Equivalence Between Noether Symmetries for the Standard and Enhanced Formalisms

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ABSTRACT: We provide an explicit proof that the Noether trasformations for the standard second order Lagrangian system with constraints, are equivalent to the Noether transformations that can be obtained from an associated enhanced space (to be defined below). We implement a change of variables between the n-tangent bundle with coordinates $q, \dot{q}, \ddot{q}, ...$ and an enhanced space with coordinates $q, p, \lambda, \ell, \dot{\lambda}, \dot{\ell}...$ where $\ell \equiv \dot{p} - \{p, H_D\}$ and λ are the Lagrange multipliers. In this space the Noether identities are self contained and this information can be used to construct the corresponding Noether symmetries. We rewrite the Dirac algorithm in this enhanced space. As a consequence we proof that the Noether symmetries of the form $\delta^L(q, \dot{q}, \ddot{q}...)$ are the most general Noether transformations that can be constructed for a gauge system, up to constraints, that vanish on shell. This result gives an answer to the question as to what extent and in what sense a general continuos Noether trasformation is canonically generated.

1. Introduction

¿From now on there are two different ways to construct Noether symmetries for gauge systems in the market [1, 3]. One of them [3, 4], based on Dirac algorithm, requires a canonical generator of the form $G_c(q, p, \lambda, \dot{\lambda}...t)$, to generate through the Poisson bracket, a symmetry transformation in an enlarged space (defined by the coordinates $(q, p, \lambda, \dot{\lambda}...)$) where λ is the set of Lagrange multipliers associated with the primary first class constraints. The symplectic structure is the same as the original phase space (q, p) structure. The original derivation [3] of the necessary and sufficient conditions under a function G_c in the enlarged space to be a generator of a Noether symmetry, was based on the extended Dirac [2] formalism, but this *tour de force* is in fact not needed. The condition can be obtained also in the standard Dirac formalism without the need to perform a gauge fixing for the Lagrange multipliers of the secondary, terciary... etc. first class constraints [1]. These Noether symmetries "live" in the enlarged space and the original symmetries of the standard second order Lagrangian formalism with Lagrangian $L(q, \dot{q})$ can be recovered by using the fact that p and λ are auxiliary variables. The elimination of these auxiliary variables produces a Lagrangian generator which "lives" in the n-tanget bundle $G^L(q, \dot{q}, \ddot{q}...t)$ that is related to the Noether symmetry

$$\delta^L q^i(q, \dot{q}, \ddot{q}...; t), \tag{1.1}$$

by

$$\delta^L L = \frac{d^L}{dt} F,$$

for some infinitesimal function $F(q, \dot{q}, \ddot{q}..., t)$, where

$$\frac{d^L}{dt} \equiv \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i} + ...$$

is the evolution operator of shell and

$$[L]_i \delta^L q^i + \frac{d^L G_L}{dt} = 0,$$

$$[L]_i \equiv \alpha_i - W_{ij} \ddot{q}^j, \quad \alpha_i \equiv -\frac{\partial L}{\partial \dot{q}^i \partial q^j} \dot{q}^j + \frac{\partial L}{\partial q^i}.$$

Here $W_{ij} = \frac{\partial L}{\partial \dot{q}^i \partial \dot{q}^j}$ is the Hessian matrix associated to L, and $[L]_i$ are the Euler-Lagrange equations of motion. The associated conserved quantity is

$$G^L = \frac{\partial L}{\partial \dot{q}^i}\delta^L q^i - F.$$

An important advantage of this method is that the symmetry can be found by purely algebraic means.

We can also construct Noether symmetries in $TQ \times R$ by tackling the problem from the very definition of a Noether symmetry ([1] an references there in). This second approach does not require the symmetry to be canonically generated. The form of the symmetry depends on the base of constraints used to find it. In some cases the resulting symmetry is not projectable to phase space. If we require projectability, the first method is the most powerful one. When the symmetry is projectable it is also canonically generated through the Poisson bracket with generator $G(q,p)$. The two methods are equivalent when the generator has the form $G_c(q,p,\lambda)$. The original Lagrangian symmetries can be obtained by the elimination of the auxiliary variables p and λ as in the previous case. The resulting generator is a function in $TQ \times R$, $G^L(q,\dot{q},t)$. For details we refer the reader to [1].

