_2$, $\mathbf{Z}_3$ are obtained from here by cyclically permuting indices. These structures possess remarkably simple superconformal transformation rules:

$$\delta\mathbf{X}_{1\,\alpha\dot{\alpha}} = \left(\omega_\alpha{}^\beta(z_1) - \delta_\alpha{}^\beta\,\sigma(z_1)\right)\mathbf{X}_{1\,\beta\dot{\alpha}} + \mathbf{X}_{1\,\alpha\dot{\beta}}\left(\bar{\omega}^{\dot{\beta}}{}_{\dot{\alpha}}(z_1) - \delta^{\dot{\beta}}{}_{\dot{\alpha}}\,\bar{\sigma}(z_1)\right) ,$$
$$\delta\Theta^i_{1\,\alpha} = \omega_\alpha{}^\beta(z_1)\Theta^i_{1\,\beta} - \mathrm{i}\Theta^j_{1\,\alpha}\Lambda_j{}^i(z_1) - \frac{1}{\mathcal{N}}\Big((\mathcal{N}-2)\sigma(z_1) + 2\bar{\sigma}(z_1)\Big)\Theta^i_{1\,\alpha}$$

and turn out to be essential building blocks for correlations functions of primary superfields. The variables $\mathbf{Z}$ with different labels are related to each other, in particular:

$$\tilde{x}_{\bar{1}3}\,\mathbf{X}_3\,\tilde{x}_{\bar{3}1} = -\bar{\mathbf{X}}_1{}^{-1} ,$$
$$\tilde{x}_{\bar{1}3}\,\tilde{\Theta}^i_3\,u_i{}^j(z_{31}) = -\mathbf{X}_1{}^{-1}\,\tilde{\Theta}^j_1 . \qquad (2.16)$$

[1] We use the notation adopted in [11, 12]. When the spinor indices are not indicated explicitly, the following matrix-like conventions are used [2]: $\psi = (\psi^\alpha)$, $\tilde{\psi} = (\psi_\alpha)$, $\bar{\psi} = (\bar{\psi}^{\dot{\alpha}})$, $\tilde{\bar{\psi}} = (\bar{\psi}_{\dot{\alpha}})$, $x = (x_{\alpha\dot{\alpha}})$, $\tilde{x} = (x^{\dot{\alpha}\alpha})$; but $x^2 \equiv x^a x_a = -\frac{1}{2}\,\mathrm{tr}\,(\tilde{x}x)$, and hence $\tilde{x}^{-1} = -x/x^2$.

With the aid of the matrices $u(z_{rs})$, $r, s = 1, 2, 3$, defined in (2.12), one can construct unitary matrices $\hat{\mathbf{u}}(\mathbf{Z}_s)$ [3], in particular

$$\hat{\mathbf{u}}(\mathbf{Z}_3) = \hat{u}(z_{31})\hat{u}(z_{12})\hat{u}(z_{23}) = \left(\frac{\bar{\mathbf{X}}_3{}^2}{\mathbf{X}_3{}^2}\right)^{1/\mathcal{N}}\left(\delta_i{}^j - 4\mathrm{i}\tilde{\bar{\Theta}}_{3\,i}\mathbf{X}_3{}^{-1}\tilde{\Theta}^j_3\right) \qquad (2.17)$$

transforming at z_3 only. Their properties are

$$\hat{\mathbf{u}}^\dagger(\mathbf{Z}_3) = \hat{\mathbf{u}}^{-1}(\mathbf{Z}_3) , \qquad \det\hat{\mathbf{u}}(\mathbf{Z}_3) = 1 . \quad (2.18)$$

The above general formalism has specific features in the case $\mathcal{N} = 2$ that is of primary interest for us. Here we have at our disposal the SU(2)–invariant tensors $\varepsilon_{ij} = -\varepsilon_{ji}$ and $\varepsilon^{ij} = -\varepsilon^{ji}$, normalized to $\varepsilon^{\underline{12}} = \varepsilon_{\underline{21}} = 1$. They can be used to raise and lower isoindices: $C^i = \varepsilon^{ij}C_j$, $C_i = \varepsilon_{ij}C^j$. For $\mathcal{N} = 2$, the condition of unimodularity of the matrix defined in (2.12) can be written as

