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Holographic Anomalies

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Abstract: The calculation of trace anomalies in the AdS/CFT correspondence is reviewed. We identify a subgroup of diffeomorphisms in odd dimension which generate the Weyl transformations. Universal results on the coefficients of trace anomalies are obtained. The mechanism for producing the loop terms responsible for anomalies in a classical calculation is described.

1. Introduction

This contribution is based on the work described in detail in [1] and [2].

The AdS/CFT correspondence offers remarkable insight into nonperturbative phenomena in gauge theories [3]. Many of the proposed tests of the correspondence rely on the symmetry algebras being isomorphic.

Among the tests going beyond the mapping of the algebraic structure the correct mapping of the trace anomalies is one of the most impressive [4, 5, 6].

On the CFT side, when the theory is put in a general gravitational background, the effective action contains specific nonlocal terms with local Weyl variations. The trace anomalies are produced by these terms.

On the supergravity side the correspondence involves a classical calculation: one solves the equation of motion using the metric at the boundary as initial condition. The action evaluated for this classical solution gives the effective action in terms of the boundary metric. An anomaly appears in a classical calculation due to the apparently infrared logarithmically divergent terms obtained when the action is evaluated with the classical solution.

In this contribution we will describe how these two aspects of the anomaly appear simultaneously in the supergravity calculation and their interrelation.

A special role in understanding this structure is played by a certain subgroup of diffeomorphisms in odd dimension (called the "PBH transformation") introduced in [1].

Due to the nonchiral nature of the Weyl transformations it is extremely convenient to use dimensional regularization. We start by reviewing how the trace anomalies appear in dimensional regularization which would be our main tool in the following.

2. The relation between Weyl cohomologies and dimensional regularization

Consider a conformal theory coupled to an external, c-number gravitational field $g_{ij}(x)$. The effective action, after integrating out the fields of the conformal theory, is $W(g)$.

In an even dimension the anomaly can be formulated as a cohomological problem [7], i.e. searching for local Weyl variations which cannot be obtained as the variation of a local action.

If we use dimensional regularization, *i.e.* we consider the theory in dimension d, there are two interrelated features which make the treatment of anomalies particularly simple:

1) In d dimensions there are no nontrivial cohomologies: there is always a local expression whose variation gives the anomaly in $2n$ dimensions. These local expressions have, however, a

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pole in $d-2n$. We distinguish between two situations:

i) The residue of the pole vanishes exactly in $d=2n$ due to a "topological" identity. This is the origin of the so called "type A anomaly"; we will continue to call more generally "type A" expressions in dimensional regularization those which have a pole and whose residue vanishes for $d=2n$ due to some special identity.

ii) The residue of the pole does not vanish for $d=2n$. We will call this situation "Type B". The effective action related to this case in $d=2n$ will have a scale.

2) In d dimensions there are no anomalies, *i.e.* the effective action is exactly Weyl invariant. It follows that the aforementioned local pole terms will be accompanied by finite, nonlocal terms such that the sum is invariant. As a consequence the variation of the pole term is equal (but opposite in sign) to the variation of the nonlocal term (the anomaly).

We illustrate the above general discussion with two typical examples.

We start with "type A". In d dimensions we expand the invariant effective action around $d=2n$. The leading term is local while the order $d-2n$ term is nonlocal. The effective action has therefore a pole term with a local residue which vanishes in $d=2n$ and a nonlocal finite term. The simplest example of this kind is in $d=2$ where the effective action $W(g)$ has the form:

$$W(g) = -\frac{1}{4}\int d^dx\sqrt{g}R\frac{1}{\Box^{1+\frac{\epsilon}{2}}}R + \frac{1}{d-2}\int d^dx\sqrt{g}R\,, \quad (2.1)$$

where $E_2=R$ is the two-dimensional Euler density and $d=2-\epsilon$. As usual in dimensional regularization, the tensors are defined in d-dimensions. $W(g)$ is Weyl invariant in d dimensions to order ϵ as is easy to verify. . The second term, with the $0/0$ structure, signals the presence in $d=2$ of a nontrivial cohomology in the Weyl transformations.

Once we have the expansion 2.1 we can discuss how to take the limit $d\to 2$: we simply drop the second, local term which has the $0/0$ structure, and therefore the limit is the first, finite, nonlocal term.