Here we will construct an enhanced space [5] with coordinates $(q,p,\lambda,\dot{\lambda}..,\ell,\dot{\ell}...)$ where λ are the Lagrange multipliers and $\ell \equiv \dot{p} + \{p, H_D\}$. Here H_D is the Dirac Hamiltonian. We construct this space in a recursive way throughout a change of variables from the n-tangent bundle (extending the usual Legendre Map) and show that it is possible to define a "Dirac algorithm" in this new enhanced space. Then we will construct the general conditions for a function G^E in this enhanced space to be a canonical generator of a Noether symmetry and describe how we can recover the Noether symmetry for the original second order Lagrangian system from the Noether symmetry defined in the enhanced space. This allow us to proof that the most general Noether transformations that can be constructed for a given Lagrangian system with gauge invariance are of the form $\delta^L q(q,\dot{q},\ddot{q},...t)$ and answer the question about to what extend and in what sense a given Noether symmetry is canonically generated. The Noether identities are self contained in the enhanced space. As a consequence the construction of Noether symmetries is straightforward.

2. The enhanced space

To construct the enhanced space in a recurrent way we start from the standard Legendre Map [5]

$$TQ \longleftrightarrow M_0 \times R^M, \qquad \text{(locally)}$$
$$q^i, \dot{q}^i \longleftrightarrow q^i, p_i, \lambda^\mu, \quad \phi^{(0)}_\mu = 0$$

where $\phi^{(0)}_\mu$ are the primary constraints and λ^μ its associated Lagrange multipliers. The explicit transformation can be constructed by noticing that the canonical momenta p and the Lagrange multipliers λ^μ are auxiliary variables. Indeed, from left to right

$$\dot{q}^i = \frac{\partial H_c}{\partial p_i} + \lambda^\mu \frac{\partial \phi_\mu}{\partial p_i} \equiv \frac{\partial H_D}{\partial p_i}, \qquad \phi^{(0)}_\mu(q,p) = 0,$$

where $H_D = H_c + \lambda^\mu \phi^{(0)}_\mu$ is the Dirac Hamiltonian. From these equations we can obtain p and λ as functions of $(q,\dot{q})$

$$\hat{p}_i = \frac{\partial L}{\partial \dot{q}^i}, \qquad \lambda^\mu = v^\mu(q,\dot{q}).$$

which are the explicit trasformations from right to left. These are in fact the transformation rules that we apply to recover the original Lagrangian function $L(q,\dot{q})$ from the Hamiltonian one defined by the Legendre Map. This procedure can be generalized to the next order

$$q^i, \dot{q}^i, \ddot{q}^i \longleftrightarrow q^i, p_i, \lambda^\mu, \ell_i, \dot{\lambda}^\mu,$$
$$\phi^{(0)}_\mu(q,p) = 0, \quad \phi^{(1)}_\mu(q,p,\lambda,\ell) = 0,$$

where

$$\phi^{(1)}_\mu(q,p,\lambda,\dot{p}) = \{\phi^{(0)}_\mu, H_D\} + \ell_i \frac{\partial \phi^{(0)}_\mu}{\partial p_i},$$

with the definition

$$\ell_i \equiv \dot{p}_i + \frac{\partial H_D}{\partial q^i}.$$

Note that $\ell_i = -[L]_i$ in the old variables, where $[L]_i$ denotes the Euler-Lagrange equations of motion. The explicit transformatios are, from left to right

$$\ddot{q}^i = \{\frac{\partial H_D}{\partial p_i}, H_D\} + \ell_j \frac{\partial H_D}{\partial p_i \partial p_j} + \dot{\lambda}^\mu \frac{\partial H_D}{\partial p_i \partial \lambda^\mu},$$

and from right to left

$$\ell_i = -[L]_i = \dot{\hat{p}}_i - \frac{\partial L}{\partial q^i}, \quad \dot{\lambda}^\mu = \dot{q}^i \frac{\partial v^\mu}{\partial q^i} + \ddot{q}^i \frac{\partial v^\mu}{\partial \dot{q}^i}.$$

We can construct in a recursive way all the enhanced phase space and interchange variables from the n-tangent bundle and this new enhanced space just constructed. In the same way as $\phi_\mu^{(0)} = 0$ restrains the number of independent momenta, the new variables $p, \ell, \dot{\ell}$... are not all independent but they are restricted by $\phi_\mu^{(0)}, \phi_\mu^{(1)}, \phi_\mu^{(2)}$.... For that reason we will call the functions $\phi_\mu^{(n)}$ *restrictions* and not constraints because they are not in the form of standard Dirac constraints. Note that the restrictions are identically zero when they are evaluated in the old coordinates (as the primary constraints in the standard Dirac analysis).