$$\hat{u}_{ji}(z_{21}) = -\,\hat{u}_{ij}(z_{12}) . \qquad (2.19)$$

The importance of this relation is that it implies that the two-point function

$$A_{i_1 i_2}(z_1, z_2) \equiv \frac{\hat{u}_{i_1 i_2}(z_{12})}{(x_{\bar{1}2}{}^2 x_{\bar{2}1}{}^2)^{\frac{1}{2}}} = -\frac{\hat{u}_{i_2 i_1}(z_{21})}{(x_{\bar{1}2}{}^2 x_{\bar{2}1}{}^2)^{\frac{1}{2}}} \qquad (2.20)$$

is analytic [13] in z_1 and z_2 for $z_1 \neq z_2$,

$$D_{1\,\alpha(j_1}A_{i_1)i_2}(z_1, z_2) = 0 ,$$
$$\bar{D}_{1\,\dot{\alpha}(j_1}A_{i_1)i_2}(z_1, z_2) = 0 . \qquad (2.21)$$

As we will see later, $A_{i_1 i_2}(z_1, z_2)$ is a building block of correlation functions of analytic primary superfields like the $\mathcal{N} = 2$ flavor currents. For $\mathcal{N} = 2$, the fact that $\hat{\mathbf{u}}(\mathbf{Z}_3)$ is unimodular and unitary, implies

$$\mathrm{tr}\ \hat{\mathbf{u}}^\dagger(\mathbf{Z}_3) = \mathrm{tr}\ \hat{\mathbf{u}}(\mathbf{Z}_3) ,$$
$$\hat{\mathbf{u}}^\dagger_{ji}(\mathbf{Z}_3) = -\hat{\mathbf{u}}_{ij}(\mathbf{Z}_3) . \qquad (2.22)$$

3. Correlation functions of $\mathcal{N} = 2$ currents

According to the general prescription of [2, 3], the two-point function of a primary superfield $\mathcal{O}_{\mathcal{I}}$, which is a Lorentz scalar and transforms in a representation T of the R–symmetry group SU($\mathcal{N}$), with its conjugate $\bar{\mathcal{O}}^{\mathcal{J}}$ reads

$$\langle \mathcal{O}_{\mathcal{I}}(z_1)\, \bar{\mathcal{O}}^{\mathcal{J}}(z_2)\rangle \;=\; C_{\mathcal{O}}\, \frac{T_{\mathcal{I}}{}^{\mathcal{J}}\Big(\hat{u}(z_{12})\Big)}{(x_{\bar{1}2}{}^2)^{\bar{q}}(x_{\bar{2}1}{}^2)^{q}}\ ,$$

where $C_{\mathcal{O}}$ is a normalization constant.

For the $\mathcal{N}=2$ supercurrent $\mathcal{J}$ and the flavor current $\mathcal{L}^{\bar{a}}_{ij}$, the above prescription gives

$$\langle \mathcal{J}(z_1)\ \mathcal{J}(z_2)\rangle = c_{\mathcal{J}}\ \frac{1}{x_{\bar{1}2}{}^2 x_{\bar{2}1}{}^2}\ , \qquad (3.1)$$

$$\langle \mathcal{L}^{\bar{a}_1}_{i_1 j_1}(z_1)\mathcal{L}^{\bar{a}_2\, i_2 j_2}(z_2)\rangle = 2c_{\mathcal{L}}\,\delta^{\bar{a}_1\bar{a}_2} \times \frac{\hat{u}_{i_1}{}^{(i_2}(z_{12})\ \hat{u}_{j_1}{}^{j_2)}(z_{12})}{x_{\bar{1}2}{}^2 x_{\bar{2}1}{}^2}\ . \qquad (3.2)$$

The relevant conservation equations prove to be satisfied at $z_1 \neq z_2$,

$$D_1{}^{ij}\langle \mathcal{J}(z_1)\ \mathcal{J}(z_2)\rangle \;=\; 0\ ,$$
$$D_{1\,\alpha(k_1}\langle \mathcal{L}_{i_1 j_1)}(z_1)\mathcal{L}^{i_2 j_2}(z_2)\rangle \;=\; 0\ . \qquad (3.3)$$

