PROCEEDINGS

M-Theory N=1 Effective Supergravity in Four Dimensions

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ABSTRACT: We present a general 'off-shell' description of the effective $N = 1$ supergravity describing the low-energy limit of M-theory compactified on (Calabi-Yau) $\times S^1/\mathbf{Z}_2$. In our formulation, the M-theory Bianchi identities are imposed by the equations of motion of four-dimensional supermultiplets. Modifications of these identities (resulting for instance from contributions localized at orbifold singularities or non-perturbative sources like five-branes) can then easily be implemented.

KEYWORDS: Effective supergravity, M-theory.

1. Introduction and conclusions

M-theory compactified on $O_7 \equiv X_6 \times S^1/\mathbf{Z}_2$, where X_6 is a Calabi-Yau (CY) three-fold, leads to a four-dimensional theory with $N = 1$ local supersymmetry. In the low-energy limit, M-theory information can be organized as an expansion in powers of the eleven-dimensional gravitational constant κ_{11} [1, 2]. The lowest order κ_{11}^{-2} is eleven-dimensional supergravity [3]. In a compactification on $S^1/\mathbf{Z}_2$ only, the next orders are known to include orbifold plane contributions as well as gauge and gravitational anomaly-cancelling terms [1, 4, 2]. Similarly, the effective four-dimensional supergravity can be formulated as an expansion in the four-dimensional gravitational constant κ, even if string theory rather suggests to use the dilaton as expansion parameter. The lowest order κ^{-2} is the $S^1/\mathbf{Z}_2$ truncation of eleven-dimensional supergravity on a CY three-fold. The next order includes super-Yang-Mills (SYM) and charged matter kinetic and superpotential contributions. Sigma-model anomaly-cancelling terms modifying in particular the gauge thresholds are then also involved. These first corrections to the low-energy limit of M-theory compactifications on O_7 are identical to those obtained from heterotic compactifica-

tions on CY. The literature gives a detailed description of these results, with particular attention paid to the 'strong-coupling heterotic limit' in which the size of the CY space is smaller than the orbifold length, supersymmetry breaking by gaugino condensation and non-standard embeddings [5]–[8].

In this note, we give the structure of the four-dimensional $N = 1$ wilsonnian effective supergravity describing the universal massless sector of M-theory compactified on O_7. We begin by writing the theory corresponding to the reduction of the bulk eleven-dimensional supergravity directly in terms of four-dimensional 'M-theory supermultiplets'. The supersymmetrized Bianchi identities for the components of the M-theory tensor field strength are promoted to equations of motion using 'Lagrange multiplets'. Within this 'off-shell' approach, we can then introduce 'source multiplets' to take into account the contributions of the $S^1/\mathbf{Z}_2$ planes which appear as modifications of the Bianchi identities. This formulation is also particularly appropriate for the inclusion of non-perturbative states (M-theory five-branes, condensates, etc.).

The material presented here is detailed in ref. [9] and a forthcoming publication [10] will contain a direct application of our approach (the coupling of five-brane moduli to the background).

2. The bulk Lagrangian

In this section, we establish our basic procedure by considering the well-known 'bulk dynamics', which follows from O_7 compactification of eleven-dimensional supergravity. The resulting Lagrangian is the lowest order in the κ-expansion and describes Kaluza-Klein (KK) massless modes of eleven-dimensional supergravity.

We will precisely describe two aspects which may be of importance in M-theory compactifications. Firstly, we will introduce chiral, linear or vector supermultiplets with constraints in order to obtain a supersymmetric version of the Bianchi identities satisfied by antisymmetric tensors. Secondly, we will use superconformal supergravity in which we can keep open the choice of gravity frame.

2.1 Superconformal formalism

We use the superconformal formulation of $N = 1$ supergravity with a chiral compensating multiplet S_0 (with conformal and chiral weights $w = 1$ and $n = 1$) to generate Poincaré theories by gauge fixing. In this formalism, a change of frame corresponds to a different Poincaré gauge condition applied on the modulus of the scalar compensator z_0, which fixes dilatation symmetry. Up to terms with more than two derivatives and up to terms which would contribute to kinetic terms in a fermionic background only [11, 12], the most general supergravity Lagrangian reads[1]

$$\mathcal{L} = \left[S_0 \overline{S}_0 \Phi \right]_D + \left[S_0^3 W \right]_F + \frac{1}{4} \left[f_{ab} \mathcal{W}^a \mathcal{W}^b \right]_F .$$
(2.1)

