Testing gauge invariance for a lattice with fermions

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ABSTRACT: We have included fermions on a 2-dimensional lattice mediated by electromagnetic field. The Dirac gamma matrices were introduced explicitly in the fermionic determinant and gauge invariance was tested even in extra terms from the integrals. Our result confirms gauge invariance, which means that no Itô terms are needed in the effective continuum action.

1. Introduction

In a previous work [1] we showed the invariance under local gauge transformations of the partition function of a lattice with electromagnetic field and fermions for the d=2 and d=3 cases, where d is the lattice space-time dimension. Similar result was found in ref. [2], but in the absence of fermions. This fact is expected due to the flatness of the $U(1)$ manifold; it is connected with the nonsymmetric Itô terms[3,4] in the effective action, which must be added to the action to maintain gauge invariance.

Here we deal with the case d=2 avoiding a simplification of [1] and obtain an 8x8 determinant, which has to be solved by computational work.

In Sec.2 we write the partition function Z, expressing then the corresponding determinant. Sec.3 is devoted to verify the gauge invariance of Z and finally in Section 4 the conclusions are presented.

2. Partition function and determinant

For a review of lattice theories see for instance [5-10]. The inclusion of fermions on a lattice is not so trivial due to renormalization difficulties [9].

We shall follow the notation of [1], and $U_{\mu n} \equiv e^{i\theta_{\mu n}}$. The associated partition function is then expressed as [7]

$$Z = \prod_{l'} \int_{-\pi}^{\pi} d\theta_{l'} \prod_{P'} \exp^{-\frac{(Ma)^{d-4}}{g^2}(1-\cos\theta_{P'})} \int_{-\pi}^{\pi} d\theta_1 (detA_{nm}) \exp^{-\frac{(Ma)^{d-4}}{g^2}\sum_{P=1}^{2d-1}(1-\cos\theta_P)} \tag{2.1}$$

Here we have chosen to integrate over the link variable $\theta_{22} = \theta_1$ and (see Fig.1):

$$\begin{aligned} \alpha_1 &= \theta_{11} + \theta_{22} - \theta_{14} - \theta_{21} = \theta_{22} + \beta_1 \\ \alpha_2 &= \theta_{12} + \theta_{21} - \theta_{13} - \theta_{22} = -\theta_{22} + \beta_2 \\ \alpha_3 &= \theta_{13} + \theta_{24} - \theta_{12} - \theta_{23} \\ \alpha_4 &= \theta_{14} + \theta_{23} - \theta_{11} - \theta_{24} \end{aligned} \tag{2.2}$$

$$\beta_1 \equiv \theta_{11} - \theta_{14} - \theta_{21} \tag{2.3}$$

$$\beta_2 \equiv \theta_{12} + \theta_{21} - \theta_{13} \tag{2.4}$$

The resulting determinant of (2.1) is shown in Table 1:

where the c's are linear combinations of the exponentials.

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$$\Delta = \begin{vmatrix} 1 & 0 & 2Ke^{-i\theta_{11}} & 0 & 0 & 0 & c_{17} & c_{18} \\ 0 & 1 & 0 & 2Ke^{i\theta_{12}} & 0 & 0 & c_{27} & c_{28} \\ 2Ke^{-i\theta_{12}} & 0 & 1 & 0 & c_{35} & c_{36} & 0 & 0 \\ 0 & 2Ke^{i\theta_{11}} & 0 & 1 & c_{45} & c_{46} & 0 & 0 \\ 0 & 0 & c_{53} & c_{54} & 1 & 0 & 2Ke^{-i\theta_{13}} & 0 \\ 0 & 0 & c_{63} & c_{64} & 0 & 1 & 0 & 2Ke^{i\theta_{14}} \\ c_{71} & c_{72} & 0 & 0 & 2Ke^{-i\theta_{14}} & 0 & 1 & 0 \\ c_{81} & c_{82} & 0 & 0 & 0 & 2Ke^{i\theta_{13}} & 0 & 1 \end{vmatrix}$$

Table 1: Determinat

The terms of $(\det A_{\mathrm{nm}})$ which contribute to θ_{22} are:

$$\det \mathrm{A_{nm}} = 1{+}64\mathrm{K}^2{+}768\mathrm{K}^6{+}\mathrm{aK}^2{+}\mathrm{bK}^4{+}\mathrm{cK}^6{+}\mathrm{dK}^8, \tag{2.5}$$

with

$$\mathrm{a} = -8\cos\phi_0 \tag{2.6}$$

$$\begin{aligned} \mathrm{b} = & -8(\cos\phi_1 - \cos\phi_2 + \cos\phi_3 - \cos\phi_4 + \\ & \cos\phi_9 + \cos\phi_{10} + \cos\phi_{11} + \cos\phi_{12}) \\ & +16(\cos\phi_5 + \cos\phi_6 + \cos\phi_7 + \cos\phi_8) + \\ & 32(\cos\phi_{13} + \cos\phi_{14}) \end{aligned}$$