According to the general prescription of [2, 3], the three-point function of primary superfields $\mathcal{O}^{(1)}_{\mathcal{I}_1}$, $\mathcal{O}^{(2)}_{\mathcal{I}_2}$ and $\mathcal{O}^{(3)}_{\mathcal{I}_3}$ reads

$$\langle \mathcal{O}^{(1)}_{\mathcal{I}_1}(z_1)\,\mathcal{O}^{(2)}_{\mathcal{I}_2}(z_2)\,\mathcal{O}^{(3)}_{\mathcal{I}_3}(z_3)\rangle = \frac{T^{(1)}{}_{\mathcal{I}_1}{}^{\mathcal{J}_1}\,(\hat{u}(z_{13}))\,T^{(2)}{}_{\mathcal{I}_2}{}^{\mathcal{J}_2}\,(\hat{u}(z_{23}))}{(x_{\bar{1}3}{}^2)^{\bar{q}_1}(x_{\bar{3}1}{}^2)^{q_1}(x_{\bar{2}3}{}^2)^{\bar{q}_2}(x_{\bar{3}2}{}^2)^{q_2}} \times\ H_{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}(\mathbf{Z}_3)\ .$$

Here $H_{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}(\mathbf{Z}_3)$ transforms as an isotensor at z_3 in the representations $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$ with respect to the indices $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{I}_3$, respectively, and possesses the homogeneity property

$$H_{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}(\Delta\bar{\Delta}\,\mathbf{X},\Delta\,\Theta,\bar{\Delta}\bar{\Theta}) = \Delta^{2p}\bar{\Delta}^{2\bar{p}}H_{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}(\mathbf{X},\Theta,\bar{\Theta})\ ,$$
$$p-2\bar{p}=\bar{q}_1+\bar{q}_2-q_3\ ,$$
$$\bar{p}-2p=q_1+q_2-\bar{q}_3\ .$$

In general, the latter equation admits a finite number of linearly independent solutions, and this can be considerably reduced by taking into account the symmetry properties, superfield conservation equations and, of course, the superfield constraints (such as chirality or analyticity [13]). Below we shall present the most general expressions for three-point functions of the $\mathcal{N}=2$ supercurrent $\mathcal{J}$ and the flavor current $\mathcal{L}^{\bar{a}}_{ij}$, which are compatible with all physical requirements. Details can be found in [1].

The three-point function of the $\mathcal{N}=2$ supercurrent is

$$\langle \mathcal{J}(z_1)\,\mathcal{J}(z_2)\,\mathcal{J}(z_3)\rangle = \frac{1}{x_{\bar{1}3}{}^2 x_{\bar{3}1}{}^2 x_{\bar{2}3}{}^2 x_{\bar{3}2}{}^2} \times\ \Big\{ A\Big(\frac{1}{\mathbf{X}_3{}^2}+\frac{1}{\bar{\mathbf{X}}_3{}^2}\Big) +\ B\,\frac{\Theta_3^{\alpha\beta}\mathbf{X}_{3\alpha\dot{\alpha}}\mathbf{X}_{3\beta\dot{\beta}}\bar{\Theta}_3^{\dot{\alpha}\dot{\beta}}}{(\mathbf{X}_3{}^2)^2}\Big\}\ , \qquad (3.4)$$

where

$$\Theta_3^{\alpha\beta} \;=\; \Theta_3^{(\alpha\beta)} \;=\; \Theta_3^{\alpha i}\,\Theta^{\beta}_{3\,i}\ ,$$
$$\bar{\Theta}_3^{\dot{\alpha}\dot{\beta}} \;=\; \bar{\Theta}_3^{(\dot{\alpha}\dot{\beta})} \;=\; \bar{\Theta}^{\dot{\alpha}}_{3\,i}\,\bar{\Theta}_3^{\dot{\alpha}i}\ , \qquad (3.5)$$

and A, B are real parameters. The second structure is nilpotent and real.

The three-point function of the $\mathcal{N}=2$ flavor current reads

$$\langle \mathcal{L}^{\bar{a}}_{i_1 j_1}(z_1)\,\mathcal{L}^{\bar{b}}_{i_2 j_2}(z_2)\,\mathcal{L}^{\bar{c}}_{i_3 j_3}(z_3)\rangle = \frac{\hat{u}_{i_1}{}^{k_1}(z_{13})\hat{u}_{j_1}{}^{l_1}(z_{13})\hat{u}_{i_2}{}^{k_2}(z_{23})\hat{u}_{j_2}{}^{l_2}(z_{23})}{x_{\bar{3}1}{}^2 x_{\bar{1}3}{}^2 x_{\bar{3}2}{}^2 x_{\bar{2}3}{}^2} \times\ f^{\bar{a}\bar{b}\bar{c}}\Big\{\frac{\varepsilon_{i_3(k_1}\hat{\mathbf{u}}_{l_1)(l_2}(\mathbf{Z}_3)\varepsilon_{k_2)j_3}}{(\mathbf{X}_3{}^2\bar{\mathbf{X}}_3{}^2)^{\frac12}}\ +\ (i_3\leftrightarrow j_3)\Big\} \qquad (3.6)$$

with $f^{\bar{a}\bar{b}\bar{c}} \;=\; f^{[\bar{a}\bar{b}\bar{c}]}$ a completely antisymmetric real tensor being proportional to the structure constants of the flavor group.