In general these restrictions are generated by

$$\phi_\mu^{(n+1)} = \frac{d^E \phi_\mu^{(n)}}{dt}$$

where

$$\frac{d^E}{dt} = \mathcal{D} + \ell \frac{\partial}{\partial p}, \tag{2.1}$$

is the evolution operator of shell in the new enhanced space, and

$$\mathcal{D} = \frac{D}{Dt} + \{-, H_D\}, \tag{2.2}$$

$$\frac{D}{Dt} := \frac{\partial}{\partial t} + \dot{\lambda}\frac{\partial}{\partial \lambda} + \ddot{\lambda}\frac{\partial}{\partial \dot{\lambda}} + ... + \dot{\ell}\frac{\partial}{\partial \ell} + \ddot{\ell}\frac{\partial}{\partial \dot{\ell}} + ..., \tag{2.3}$$

It is worth noticing that if we set $\ell, \dot{\ell}, \ddot{\ell}$ to zero we obtain the relations

$$\mathcal{D}^n \phi_\mu^{(0)}(q, p, \lambda, \dot{\lambda}...) = 0. \tag{2.4}$$

These are not the standard Dirac constraints, but coincide with the Lagrangian constraints when $p = \hat{p}(q, \dot{q})$ and $\lambda = v(q, \dot{q})$ are used. In fact for $n = 0$ we have the identity $\phi^{(0)}(q, \hat{p}) = 0$, and for $n = 1$

$$\mathcal{D}\phi_\mu^{(0)}(q, \hat{p}, v^\mu) = [L]_i \frac{\partial \phi_\mu}{\partial p_i} = \alpha_i \frac{\partial \phi_\mu}{\partial p_i}\Big|_{p=\hat{p}}.$$

3. Relation to Dirac constraint analysis

The Lagrangian constraints $\mathcal{D}^n \phi_\mu^{(0)} = 0$ are not in the form of Dirac constraints. Dirac algorithm is quite more refined, at least, for two reasons: a) The Dirac constraints are of the form $\phi(q, p) = 0$ and b) the determination of some Lagrange multipliers λ in terms of q, p variables (for the associated second class constraints) and the construction of the Dirac Bracket are basic steps in the Dirac algorithm.

The Dirac algorithm starts from primary constraints ϕ_μ and the cleaver trick is to split these constraints into first class ϕ_{μ_0} and second class ϕ_{μ_1}. Then

$$\mathcal{D}\phi_{\mu_0} = \{\phi_{\mu_0}, H_c\} + \lambda^\nu \{\phi_{\mu_0}, \phi_\nu\} = \{\phi_{\mu_0}, H_c\},$$

and

$$\mathcal{D}\phi_{\mu_1} = \{\phi_{\mu_1}, H_c\} + \lambda^\nu \{\phi_{\mu_1}, \phi_\nu\} = 0 \to \lambda^{\mu_1}(q, p),$$

so the operator $\mathcal{D}$ is now modified (adapted) to

$$\mathcal{D}' = \frac{\mathcal{D}'}{\mathcal{D}t} + \{-, H'_D\},$$

where

$$H'_D = H'_c + \lambda^{\mu_0}\phi_{\mu_0} \quad \text{and} \quad H'_c = H_c + \lambda^{\mu_1}\phi_{\mu_1},$$

λ^{μ_0} are the remaining undetermined Lagrange multipliers. In this way we recover the Dirac algorithm from the previous one in the enhanced space.

4. Noetherian symmetries in the enhanced space

Here we will find the general conditions under a function G^E in the enhanced space to be a canonical generator of a Noether symmetry for a first order Lagrangian

$$L_c(q, p, \lambda; \dot{q}, \dot{p}, \dot{\lambda}) := p_i \dot{q}^i - H_c(q, p) - \lambda^\mu \phi_\mu(q, p),$$

where ϕ_μ are the primary constraints. The new configuration space for L_c is the old phase space enlarged with the Lagrange multipliers λ^μ as new independent variables. The dynamics given by L_c is nothing but the constrained Dirac Hamiltonian dynamics for a system with canonical Hamiltonian H_c and a number of primary constraints ϕ_μ.