$$\begin{aligned} \mathrm{c} = & \ 128(-\cos\phi_0 + \cos\phi_2 + \cos\phi_4 + \cos\phi_9 + \\ & \cos\phi_{10} - \cos\phi_{13} + \cos\phi_{14}) - 32(\cos\phi_{15} + \\ & \cos\phi_{16} + \cos\phi_{17} + \cos\phi_{18} + \cos\phi_{19} + \\ & \cos\phi_{20} + \cos\phi_{21} + \cos\phi_{22}) \end{aligned} \tag{2.7}$$

$$\begin{aligned} \mathrm{d} = & -256(\cos\phi_1 - \cos\phi_2 + \cos\phi_3 - \cos\phi_4 + \\ & \cos\phi_9 + \cos\phi_{10} + \cos\phi_{11} + \cos\phi_{12}) \\ & +128(\cos\phi_{13} + \cos\phi_{14}) + 64(\cos\phi_{19} + \\ & cos\phi_{20} + \cos\phi_{21} + \cos\phi_{22} - \cos\phi_{23} \\ & -\cos\phi_{24} - \cos\phi_{25} - \cos\phi_{26}) + \\ & 32(\cos\phi_{27} + \cos\phi_{28} + \cos\phi_{29} + \cos\phi_{30} \\ & + \cos\phi_{31} + \cos\phi_{32} + \cos\phi_{33} + \cos\phi_{34}) + \\ & 512\cos\phi_{35} \end{aligned}$$

and

$$\phi_0 = \theta_{22} + \theta_{23},$$
$$\phi_1 = \theta_{11} + \theta_{13} - \theta_{21} + \theta_{22}, etc.$$

3. Gauge invariance

For $d = 2$ and $a \to 0$, Eq. (2.1) gives

$$Z_2 = \int_{-\infty}^{\infty} d\theta_1' (\det \mathrm{A_{nm}}) \exp^{-\mathrm{K}'(2\theta_1'^2 + \frac{1}{2}(\beta_1+\beta_2)^2)} \tag{3.1}$$

Other contributions are:

$$\begin{aligned} \cos(\theta_{22} + \theta_{23}) &= \cos(\alpha_1 + \delta_0) \\ \delta_0 &= \theta_{14} + \theta_{21} - \theta_{11}. \end{aligned} \tag{3.2}$$

We require local gauge invariance of under

$$\theta_{\mu n} \to \theta_{\mu n} + g[\Lambda(n+\mu) - \Lambda(n)], \tag{3.3}$$

where $\Lambda(n)$ is an arbitrary function defined on the lattice sites n.

From here one finds that δ_0 is invariant under (3.3). $I^0_{Z_2}$ becomes in turn:

$$I^0_{Z_2} = \exp^{-\frac{1}{2}K'(\beta_1+\beta_2)^2]} \left[\begin{array}{c} -N(\beta_1+\beta_2)\sin\delta_0 + \\ \left(\begin{array}{c} 2N - M - \\ \frac{1}{4}(\beta_1+\beta_2)^2 \end{array} \right) \cos\delta_0 \end{array} \right] \tag{3.4}$$

where N and M are some constants, and fulfils gauge invariance.

Similarly:

$$\cos(\theta_{22} - \theta_{21} + \theta_{13} + \theta_{11}) = \cos(\alpha_1 + \delta_a) \tag{3.5}$$

with $\delta_1 = \alpha_4 - \theta_{23} - \theta_{24}$. Therefore

$$I^1_{Z_2} = \exp^{2[-\frac{K'}{2}(\beta_1+\beta_2)^2][N-M+\frac{1}{2}N(\beta_1+\beta_2)^2]} \tag{3.6}$$

results invariant too.

Proceeding in the same way with the other terms of (2.7), one can show that those terms that contribute to (2.1) are gauge invariant and then the partition function satisfies the same property, which in general is true since

$$\cos\alpha_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(\alpha_1)^{2n}$$

and

$$\sin \alpha_1 = \sum_{n=o}^{\infty} \frac{(-1)^n}{(2n+1)!} (\alpha_1)^{2n+1}$$

.

4. Conclusions

We have found that the effective continuum action for fermions on a lattice with electromagnetic field is gauge invariant. This result, which also means that the Itô terms are not necesary in the continuum action, could be expected since we have in first instance a flat manifold, but the insertion of fermions avoids such expectations. We have performed the calculations explicitly at least for the 2-dimensional case. For larger dimensions one has to do more computational work.

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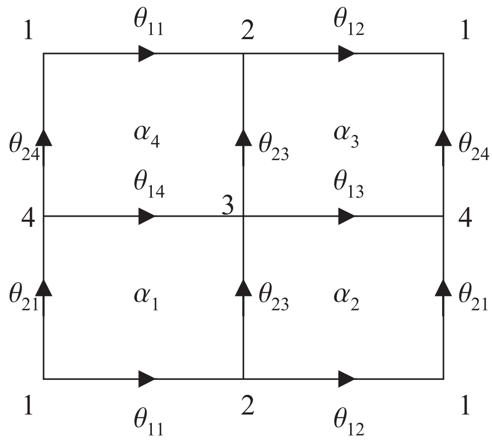

Figure 1: Two-dimensional lattice with plaquettes $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.