For mixed correlation functions of the $\mathcal{N}=2$ supercurrent and the flavor current, we get

$$\langle \mathcal{J}(z_1)\ \mathcal{J}(z_2)\ \mathcal{L}^{\bar{a}}_{ij}(z_3)\rangle = 0\ , \qquad (3.7)$$

$$\langle \mathcal{L}^{\bar{a}}_{i_1 j_1}(z_1)\,\mathcal{L}^{\bar{b}}_{i_2 j_2}(z_2)\,\mathcal{J}(z_3)\rangle = d\,\delta^{\bar{a}\bar{b}} \times\ \frac{\hat{u}_{i_1}{}^{k_1}(z_{13})\hat{u}_{j_1}{}^{l_1}(z_{13})\hat{u}_{i_2}{}^{k_2}(z_{23})\hat{u}_{j_2}{}^{l_2}(z_{23})}{x_{\bar{3}1}{}^2 x_{\bar{1}3}{}^2 x_{\bar{3}2}{}^2 x_{\bar{2}3}{}^2} \times\ \frac{\varepsilon_{k_2(k_1}\hat{\mathbf{u}}_{l_1)l_2}(\mathbf{Z}_3)+\varepsilon_{l_2(k_1}\hat{\mathbf{u}}_{l_1)k_2}(\mathbf{Z}_3)}{(\mathbf{X}_3{}^2\bar{\mathbf{X}}_3{}^2)^{\frac12}}\ , \qquad (3.8)$$

with d a real parameter which can be related, via supersymmetric Ward identities, to the parameter $c_{\mathcal{L}}$ in the two-point function (3.2),

$$d \;=\; \frac{1}{4\pi^2}\,c_{\mathcal{L}}\ . \qquad (3.9)$$

It is worth pointing out that eq. (3.7) is one of the important consequences of $\mathcal{N}=2$ superconformal symmetry and has no direct analog in the

$\mathcal{N}=1$ case. In a generic $\mathcal{N}=1$ superconformal theory with a flavor current L, the correlation function $\langle J_{\alpha\dot\alpha}\, J_{\beta\dot\beta}\, L\rangle$ is not restricted by $\mathcal{N}=1$ superconformal symmetry to vanish [2].

4. Reduction to $\mathcal{N}=1$ superfields

From the point of view of $\mathcal{N}=1$ superconformal symmetry, any $\mathcal{N}=2$ primary superfield consists of several $\mathcal{N}=1$ primary superfields. Having computed the correlation functions of $\mathcal{N}=2$ primary superfields, one can read off all correlators of their $\mathcal{N}=1$ superconformal components. Since any $\mathcal{N}=2$ superconformal theory is a particular $\mathcal{N}=1$ superconformal theory, one can then simply make use of $\mathcal{N}=1$ superconformal Ward identities [2] to relate the coefficients of various correlators.

Using the explicit form (3.1) of the $\mathcal{N}=2$ supercurrent two-point function, one can read off the two-point functions of the $\mathcal{N}=1$ primary superfields contained in $\mathcal{J}$, in particular[2]

$$\langle J(z_1)\; J(z_2)\rangle \;=\; c_{\mathcal{J}}\; \frac{1}{x_{\bar 12}{}^2 x_{\bar 21}{}^2}\,, \qquad (4.1)$$

$$\langle J_{\alpha\dot\alpha}(z_1)\; J_{\beta\dot\beta}(z_2)\rangle \;=\; \frac{64}{3}\, c_{\mathcal{J}}\; \frac{(x_{1\bar 2})_{\alpha\dot\beta}(x_{2\bar 1})_{\beta\dot\alpha}}{(x_{\bar 12}{}^2 x_{\bar 21}{}^2)^2}\,.$$