On the other hand, we can still use the local term if we are interested in the Weyl variation (anomaly) of the limit: since the total Weyl variation of 2.1 is 0, the variation of the local term is equal and opposite in sign to the variation of the limit, both variations being finite. Indeed the Weyl variation of the pole term gives $\sqrt{g}R\sigma$.

Summarizing the lessons learned from this simple example:

a) The pole term in d dimensions with a vanishing residue signals the presence of a nontrivial cohomology in integer (even) dimensions.

b) If a certain variation of the pole term has a well-defined limit then the pole term indicates that in integer dimension we should take a nonlocal expression which has the same variation. The functional dependence of the nonlocal term can be inferred by "completing" the pole term to a Weyl invariant expression in d dimensions.

Now we discuss the "type B" case, again in the simplest situation where it occurs, *i.e.* the second Weyl anomaly in $d=4$:

Considering a CFT around $d=4$ in a general background, the correlator of two energy-momentum tensors gives rise to a term in the effective action of the form:

$$W_d(g) = \frac{1}{d-4}\int d^dx\sqrt{g}C^{(d)}_{ijkl}\Box^{\frac{d-4}{2}}C^{(d)ijkl}\,, \quad (2.2)$$

where $C^{(d)}$ is the Weyl tensor in d dimensions.

The expression 2.2 is Weyl invariant to order $d-4$.[1]

The expression 2.2 does not have a well-defined limit for $d\to 4$. One should modify it by a "subtraction":

$$W^{(\mathrm{sub})}_d(g) = W_d(g) - \frac{\mu^{d-4}}{d-4}\int d^dx\sqrt{g}C^{(d)}_{ijkl}C^{(d)ijkl}\,. \quad (2.3)$$

It is important to remark that the "subtraction" introduces a scale μ. Now the limit can be taken, giving in $d=4$:

$$W_4(g) = \frac{1}{2}\int d^4x\sqrt{g}C_{ijkl}\log\left(\frac{\Box}{\mu^2}\right)C^{ijkl}\,. \quad (2.4)$$

We can rewrite W_d as :

$$W_d(g) = \frac{1}{2}\int d^dx\sqrt{g}C^{(d)}_{ijkl}\log\left(\frac{\Box}{\mu^2}\right)C^{(d)ijkl}$$

[1] For recent proposals to extend this expression to an exact one see [8].

$$+ \frac{\mu^{d-4}}{d-4} \int d^d x \sqrt{g} C^{(d)}_{ijkl} C^{(d)ijkl} . \quad (2.5)$$

The expression 2.5 is analogous to 2.1, *i.e.* we have a local pole term and a nonlocal finite term, the finite term being the limit in the integer dimension. Similarly to the "type A" situation the finite, local Weyl variation of the nonlocal term can be calculated by using the fact that W_d is invariant and therefore the variation of W_4 is equal and opposite in sign to the variation of the pole term. In this particular case this gives $\sqrt{g} C_{ijkl} C^{ijkl} \sigma$.

The general conclusions we want to draw from this example for the general "type B" situation are as follows:

a) The presence of a pole term with a nonvanishing residue in dimension d indicates that the theory in integer dimension has to be modified by defining it after a "subtraction".

b) After the subtraction the relevant expression can be rewritten as a sum of two terms, one finite and nonlocal and the other a local pole term with nonvanishing residue. The limit is the nonlocal term.

c) The local variation can be calculated directly from the pole term.

3. The PBH transformation

Consider a manifold in $d+1$ dimensions with a boundary which is topologically S_d. Following Fefferman and Graham [9], one can choose a set of coordinates in which the $d+1$ dimensional metric has the form:

$$ds^2 = G_{\mu\nu} dX^\mu dX^\nu = \frac{l^2}{4} \left(\frac{d\rho}{\rho} \right)^2 + \frac{1}{\rho} g_{ij}(x,\rho) dx^i dx^j . \quad (3.1)$$

Here $\mu, \nu = 1, \ldots, d+1$ and $i, j = 1, \ldots, d$. The coordinates are chosen such that $\rho = 0$ corresponds to the boundary. We will assume that g_{ij} is regular at $\rho = 0$, $g_{ij}(x, \rho = 0)$ being the boundary metric.