We will now look for Noether transformations for L_c that may depend on the Lagrange multipliers, the ℓ variables and their time derivatives to any finite order and that are canonically generated under the definitions

$$\delta^E q^i = \{q^i,\, G^E\}, \quad \delta^E p_i = \{p_i,\, G^E\}, \qquad (4.1)$$

and with $\delta^E \lambda^\mu$ to be determined below. We have for $\delta^E L_c$,

$$\delta^E L_c = \frac{dM}{dt} + \frac{DG^E}{Dt} + \{G^E, H_D\} - (\delta^E \lambda^\mu)\phi_\mu,$$

where $M = (p_i\, \delta^E q^i - G^E)$ and with the definition

$$\frac{d}{dt} = \frac{D}{Dt} + \dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}_i \frac{\partial}{\partial p_i}$$

where D/Dt is the operator defined in (2.3). If we require

$$\frac{DG^E}{Dt} + \{G^E, H_D\} = pc, \qquad (4.2)$$

where pc stands for a combination of first class constraints, then $\delta^E L_c = \frac{dM}{dt}$, that is, a Noether transformation for the enhanced space. Equation (4.2) is important: it is the equation characterizing the generators of Noether transformations in the enhanced space that are canonically generated. It applies both to rigid and gauge Noether symmetries. It generalizes in a compact way the theoretical setting of the results given in [3] (see also [4]) to find an algorithm to produce gauge generators for theories with only first class constraints. We have arrived at the following result: *The necessary and sufficient condition for a function $G^E(q, p, \lambda, \dot{\lambda}, \ell, \ddot{\lambda}, \dot{\ell} ...; t)$ to generate through (4.1) a Noether symmetry in the enhanced space is that G^E must fulfill equation (4.2).*

Notice that this result has been obtained with no assumptions concerning the first and second class structure of the primary constraints. Note also that in (4.2) only the primary constraints are relevant.

It is easy to see that the condition (4.2) is quite strong. Its only possible solutions are canonical generators G^c. So we conclude that the only Noether symmetries that are canonically generated are the symmetries that can be obtained in the enlarged formalism, up to trivial symmetries. Nevetheless there are a lot of Noether transformations in the enhanced space that are not canonically generated. The condition for a symmetry $\delta^E q^i$ to be a Noether symmetry in the enhanced space is

$$-\ell_i \delta^E q^i + \frac{d^E G^E}{dt} = pc. \qquad (4.3)$$

We can find solutions to this condition that are not canonically generated. For example $G^E = \frac{1}{2}\ell^2 C(q, p, t) + G_0^E$ where $C(q, p, t)$ an arbitrary function and G_0^E a solution to

$$\frac{d^E G_0^E}{dt} = pc. \qquad (4.4)$$

This conserved quantity G^E is associated with the Noether symmetry

$$\delta^E q^i = \dot{\ell}_i C + \frac{1}{2}\ell_i(\{C, H_D\} + \frac{\partial C}{\partial t}) + \frac{1}{2}\ell_i \frac{\partial C}{\partial p_j}\ell_j$$

for any system with Dirac Hamiltonian H_D. This symmetry is not canonically generated by the given G^E. G^E is not a solution of (4.2).

The condition (4.4) can be solved for a function G_0^E proportional to the restrictions $\phi^{(n)}$. Using this solution we can back to the formalism of the enlarged space [3] by writting our results *on shell*, that is by making $\ell = 0$. In that case G^E coincides with G^c the canonical generator in the enlarged formalism. The canonical generator G^c is, by construction, a solution of the basic condition (4.2) and generates canonical Noether symmetries $\delta^c q^i$ up to trivial transformations. This is so because the condition (4.4) reduce to the condition (4.2) for a function G^c to generate a canonical Noether transformation when $\ell = 0$. ¿From the enlarged formalism the original Lagrangian formalism can be recovered by noticing that the variables p and λ are auxiliary variables. Then we can recover a Noether symmetry for the original Lagrangian $L(q, \dot{q})$ given by

$$\delta^L q(q, \dot{q}, \ddot{q}, ...; t) = \delta^c q(q, \hat{p}, v^\mu, \dot{v}^\mu ...; t),$$

$$G^L(q, \dot{q}, \ddot{q} ...; t) = G^c(q, \hat{p}, v^\mu, \dot{v}^\mu ...; t).$$

We can also retrive these symmetries from the extended symmetries and the extended generator in the enhanced space by performing the change of variables, just described, to the n-tanget bundle.

In general, the procedure of the projection to the surface $\ell = 0$ and the elimination of the auxiliary variables does not commute with the change of variables.