Similarly, the two-point function of the $\mathcal{N}=1$ flavor current follows from (3.2)

$$\langle L^{\bar a_1}(z_1)\, L^{\bar a_2}(z_2)\rangle \;=\; c_{\mathcal{L}}\; \frac{\delta^{\bar a_1\,\bar a_2}}{x_{\bar 12}{}^2 x_{\bar 21}{}^2}\,. \qquad (4.2)$$

We now present several $\mathcal{N}=1$ three-point functions which are encoded in that of the $\mathcal{N}=2$ supercurrent, given by eq. (3.4).

$$\langle J(z_1)\, J(z_2)\, J(z_3)\rangle = \frac{A}{x_{\bar 13}{}^2 x_{\bar 31}{}^2 x_{\bar 23}{}^2 x_{\bar 32}{}^2} \times\left(\frac{1}{\mathbf{X}_3{}^2}+\frac{1}{\bar{\mathbf{X}}_3{}^2}\right)\,, \qquad (4.3)$$

$$\begin{aligned}\langle J(z_1)\, J(z_2)\, J_{\alpha\dot\alpha}(z_3)\rangle &= -\frac{1}{12}(8A-3B) \times\frac{1}{x_{\bar 13}{}^2 x_{\bar 31}{}^2 x_{\bar 23}{}^2 x_{\bar 32}{}^2}\\ &\times\Bigg\{\frac{2(\mathbf{P}_3\cdot\mathbf{X}_3)\mathbf{X}_{3\,\alpha\dot\alpha}+\mathbf{X}_3{}^2\mathbf{P}_{3\,\alpha\dot\alpha}}{(\mathbf{X}_3{}^2)^2} + (\mathbf{X}_3\leftrightarrow -\bar{\mathbf{X}}_3)\Bigg\}\,, \qquad (4.4)\end{aligned}$$

$$\begin{aligned}\langle J_{\alpha\dot\alpha}(z_1)\, J_{\beta\dot\beta}(z_2)\, J(z_3)\rangle &= -\frac{4}{9}(8A+3B) \times\frac{(x_{1\bar 3})_{\alpha\dot\gamma}(x_{3\bar 1})_{\gamma\dot\alpha}(x_{2\bar 3})_{\beta\dot\delta}(x_{3\bar 2})_{\delta\dot\beta}}{(x_{\bar 13}{}^2 x_{\bar 31}{}^2 x_{\bar 23}{}^2 x_{\bar 32}{}^2)^2}\\ &\times\Bigg\{\frac{\mathbf{X}_3{}^{\gamma\dot\gamma}\mathbf{X}_3{}^{\delta\dot\delta}}{(\mathbf{X}_3{}^2)^3}+\frac{1}{2}\,\frac{\varepsilon^{\gamma\delta}\,\varepsilon^{\dot\gamma\dot\delta}}{(\mathbf{X}_3{}^2)^2} + (\mathbf{X}_3\leftrightarrow -\bar{\mathbf{X}}_3)\Bigg\}\,, \qquad (4.5)\end{aligned}$$

with $\mathbf{P}_a$ defined by [2]

$$\bar{\mathbf{X}}_a-\mathbf{X}_a = \mathrm{i}\,\mathbf{P}_a\,, \qquad \mathbf{P}_a = 2\,\Theta\,\sigma_a\,\bar\Theta\,. \qquad (4.6)$$

The most interesting correlator and by far the most laborious to compute is

$$\begin{aligned}&\langle J_{\alpha\dot\alpha}(z_1)\, J_{\beta\dot\beta}(z_2)\, J_{\gamma\dot\gamma}(z_3)\rangle \qquad (4.7)\\ &\quad= \frac{(x_{1\bar 3})_{\alpha\dot\sigma}(x_{3\bar 1})_{\sigma\dot\alpha}(x_{2\bar 3})_{\beta\dot\delta}(x_{3\bar 2})_{\delta\dot\beta}}{(x_{\bar 13}{}^2 x_{\bar 31}{}^2 x_{\bar 23}{}^2 x_{\bar 32}{}^2)^2} \times H^{\dot\sigma\sigma,\dot\delta\delta}{}_{\gamma\dot\gamma}(\mathbf{X}_3,\bar{\mathbf{X}}_3)\,,\\ &H^{\dot\sigma\sigma,\dot\delta\delta}{}_{\gamma\dot\gamma}(\mathbf{X}_3,\bar{\mathbf{X}}_3) = h^{\dot\sigma\sigma,\dot\delta\delta}{}_{\gamma\dot\gamma}(\mathbf{X}_3,\bar{\mathbf{X}}_3) + h^{\dot\delta\delta,\dot\sigma\sigma}{}_{\gamma\dot\gamma}(-\bar{\mathbf{X}}_3,-\mathbf{X}_3)\,,\end{aligned}$$