The symbols $[\ldots]_D$ and $[\ldots]_F$ denote the invariant D- and F-density formulas given by (all fermion contributions are omitted)

$$[\mathcal{V}]_D = e(d + \frac{1}{3}cR) \ \text{ and } \ [\mathcal{S}]_F = e(f + \overline{f}), \ (2.2)$$

where $\mathcal{V}$ is a vector multiplet with components $(c, \chi, m, n, b_\mu, \lambda, d)$ and $\mathcal{S}$ a chiral multiplet with components (z, ψ, f). The real vector multiplet

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[1]Except otherwise mentioned, our notations for superconformal expressions are as in refs. [13], from where the original literature can also be traced back. The appendix of ref. [9] displays the conventions we follow through this note.

Φ (zero weights) is a function (in the sense of tensor calculus) of the multiplets present in the theory, including in general the compensating multiplet. The holomorphic function W of the chiral multiplets is the superpotential. The chiral multiplet $\mathcal{W}$ is the gauge field strength for the gauge multiplets and f_{ab} is the holomorphic gauge kinetic function of the chiral multiplets. Besides S_0 and $\mathcal{W}$, we will use chiral multiplets with zero weights and neither W nor f_{ab} will depend on the compensator.

Using a $U(1)/$Kähler gauge fixing the supergravity Lagrangian (2.1) can also take the form

$$\mathcal{L} = \left[S_0 \overline{S}_0 \Phi \right]_D + c \left[S_0^3 \right]_F + \frac{1}{4} \left[f_{ab} \mathcal{W}^a \mathcal{W}^b \right]_F ,$$
(2.3)

with an arbitrary constant c as superpotential and two arbitrary functions Φ and f_{ab}.

2.2 Supermultiplets with constraints

The Lagrangian of eleven-dimensional supergravity can be written as [3]

$$e^{-1} \mathcal{L}_{\text{CJS}} = -\frac{1}{2\kappa_{11}^2} R$$

$$-\frac{1}{4\kappa_{11}^2} \frac{1}{4!} G_{M_1 M_2 M_3 M_4} G^{M_1 M_2 M_3 M_4}$$

$$-\frac{1}{12\kappa_{11}^2} \frac{1}{4!4!3!} e^{-1} \epsilon^{M_1 \ldots M_{11}} G_{M_1 M_2 M_3 M_4}$$

$$\times G_{M_5 M_6 M_7 M_8} C_{M_9 M_{10} M_{11}} + \text{ fermionic terms.}$$
(2.4)

Omitting all fields related to the detailed geometry of the CY manifold, the particle content of the four-dimensional theory is the $N = 1$ supergravity multiplet, with metric tensor $g_{\mu\nu}$, and matter multiplets including on-shell four bosons and four fermions. Two bosons are scalars and correspond to the dilaton and the 'universal modulus' of the CY space, the massless volume mode. Two bosons are KK modes of the field strength G, with Bianchi identity $dG = 0$. Explicitly, these two last fields and their Bianchi identities read[2]

$$G_{\mu\nu\rho 4}, \qquad\qquad \partial_{[\mu} G_{\nu\rho\sigma 4]} = 0,$$
$$G_{\mu j \overline{k} 4} = i\, T_\mu\, \delta_{j\overline{k}}, \qquad \partial_{[\mu} T_{\nu]} = 0.$$
(2.5)

It will prove useful to identify these fields with the vector components of two real vector multiplets V ($w = 2$, $n = 0$) and V_T ($w = n = 0$), and

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[2]In our notations, x^4 is the orbifold coordinate.

to impose the Bianchi identities as field equations using a chiral multiplet S ($w = n = 0$) and a real linear multiplet L_T ($w = 2$, $n = 0$) as Lagrange multipliers. The bulk supergravity Lagrangian takes then the form

$$\mathcal{L}_{\mathrm{B}} = \left[-(S_0 \overline{S}_0 V_T)^{3/2} (2V)^{-1/2} \right.$$
$$\left. -(S + \overline{S})V + L_T V_T \right]_D . \quad (2.6)$$