We now look for those $(d+1)$-dimensional diffeomorphisms which leave the form of the metric invariant. We make the ansatz

$$\begin{aligned} \rho &= \rho' e^{-2\sigma(x')} \simeq \rho'(1 - 2\sigma(x')) , \\ x^i &= x'^i + a^i(x', \rho') . \end{aligned} \quad (3.2)$$

The $a^i(x', \rho')$ are infinitesimal, and are restricted by the requirement of form invariance of the metric. We will work to $\mathcal{O}(\sigma, a^i)$. We insert 5.10 in 3.1 and require that the $dx' d\rho'$ components of the metric vanish. This gives

$$\partial_\rho a^i = \frac{l^2}{2} g^{ij} \partial_j \sigma . \quad (3.3)$$

With the boundary condition $a^i(x, \rho = 0) = 0$ this integrates to

$$a^i(x, \rho) = \frac{l^2}{2} \int_0^\rho d\rho' g^{ij}(x, \rho') \partial_j \sigma(x) . \quad (3.4)$$

Performing the diffeomorphism defined by 5.10, g_{ij} will generally transform:

$$\begin{aligned} \delta g_{ij}(x,\rho) &= 2\sigma(1 - \rho\partial_\rho) g_{ij}(x,\rho) \\ &+ \nabla_i a_j(x,\rho) + \nabla_j a_i(x,\rho) . \end{aligned} \quad (3.5)$$

The covariant derivatives are with respect to the metric $g_{ij}(x,\rho)$ where ρ is considered a parameter.

The equations 5.10,3.8,3.5 define the BPH transformation, i.e. a subgroup of bulk diffeomorphisms which leave the metric in the form 3.1 and which on the boundary reduce to a Weyl transformation.

We assume that $a^i(x,\rho)$ and $g_{ij}(x,\rho)$ have power series expansions in the vicinity of $\rho = 0$, i.e. we write

$$a^i(x,\rho) = \sum_{n=1}^{\infty} a^i_{(n)}(x) \rho^n \quad (3.6)$$

and

$$g_{ij}(x,\rho) = \sum_{n=0}^{\infty} g_{(n)ij}(x) \rho^n . \quad (3.7)$$

For a general bulk metric the coefficients in the expansion are arbitrary. If however, the metric is the solution of an equation of motion, the coefficients of the expansion are all expressible in terms of the boundary metric $g_{(0)ij}$. This result was proven by Fefferman and Graham (FG) for the equation of motion following from the Einstein action with cosmological term; we will assume that it holds for any action which admits an AdS solution.

The first few terms are

$$a^i_{(1)} = \frac{l^2}{2} g^{ij}_{(0)} \partial_j \sigma ,$$

$$\begin{aligned} a^i_{(2)} &= -\frac{l^2}{4} g^{ij}_{(1)} \partial_j \sigma \,, \\ a^i_{(3)} &= \frac{l^2}{6} \left[g^{ik}_{(1)} g_{(1)k}{}^j - g^{ij}_{(2)} \right] \partial_j \sigma \,, \\ a^i_{(4)} &= \frac{l^2}{8} \left[-g^{ij}_{(3)} + g^{ik}_{(1)} g_{(2)k}{}^j \right. \\ &\quad \left. + \, g^{ik}_{(2)} g_{(1)k}{}^j - g_{(1)}{}^i{}_k g_{(1)}{}^k{}_l g^{lj}_{(1)} \right] \nabla_j \sigma \,. \end{aligned} \tag{3.8}$$

The variations of $g_{(n)}$ under a Weyl transformation of $g_{(0)ij}$ are easily obtained by combining 5.10, 3.8 and 3.5 with the expansion 3.7. We give again just the first few terms:

$$\begin{aligned} \delta g_{(0)ij} &= 2\sigma g_{(0)ij} \,, \\ \delta g_{(1)ij} &= \overset{(0)}{\nabla}_i \, a_{(1)j} + \overset{(0)}{\nabla}_j \, a_{(1)i} \,, \\ \delta g_{(2)ij} &= -2\sigma g_{(2)ij} \\ &\quad + \, g_{(1)ik} \overset{(0)}{\nabla}_j \, a^k_{(1)} + g_{(1)jk} \overset{(0)}{\nabla}_i \, a^k_{(1)} \,. \end{aligned} \tag{3.9}$$