where

$$\begin{aligned}h^{abc}(\mathbf{X},\bar{\mathbf{X}}) &\equiv -\frac{1}{8}\,(\sigma^a)_{\alpha\dot\alpha}\,(\sigma^a)_{\beta\dot\beta}\,(\tilde\sigma^c)^{\dot\gamma\gamma} \times h^{\dot\alpha\alpha,\dot\beta\beta}{}_{\gamma\dot\gamma}(\mathbf{X},\bar{\mathbf{X}}) \qquad (4.8)\\ &= -\frac{16}{27}\,(26A-\frac{9}{4}B)\,\frac{\mathrm{i}}{(\mathbf{X}^2)^2} \times\Big(\mathbf{X}^a\eta^{bc}+\mathbf{X}^b\eta^{ac}-\mathbf{X}^c\eta^{ab}+\mathrm{i}\,\varepsilon^{abcd}\mathbf{X}_d\Big)\\ &\quad -\frac{8}{27}\,(8A-9B)\,\frac{1}{(\mathbf{X}^2)^3} \times\Bigg\{2\Big(\mathbf{X}^a\mathbf{P}^b+\mathbf{X}^b\mathbf{P}^a\Big)\mathbf{X}^c\\ &\quad -3\mathbf{X}^a\mathbf{X}^b\Big(\mathbf{P}^c+2\frac{(\mathbf{P}\cdot\mathbf{X})}{\mathbf{X}^2}\mathbf{X}^c\Big)\\ &\quad -(\mathbf{P}\cdot\mathbf{X})\Big(3(\mathbf{X}^a\eta^{bc}+\mathbf{X}^b\eta^{ac})-2\mathbf{X}^c\eta^{ab}\Big)\\ &\quad +\frac{1}{2}\mathbf{X}^2\Big(\mathbf{P}^a\eta^{bc}+\mathbf{P}^b\eta^{ac}+\mathbf{P}^c\eta^{ab}\Big)\Bigg\}\,.\end{aligned}$$

Our final relations (4.7) and (4.8) perfectly agree with the general structure of the three-point function of the supercurrent in $\mathcal{N}=1$ superconformal field theory [2].

[2] Here and below, all building blocks are expressed in $\mathcal{N}=1$ superspace.

Using the results of [2], one may express A and B in terms of the anomaly coefficients [14]

$$a = \frac{1}{24}(5n_V + n_H) \ , \qquad c = \frac{1}{12}(2n_V + n_H) \ , \tag{4.9}$$

where n_V and n_H denote the number of free $\mathcal{N} = 2$ vector multiplets and hypermultiplets, respectivley. We get[3]

$$A = \frac{3}{64\pi^6}(4a - 3c) \ , \qquad B = \frac{1}{8\pi^6}(4a - 5c) \ . \tag{4.10}$$

In $\mathcal{N} = 1$ supersymmetry, a superconformal Ward identity relates the coefficient in the two-point function of the supercurrent (4.1) to the anomaly coefficient c as follows [2]

$$c_{\mathcal{J}} = \frac{3}{8\pi^4}\, c \ . \tag{4.11}$$

In terms of the coefficients A and B this relation reads

$$\frac{2}{\pi^2}\, c_{\mathcal{J}} = 8A - 3B \ . \tag{4.12}$$

Let us turn to the three-point function of the $\mathcal{N} = 2$ flavor current given by eq. (3.6). From it one reads off the three-point function of the $\mathcal{N} = 1$ flavor current

$$\langle L^{\bar a}(z_1)\, L^{\bar b}(z_2)\, L^{\bar c}(z_3)\rangle = \frac{1}{4} f^{\bar a \bar b \bar c} \frac{\mathrm{i}}{x_{\bar 1 3}{}^2 x_{\bar 3 1}{}^2 x_{\bar 2 3}{}^2 x_{\bar 3 2}{}^2} \times \left(\frac{1}{\bar{\mathbf{X}}_3{}^2} - \frac{1}{\mathbf{X}_3{}^2} \right) \ . \tag{4.13}$$

It is worth noting that the Ward identities allow one to represent $f^{\bar a \bar b \bar c}$ as a product of $c_{\mathcal{L}}$ and the structure constants of the flavor symmetry group, see [2] for more details.