The various superconformal multiplets appearing in this Lagrangian have the following components expressions[3]

$$V = (C, 0, H, K, v_\mu, 0, d - \Box C - \tfrac{1}{3}CR),$$

$$V_T = (C_T, 0, H_T, K_T, T_\mu, 0, d_T - \Box C_T),$$

$$S = (s, 0, -f, if, i\partial_\mu s, 0, 0),$$

$$L_T = (\ell_T, 0, 0, 0, t_\mu, 0, -\Box \ell_T - \tfrac{1}{3}\ell_T R),$$

$$S_0 = (z_0, 0, -f_0, if_0, iD^c_\mu z_0, 0, 0). \quad (2.7)$$

The role of the Lagrange multipliers S and L_T follows from the two relations

$$e^{-1}[(S + \overline{S})V]_D = -2\,\mathrm{Im}\,s\,\partial^\mu v_\mu + 2d\,\mathrm{Re}\,s$$

$$-f(H - iK) - \overline{f}(H + iK) + \text{derivative},$$

$$e^{-1}[L_T V_T]_D = \ell_T(d_T - \Box C_T)$$

$$-\tfrac{e}{2}\epsilon_{\mu\nu\rho\sigma}(\partial^\mu T^\nu)\,t^{\rho\sigma} + \text{derivative}. \quad (2.8)$$

In the last equality, we have used the constraint imposed to the linear multiplet L_T, $\partial^\mu t_\mu = 0$, to write $t_\mu = \tfrac{e}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu t^{\rho\sigma}$. Solving for the components of S leads to $\partial^\mu v_\mu = d = H = K = 0$, and V is a linear multiplet L ($w = 2$, $n = 0$). Solving for the components of L_T leads to $d_T - \Box C_T = \partial_{[\mu} T_{\nu]} = 0$, and V_T can be written as $T + \overline{T}$, with a chiral weightless multiplet T[4]. Since one can always write $v_\mu = \tfrac{e}{6}\epsilon_{\mu\nu\rho\sigma}v^{\nu\rho\sigma}$, we have generated with $\mathrm{Im}\,s$ and $t_{\mu\nu}$ the Bianchi identities $\partial_{[\mu} v_{\nu\rho\sigma]} = \partial_{[\mu} T_{\nu]} = 0$. A modification of these

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[3]We only explicitly consider the bosonic sector of the theory and omit all fermions in the $N = 1$ supermultiplets. We gauge-fix the superconformal symmetries not contained in $N = 1$ Poincaré supersymmetry, except dilatation symmetry. Notice also that our component expansion of vector multiplets differs in its highest component from refs. [13].

[4]With components: $C_T = 2\,\mathrm{Re}\,T$, $T_\mu = -2\partial_\mu\,\mathrm{Im}\,T$, $H_T = -2\,\mathrm{Re}\,f_T$, $K_T = -2\,\mathrm{Im}\,f_T$.

Bianchi identities, as induced by $S^1/\mathbf{Z}_2$ compactification or by five-brane couplings will then be phrased as a modification of the supermultiplets appearing multiplied by $S + \overline{S}$ or L_T in Eqs. (2.8).

The structure of the Lagrangian (2.6) reflects the familiar duality relating scalars and antisymmetric tensors or, for superfields, chiral and linear multiplets.

Solving in Eq. (2.6) for the Lagrange multipliers S and L_T leads to the 'standard form' of the bulk four-dimensional Lagrangian [14, 15]

$$\mathcal{L}_{\mathrm{B,l}} = -\left[\left(S_0 \overline{S}_0\, e^{-\hat{K}/3} \right)^{3/2} (2L)^{-1/2} \right]_D , \quad (2.9)$$

with the Kähler potential $\hat{K} = -3\log(T + \overline{T})$ for the volume modulus T. We will see again below that this standard form is naturally obtained by direct reduction of the Cremmer, Julia and Scherk version of eleven-dimensional supergravity on O_7. Clearly, theory (2.9) is also the CY truncation of ten-dimensional $N = 1$ pure supergravity [14].

Solving for V and L_T in Eq. (2.6) leads to the familiar chiral form [16]

$$\mathcal{L}_{\mathrm{B,c}} = -\frac{3}{2}\left[S_0 \overline{S}_0\, e^{-K/3} \right]_D , \quad (2.10)$$

with $K = -\log(S + \overline{S}) + \hat{K}$.