It is interesting to notice that the restrictions $\phi^{(n)} = 0$ contain all the information about the gauge symmetries of the system because the Dirac first class constraints can be written in terms of a linear combination of the equations of motion (the second class constraints allows to determine some Lagrange multipliers in the enhanced space). This allow us to construct the Noether identities and recover the gauge Noether symmetries of the system. We conclude that in general Noether symmetries in the enhanced space are not canonically generated.

5. Example

In this section we will consider a simple and transparent example defined by the Lagrangian,

$$L = \frac{1}{2}[(\dot{q}_4 - q_3)^2 - (e^{-q_1}\dot{q}_3)^2] + \dot{q}_3\dot{q}_2 e^{-q_1}.$$

with $p_2 \neq 0$. The associate Dirac Hamiltonian is

$$H_D = \frac{1}{2}(p_2^2 + p_4^2) + e^{q_1}p_2p_3 + q_3p_4 + \lambda p_1,$$

where λ is the Lagrange multiplier associated with the primary constraint $p_1 = 0$. In the enhanced space the restrictions are

$$\phi^{(0)} = p_1, \quad \phi^{(1)} = -e^{q_1}p_2p_3 + \ell_1,$$

$$\phi^{(2)} = e^{q_1}p_2p_4 - \lambda e^{q_1}p_2p_3 - e^{q_1}\ell_2p_3 - e^{q_1}p_2\ell_3 + \dot{\ell}_1,$$

that are identically zero in the n-tanget bundle. We can prove this assertion by using our change of variables. It is also easy to recover the Dirac constraints from these expressions by enforcing $\ell = 0$ and discard all the terms proportional to λ. This can be done in this case because all the Dirac constraints are first class. Now, the secondary Dirac constraints $e^{q_1}p_2p_3$ and $e^{q_1}p_2p_4$ can be written in terms of the equations of motion

$$\begin{aligned} e^{q_1}p_2p_3 &= \ell_1, \\ e^{q_1}p_2p_4 &= \lambda\ell_1 + e^{q_1}\ell_2p_3 + e^{q_1}p_2\ell_3 - \dot{\ell}_1. \end{aligned}$$

The evolution in the enhanced space of this last equation gives the Noether identity

$$\ddot{\ell}_1 - \dot{\lambda}\ell_1 - 2\lambda\dot{\ell}_1 + \lambda^2\ell_1 - e^{q_1}\dot{\ell}_2p_3 - 2e^{q_1}\ell_2\ell_3 -$$

$$e^{q_1}p_2\dot{\ell}_3 + 2e^{q_1}\ell_2p_4 + e^{q_1}p_2\ell_4 = 0,$$

where we have used our change of variables $\dot{p}_2 = \ell_2$, $\dot{p}_3 = \ell_3 - p_4$ and $\dot{p}_4 = \ell_4$. ¿From here we can read a Noether symmetry for the system

$$\begin{aligned} \delta^E q_1 &= \ddot{\epsilon} + \lambda^2\epsilon + \dot{\lambda}\epsilon + 2\dot{\epsilon}\lambda, \\ \delta^E q_2 &= e^{q_1}p_3(\lambda\epsilon + \dot{\epsilon}) + \epsilon e^{q_1}(p_4 - \ell_3), \\ \delta^E q_3 &= (\lambda\epsilon + \dot{\epsilon})e^{q_1}p_2 + e^{q_1}\ell_2\epsilon, \\ \delta^E q_4 &= e^{q_1}p_2\epsilon, \end{aligned} \tag{5.1}$$

where ϵ is an arbitrary parameter. Now, a solution to the condition (4.4) for G_0^E in the enhanced space is

$$G_0^E = \mu_0\phi^{(0)} + \mu_1\phi^{(1)} + \mu_2\phi^{(2)},$$

where $\mu_0 = \lambda^2\epsilon + \dot{\lambda}\epsilon + 2\lambda\dot{\epsilon} + \ddot{\epsilon}$, $\mu_1 = 2\lambda\epsilon + \dot{\epsilon}$ and $\mu_2 = \epsilon$. This function is a trivial solution of the Noether condition (4.3). Nevertheless it coincides with the canonical generator by enforcing $\ell = 0$ and generates through the Poisson bracket the correct Noether symmetries up to trivial transformations. The Lagrange Noether symmetries and the associated generator can be recovered by the elimination of p and λ as auxiliary variables.

Acknowledgments

The author is grateful to J.M. Pons for useful discussions. Also acknowledges partial support from the grants DGPA-IN100397 and CONACyT-32431-E.

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