In $\mathcal{N} = 1$ superconformal field theory, the three-point function of the flavor current superfield L contains, in general, two linearly independent forms [2]:

$$\langle L^{\bar a}(z_1)\, L^{\bar b}(z_2)\, L^{\bar c}(z_3)\rangle = \frac{1}{x_{\bar 1 3}{}^2 x_{\bar 3 1}{}^2 x_{\bar 2 3}{}^2 x_{\bar 3 2}{}^2} \times \left\{ \mathrm{i}\, f^{[\bar a \bar b \bar c]} \left(\frac{1}{\mathbf{X}_3{}^2} - \frac{1}{\bar{\mathbf{X}}_3{}^2} \right) + d^{(\bar a \bar b \bar c)} \left(\frac{1}{\mathbf{X}_3{}^2} + \frac{1}{\bar{\mathbf{X}}_3{}^2} \right) \right\} \ .$$

The second term, involving a completely symmetric group tensor $d^{\bar a \bar b \bar c}$, reflects the presence of chiral anomalies in the theory. The field-theoretic origin of this term is due to the fact that the $\mathcal{N} = 1$ conservation equation $\bar D^2 L = D^2 L = 0$ admits a non-trivial deformation

$$\bar D^2 \langle L^{\bar a} \rangle \ \propto \ d^{\bar a \bar b \bar c}\, W^{\bar b\, \alpha}\, W^{\bar c}_\alpha$$

when the chiral flavor current is coupled to a background vector multiplet. Eq. (4.13) tells us that the flavor currents are anomaly-free in $\mathcal{N} = 2$ superconformal theory. This agrees with the facts that (i) $\mathcal{N} = 2$ super Yang-Mills models are non-chiral; (ii) the $\mathcal{N} = 2$ conservation equation (1.4) does not possess non-trivial deformations.

[3] Our definition of the $\mathcal{N} = 1$ supercurrent corresponds to that adopted in [12] and differs in sign from Osborn's convention [2].

Acknowledgements
Support from DFG, the German National Science Foundation, from GIF, the German-Israeli foundation for Scientific Research and from the EEC under TMR contract ERBFMRX-CT96-0045 is gratefully acknowledged. This work was also supported in part by the NATO collaborative research grant PST.CLG 974965, by the RFBR grant No. 99-02-16617, by the INTAS grant No. 96-0308 and by the DFG-RFBR grant No. 99-02-04022.

References

[1] S.M. Kuzenko and S. Theisen, Class. Quant. Grav. **17** (2000) 665, hep-th/9907107.

[2] H. Osborn, Ann. Phys. **272** (1999) 243, hep-th/9808041.

[3] J-H. Park, Nucl. Phys. **B539** (1999) 599, hep-th/9807186; Nucl. Phys. **B559** (1999) 455, hep-th/9903230.

[4] J. Maldacena, Adv. Theor. Math. Phys. **2** (1998) 231, hep-th/9711200.

[5] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, Phys. Rep. **323** (2000) 183, hep-th/9905111.

[6] S. Ferrara and B. Zumino, Nucl. Phys. **87B** (1975) 207.

[7] J. Wess and B. Zumino, Nucl. Phys. **B78** (1974) 1; S. Ferrara and B. Zumino, Nucl. Phys. **B79** (1974) 413; A. Salam and J. Strathdee, Phys. Rev. **D11** (1975) 1521.

[8] M.F. Sohnius, Phys. Lett. **81B** (1979) 8.

[9] L. Mezincescu, *On the superfield formulation of O(2) supersummetry*, JINR preprint P2-12572, 1979 (unpublished).

[10] P. Howe, K.S. Stelle and P.K. Townsend, Nucl. Phys. **B192** (1981) 332.

[11] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, 2nd ed. (Princeton University Press, 1992).

[12] I.L. Buchbinder and S.M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity*, 2nd ed. (IOP, 1998).

[13] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, Class. Quantum Grav. **1** (1984) 469.

[14] D. Anselmi, D.Z. Freedman, M.T. Grisaru and A.A. Johansen, Nucl. Phys. **B526** (1998) 543, hep-th/9708042.