2.2.1 Choice of Poincaré frame

According to the component expression for the D-density and the tensor calculus of superconformal multiplets [13], the Einstein term included in the bulk Lagrangian (2.6) is [17, 15]

$$\mathcal{L}_{\mathrm{E}} = -\frac{1}{2}eR\left[(z_0 \overline{z}_0 C_T)^{3/2} (2C)^{-1/2} \right]. \quad (2.11)$$

As they should, the terms introduced to impose Bianchi identities do not contribute. We then select the Einstein frame, in which the gravitational Lagrangian is $-\frac{1}{2\kappa^2}eR$, by the dilatation gauge condition

$$\kappa^{-2} = (z_0 \overline{z}_0 C_T)^{3/2} (2C)^{-1/2}. \quad (2.12)$$

It will be convenient to introduce the (composite) real vector multiplet

$$\Upsilon = (S_0 \overline{S}_0 V_T)^{3/2} (2V)^{-1/2}, \quad (2.13)$$

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with conformal weight two. In the Poincaré theory and in the Einstein frame, its lowest component is equal to κ^{-2}.

2.2.2 Identification of the components

Choosing the Einstein frame, $\Upsilon = \kappa^{-2}$, and solving for the components of S and L_T, the complete bosonic expansion of the four-dimensional supergravity (2.6) is

$$e^{-1}\mathcal{L}_{\mathrm{B}} = -\frac{1}{2\kappa^2}R$$
$$-\frac{1}{4\kappa^2}C^{-2}[(\partial_\mu C)(\partial^\mu C) - v_\mu v^\mu] \qquad (2.14)$$
$$-\frac{3}{4\kappa^2}C_T^{-2}[(\partial_\mu C_T)(\partial^\mu C_T) + T_\mu T^\mu],$$

with $v_\mu = \frac{e}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu b^{\rho\sigma}$ since V is a linear multiplet, $C_T = 2\,\mathrm{Re}\,T$ and $T_\mu = -2\partial_\mu\,\mathrm{Im}\,T$ since $V_T = T + \overline{T}$.

This Lagrangian is to be compared with the one we obtain from the reduction of eleven-dimensional supergravity (2.4). The $\mathbf{Z}_2$ orbifold projection eliminates all states which are odd under $x^4 \to -x^4$, and the reduction of the eleven-dimensional space-time metric is

$$g_{MN} = \begin{pmatrix} e^{-\gamma}e^{-2\sigma}g_{\mu\nu} & 0 & 0 \\ 0 & e^{2\gamma}e^{-2\sigma} & 0 \\ 0 & 0 & e^{\sigma}\delta_{i\bar{j}} \end{pmatrix}. \qquad (2.15)$$

The surviving components of the field strength G_{MNPQ} are only $G_{\mu\nu\rho4}$ and $G_{\mu i\bar{j}4}$, with

$$G_{\mu\nu\rho4} = 3\partial_{[\mu}C_{\nu\rho]4}, \quad G_{\mu i\bar{j}4} = \partial_\mu C_{i\bar{j}4},$$
$$C_{i\bar{j}4} = ia(x)\,\delta_{i\bar{j}}. \qquad (2.16)$$

The resulting four-dimensional Lagrangian is

$$e^{-1}\mathcal{L}_{\mathrm{CJS}} = -\frac{1}{2\kappa^2}R$$
$$-\frac{1}{4\kappa^2}\left[9(\partial_\mu\sigma)(\partial^\mu\sigma) + \frac{1}{6}e^{6\sigma}G_{\mu\nu\rho4}G^{\mu\nu\rho4}\right]$$
$$-\frac{3}{4\kappa^2}\left[(\partial_\mu\gamma)(\partial^\mu\gamma) + e^{-2\gamma}(\partial_\mu a)(\partial^\mu a)\right]. \qquad (2.17)$$

In this expression, κ is the four-dimensional gravitational coupling with $\kappa^2 = \kappa_{11}^2/V_7$, $V_7 = V_1 V_6$ being the volume of the compact space $S^1 \times X_6$.

At this stage, the identification of the bosonic components C, $b_{\mu\nu}$, C_T and T_μ with the bulk fields σ, $C_{\mu\nu4}$, γ and a can only be determined up to two proportionality constants (one for each 'M-theory multiplet' V and V_T). These constants

can however be determined from the couplings of C and C_T to charged matter and gauge fields [9]. The result is

$$4\kappa^2 C = \frac{\lambda^2}{V_6}e^{-3\sigma}, \qquad 4\kappa^2 b_{\mu\nu} = \frac{\lambda^2}{V_6}C_{\mu\nu4},$$
$$C_T = 2\frac{\lambda^2}{V_6}e^{\gamma}, \qquad T_\mu = -2\frac{\lambda^2}{V_6}\partial_\mu a. \qquad (2.18)$$

The quantity λ is the gauge coupling constant on the $\mathbf{Z}_2$ fixed planes. The dimensionless number λ^2/V_6 actually never appears in the four-dimensional effective theory.

2.2.3 Addition of a superpotential

The standard reduction of eleven-dimensional supergravity with unbroken $N = 1$ supersymmetry does not generate a superpotential. This fact is however not a direct consequence of the eleven-dimensional Bianchi identity or of the CY and $S^1/\mathbf{Z}_2$ symmetries. In principle, the Bianchi identity $\partial_{[M}G_{NPQR]} = 0$ allows a solution

$$G_{ijk4} = 2i\kappa^{-1}h\epsilon_{ijk}, \quad G_{\overline{ijk}4} = -2i\kappa^{-1}h\epsilon_{\overline{ijk}}, \qquad (2.19)$$

where h is a real constant and ϵ_{ijk} is the $SU(3)$-invariant CY tensor. The second term in the Lagrangian (2.4) generates then an extra contribution in the effective supergravity which corresponds to the addition of a superpotential term $[ihS_0^3]_F$ to the bulk Lagrangian. This contribution however breaks supersymmetry [18]. Since we have insisted in writing Lagrangians in which all Bianchi identities are field equations, we prefer to consider

$$[U(W + \overline{W})]_D + [S_0^3\,W]_F. \qquad (2.20)$$

In this way, the fact that the chiral multiplet W ($w = n = 0$) is an arbitrary imaginary constant is imposed by the field equation of the vector multiplet U ($w = 2$, $n = 0$).

With the addition of a superpotential, the bulk Lagrangian takes its final 'off-shell' form

$$\mathcal{L}_{\mathrm{B}} = \left[-\Upsilon - (S + \overline{S})V + L_T V_T\right]_D$$
$$+ \left[U(W + \overline{W})\right]_D + [S_0^3 W]_F, \qquad (2.21)$$

in which the Bianchi identities of eleven-dimensional supergravity are translated into field equations of the Lagrange multipliers S, L_T and U.

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2.3 Modified Bianchi identities and κ-expansion

Compactification of M-theory on $S^1/\mathbf{Z}_2$ is usually discussed in an expansion in powers of κ_{11}. Compactification on O_7 can similarly be formulated with κ as expansion parameter. In the upstairs version, Bianchi identities are modified at the ten-dimensional planes fixed by $S^1/\mathbf{Z}_2$. Suppose now that we modify the four-dimensional supersymmetric Bianchi identities of the bulk Lagrangian in the following way (we set $h = 0$):

$$\mathcal{L}_{\mathrm{B}} \quad \longrightarrow$$
$$\left[-\Upsilon - (S + \overline{S})(V + \Delta_V) + L_T(V_T + \Delta_T)\right]_D , \tag{2.22}$$

with two composite vector multiplets Δ_V ($w = 2$, $n = 0$) and Δ_T ($w = n = 0$). Solving for the Lagrange multipliers now leads to

$$V = L - \Delta_V, \quad V_T = T + \overline{T} - \Delta_T.$$

The Lagrangian to first order in these modifications is then

$$\mathcal{L} = \mathcal{L}_{\mathrm{B}} - \left[\frac{\Upsilon}{2V}\Delta_V - \frac{3}{2V_T}(\Upsilon\Delta_T)\right]_D , \tag{2.23}$$

with V and V_T respectively replaced by L and $T + \overline{T}$. The multiplets Δ_V and $\Upsilon\Delta_T$, with 'canonical' dimension $w = 2$, appear at order $\Upsilon^0 \sim \kappa^0$, in comparison with bulk terms of order $\Upsilon \sim \kappa^{-2}$. This is the relation with the expansion in powers of κ_{11} of M-theory in the low-energy limit. In M-theory compactification, the multiplets Δ_V and Δ_T can thus be obtained either by considering the modified Bianchi identities on O_7, formulated as in Eq. (2.22), or from corrections to the Lagrangian of eleven-dimensional supergravity on O_7, as in expression (2.23).

3. Gauge and matter contributions from the two $\mathbf{Z}_2$ fixed planes

In this section, we show that the introduction of the next to lowest order corrections (gauge multiplets and charged matter contributions) is controlled by a simple modification of the four-dimensional Bianchi identities, in analogy with the appearance of $\mathbf{Z}_2$ fixed planes contributions in the M-theory Bianchi identities.

We start by considering the well-known dependence on charged matter (in chiral multiplets collectively denoted by M, with $w = n = 0$) and gauge multiplets (vector multiplet A, in the adjoint representation, with $w = n = 0$) of the effective $N = 1$ four-dimensional supergravity for CY compactifications of heterotic strings [16, 19, 20]. The Lagrangian in the chiral formulation (2.10) becomes

$$\mathcal{L}_{\mathrm{c}} = -\tfrac{3}{2}\left[S_0\overline{S}_0 e^{-K/3}\right]_D + \left[S_0^3 W\right]_F$$
$$+ \tfrac{1}{4}\left[S\mathcal{WW}\right]_F , \tag{3.1}$$

with

$$K = -\log(S + \overline{S}) - 3\log(T + \overline{T} - 2\overline{M}e^A M) \tag{3.2}$$

and $W = \alpha M^3$. The superpotential should be understood as a gauge invariant trilinear interaction with coupling constant α defined as an integral over the CY space. The chiral multiplet $\mathcal{W}$ ($w = n = 3/2$) is the gauge field-strength for A. The gauge group is in general not simple, and

$$\mathcal{WW} = \sum_a c^a \mathcal{W}^a \mathcal{W}^a, \tag{3.3}$$

with a real coefficient c^a for each simple or abelian factor. In the linear equivalent version of the theory, the Lagrangian (2.9) reads now [14, 15]

$$\mathcal{L}_{\mathrm{l}} = -\left[(S_0\overline{S}_0 e^{-\hat{K}/3})^{3/2}(2\hat{L})^{-1/2}\right]_D$$
$$+ [\alpha S_0^3 M^3]_F , \tag{3.4}$$

where the new modulus and matter Kähler potential is

$$\hat{K} = -3\log(T + \overline{T} - 2\overline{M}e^A M). \tag{3.5}$$

The linear multiplet L is replaced by

$$\hat{L} = L - 2\Omega, \tag{3.6}$$

with the Chern-Simons vector multiplet Ω ($w = 2$, $n = 0$) defined by[5]

$$\Omega = \sum_a c^a \Omega^a, \quad \Sigma(\Omega^a) = \frac{1}{16}\mathcal{W}^a \mathcal{W}^a. \tag{3.7}$$

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[5]In global Poincaré supersymmetry, $\Sigma(\Omega) = -\tfrac{1}{4}\overline{DD}\Omega$.

Insisting as before on Bianchi identities, both forms (3.1) and (3.4) are equivalent to

$$\mathcal{L} = \big[-\Upsilon - (S + \overline{S})(V + 2\Omega)$$

$$+ L_T(V_T + 2\overline{M}e^A M)\big]_D$$

$$+ \big[U(W - \alpha M^3) + \text{c.c.}\big]_D + \big[S_0^3 W\big]_F$$

$$= \big[-\Upsilon - (S + \overline{S})(V + 2\Omega)$$

$$+ L_T(V_T + 2\overline{M}e^A M)\big]_D + \big[S_0^3(ih + \alpha M^3)\big]_F .$$
(3.8)

Supersymmetric vacua have $h = 0$. As before, solving for S and L_T imposes respectively $V = L - 2\Omega = \hat{L}$ and $V_T = T + \overline{T} - 2\overline{M}e^A M$, leading to Eq. (3.4). Alternatively, with the tensor calculus identity (and up to an irrelevant total derivative)

$$-2[(S + \overline{S})\Omega]_D = \frac{1}{4}\sum_a c^a [S\mathcal{W}^a\mathcal{W}^a]_F, \quad (3.9)$$

the resolution for V and L_T leads back to the chiral form (3.1).

This reformulation of the gauge invariant Lagrangian suggests some remarks. Firstly, it enhances the importance of gauge *and* matter Chern-Simons multiplets in superstring effective actions. Secondly, the Chern-Simons vector multiplet $\Omega(A)$ is not gauge invariant: its variation is a linear multiplet. The variation of $[(S + \overline{S})2\Omega]_D$ is then a derivative and V remains gauge invariant. When solving for S, it simply follows that $\hat{L}$ is gauge invariant and that the linear multiplet transforms as $\delta L = 2\delta\Omega$. Finally, expression (3.8) shows that all gauge and chiral matter contributions can be viewed as the supersymmetrization of modified Bianchi identities imposed by S, L_T and U. This observation provides the link to the approach based on M-theory on O_7, in which the $\mathbf{Z}_2$ fixed planes carrying the Yang-Mills fields induce because of supersymmetry modifications to the Bianchi identity of the four-form field strength of eleven-dimensional supergravity.

In the effective supergravity of M-theory on O_7 ('upstairs formulation'), the various components of the Lagrangian (3.8) have the following origin. The first term is the bulk supergravity contribution. The second term, $[(S + \overline{S})(V + 2\Omega)]_D$, is the supersymmetrization of the Bianchi identity verified by the component $G_{\mu\nu\rho 4}$

of the field G, modified by gauge contributions on the fixed planes. Similarly, the two last terms, $[L_T(V_T + 2\overline{M}e^A M)]_D$ and $[U(W - \alpha M^3) + \text{c.c.}]_D$, are respectively the supersymmetric extensions of the Bianchi identities of $G_{\mu j\overline{k}4}$ and G_{ijk4}. All the fixed plane contributions are then given at this order by the supersymmetrization of Bianchi identities, as obtained by direct O_7 truncation of the eleven-dimensional identities [1, 2].

At this point, the gauge coupling constant for each simple or abelian factor a in the gauge group appears to be

$$\frac{1}{g_a^2} = c^a \operatorname{Re} s. \qquad (3.10)$$

At this order, g_a is the tree-level wilsonnian and physical[6] gauge coupling.

It is clear, as already observed [5]–[7], that as far as the structure of the four-dimensional effective supergravity is concerned, the same information follows from O_7 compactification of M-theory at the next to lowest order in the κ-expansion and from CY compactifications of the heterotic strings, at zero string loop order.

4. Anomaly-cancelling terms

In the ten-dimensional heterotic string, cancellation of gauge and gravitational anomalies is a one-loop effect in string or effective supergravity perturbation theory. In four space-time dimensions, the nature of the cancelled anomalies is known from studies of $(2, 2)$ compactifications of heterotic strings in the Yang-Mills sector [21, 12, 22]: target-space duality of the modulus T has a one-loop anomaly which is cancelled by a counterterm in the one-loop Wilson Lagrangian $\mathcal{L}_W^{(1)}$[7], in a generalization to sigma-model anomalies of the Green-Schwarz mechanism [23]. The derivation of the complete counterterm requires a calculation to all orders in the modulus T [21]. However, at the present stage of understanding, the M-theory approach should be regarded as a large-T limit in which T-duality reduces to a shift symmetry in the imaginary part of T.

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[6]The coefficient of $-\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu}$ in the generating functional of one-particle irreducible Green's functions.

[7]The expressions given in the previous sections were for $\mathcal{L}_W^{(0)}$, or for the tree-level standard effective Lagrangian $\mathcal{L}_\Gamma$.

In the large-T limit, the T-dependent corrections to gauge kinetic terms are of the form (see ref. [9] and citations therein)

$$\frac{1}{4}\sum_a \beta^a \left[T\mathcal{W}^a\mathcal{W}^a\right]_F,\qquad (4.1)$$

where the coefficients β^a are in principle calculable in heterotic strings. Taking also into account the D-density $\left[L_T(V_T + 2\overline{M}e^A M)\right]_D$ present in Lagrangian (3.8), we can rewrite expression (4.1) in terms of the 'M-theory multiplets':

$$\left[(L_T - 2\sum_a \beta^a\Omega^a)(V_T + 2\overline{M}e^A M)\right]_D.\quad (4.2)$$

The correction (4.1) to the SYM Lagrangian is independent of the matter fields and can be seen as a correction to the holomorphic gauge kinetic function f_{ab}. A possible matter-dependent contribution to gauge kinetic terms is the gauge invariant real density[8]:

$$-2\delta\left[\overline{M}e^A M(L - 2\sum_{a=1}^2 \Omega^a)\right]_D,\qquad (4.3)$$

or

$$-2\delta\left[\overline{M}e^A M\, V\right]_D,\qquad (4.4)$$

using the 'M-theory multiplet' V.

The M-theory anomaly-cancelling terms generate a further contribution of the form [9]

$$\epsilon\left[V|\alpha M^3|^2\right]_D.\qquad (4.5)$$

In summary, the Wilson Lagrangian up to string one-loop order is expected to become

$$\begin{aligned}
\mathcal{L} = &\left[-\Upsilon - (S+\overline{S})(V+2\Omega)\right.\\
&\left.+ (L_T - 2\sum_{a=1}^2 \beta^a\Omega^a)(V_T + 2\overline{M}e^A M)\right]_D\\
&+ \left[U(W - \alpha M^3) + \text{c.c.}\right]_D + \left[S_0^3 W\right]_F\\
&+ \left[V(\epsilon|\alpha M^3|^2 - 2\delta\overline{M}e^A M)\right]_D.
\end{aligned}$$
$$(4.6)$$

Each of the one-loop corrections, with coefficients β^1, β^2, ϵ and δ is related to a well-defined counterterm which can be easily identified in the KK

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[8]For simplicity, we consider the standard embedding with a gauge group $E_6 \times E_8$, with the notation $\Omega = \Omega^1 + \Omega^2$, and with a matter multiplet M transforming as $(\mathbf{27},\mathbf{1})$ of $E_6 \times E_8$.

reduction of the ten-dimensional Green-Schwarz counterterms arising from M-theory on $S^1/\mathbf{Z}_2$ [1, 2, 24]. An explicit computation predicts in particular the relations $\beta^1 = -\beta^2 = \delta$ [9].

From the general expression (4.6), we can derive various equivalent forms. For instance, solving for S, L_T and U gives the version of the effective supergravity in which the dilaton is described by a linear multiplet:

$$\mathcal{L}_l =$$

$$\left[-(S_0\overline{S}_0)^{3/2}\left(T + \overline{T} - 2\overline{M}e^A M\right)^{3/2}(2\hat{L})^{-1/2}\right]_D$$

$$+ \left[S_0^3(ih + \alpha M^3)\right]_F + \frac{1}{4}\left[T\sum_{a=1}^2 \beta^a\mathcal{W}^a\mathcal{W}^a\right]_F$$

$$+ \left[\hat{L}\left(\epsilon|\alpha M^3|^2 - 2\delta\overline{M}e^A M\right)\right]_D.$$
$$(4.7)$$

The threshold corrections are the holomorphic T-dependent terms controlled by β^1 and β^2.

We can also solve for V, L_T and U in Eq. (4.6) to get the version with a chiral dilaton multiplet:

$$\mathcal{L}_c = -\frac{3}{2}\left[S_0\overline{S}_0\, e^{-K/3}\right]_D + \left[S_0^3(ih + \alpha M^3)\right]_F$$

$$+ \frac{1}{4}\left[\sum_{a=1}^2 (S + \beta^a T)\mathcal{W}^a\mathcal{W}^a\right]_F,$$
$$(4.8)$$

with the Kähler potential

$$\begin{aligned}
K = &-\log\left(S + \overline{S} + 2\delta\overline{M}e^A M - \epsilon|\alpha M^3|^2\right)\\
&-3\log\left(T + \overline{T} - 2\overline{M}e^A M\right),
\end{aligned}$$
$$(4.9)$$

and the gauge kinetic functions $f^a = S + \beta^a T$. The term with coefficient δ has been obtained in direct CY reductions of M-theory on $S^1/\mathbf{Z}_2$ (see for instance [6, 7]). The charged matter contribution with coefficient ϵ was not included in these analyses. Observe however that an ambiguity exists because of the possibility to perform a holomorphic redefinition of the two chiral multiplets S and T. To remove this ambiguity, we can use information from M-theory compactification [9], or choose the unequivocal linear version.

The gauge contributions appearing in Eq. (4.6) read

$$-2\sum_{a=1}^2 \left[\left(S + \overline{S} + \beta^a(V_T + 2\overline{M}e^A M)\right)\Omega^a\right]_D,$$

so that the gauge coupling constants are given by

$$\frac{1}{g_a^2} = \text{Re}\, s + \frac{1}{2}\beta^a \left(C_T + 2\overline{M}M \right).$$ (4.10)

This expression becomes harmonic once the Bianchi identity imposing $C_T + 2\overline{M}M = 2\,\text{Re}\,T$ has been used.

Acknowledgments

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