The covariant derivative $\overset{(0)}{\nabla}_i$ is w.r.t. $g_{(0)}$. These equations can be integrated w.r.t. σ. For the first two non-trivial terms in the expansion 3.7 we find

$$g_{(1)ij} = \frac{l^2}{d-2} \Big[R_{ij} - \frac{1}{2(d-1)} R g_{(0)ij} \Big] \,, \tag{3.10}$$

$$\begin{aligned} g_{(2)ij} &= c_1 l^4 C_{ijkl} C^{ijkl} g_{(0)ij} + c_2 l^4 C_{iklm} C_j{}^{klm} \\ &+ \frac{l^4}{d-4} \Big\{ -\frac{1}{8(d-1)} \nabla_i \nabla_j R + \frac{1}{4(d-2)} \nabla^2 R_{ij} \\ &- \frac{1}{8(d-1)(d-2)} \nabla^2 R g_{(0)ij} - \frac{1}{2(d-2)} R^{kl} R_{ikjl} \\ &+ \frac{d-4}{2(d-2)^2} R_i{}^k R_{jk} + \frac{1}{(d-1)(d-2)^2} R R_{ij} \\ &+ \frac{1}{4(d-2)^2} R^{kl} R_{kl} g_{(0)ij} \\ &- \frac{3d}{16(d-1)^2(d-2)^2} R^2 g_{(0)ij} \Big\} \,. \end{aligned}$$

Here the curvature R, the Weyl–tensor in d dimensions C and the covariant derivative ∇ are all those of the boundary metric $g_{(0)}$.[2]

Starting from $g_{(2)}$, there are solutions to the homogeneous equations, i.e. curvature invariants which transform homogeneously under Weyl transformations. E.g. for $g_{(2)}$ there are two free parameters. Of course we have complete agreement with the explicit calculation of ref.[5].

[2] We use the following curvature conventions: $[\nabla_i, \nabla_j] V_k = -R_{ijk}{}^l V_l$ and $R_{ij} = R_{ikj}{}^k$.

We stress that the above expressions do not assume any specific form of the action. The dependence on the action enters only through the arbitrary coefficients of the homogeneous terms.

4. The PBH transformation , the effective boundary action ad a universal formula for the coefficient of the type A anomaly

We study now the implications of the symmetry defined above for a general gravitational action S which is invariant under $d+1$ dimensional diffeomorphisms:

$$S = \frac{1}{2\kappa^2_{d+1}} \int d^{d+1} X \sqrt{G} f(R(G)) \,, \tag{4.1}$$

where f is a local function of the curvature and its covariant derivatives. For the application we have in mind, we must choose $f(R)$ such that AdS_{d+1} with radius l is a solution of the equations of motion. If we insert 3.1 into 4.1 we get an expression of the form

$$2\kappa^2_{d+1} S = \frac{l}{2} \int d\rho \, d^d x \, \rho^{-\frac{d}{2}-1} \sqrt{g_{(0)}(x)} b(x,\rho) \,, \tag{4.2}$$

where the specific form of $b(x,\rho)$ does depend on $f(R)$.

Since the integrand is a scalar under diffeomorphisms, the action is invariant under the transformation:

$$\begin{aligned} \delta b(x,\rho) &= b'(x,\rho) - b(x,\rho) \\ &= -2\sigma(x) \, \rho \, \partial_\rho b(x,\rho) + \overset{(0)}{\nabla}_i \left(b(x,\rho) a^i(x,\rho) \right) . \end{aligned} \tag{4.3}$$

Expanding the generating function $b(x,\rho)$ in a power series in ρ:

$$b(x,\rho) = \sum_{n=0}^{\infty} b_n(x) \rho^n \,. \tag{4.4}$$

the Weyl transformations of b_n can be integrated to give:

$$\begin{aligned} b_0 &= \text{const} \,, \\ b_1 &= b_0 \frac{l^2}{4(d-1)} R \,, \\ b_2 &= \frac{l^4 b_0}{32(d-2)(d-3)} E_4 + c l^4 C_{ijkl} C^{ijkl} \,, \end{aligned} \tag{4.5}$$

where as before C_{ijkl} is the Weyl-tensor corresponding to $g_{(0)}$ in d dimensions and E_{2n} is the Euler density, which is defined in $d = 2n$ dimensions as

$$E_{2n} = \frac{1}{2^n} R_{i_1 j_1 k_1 l_1} \cdots R_{i_n j_n k_n l_n} \epsilon^{i_1 j_1 i_2 j_2 \ldots i_n j_n} \epsilon^{k_1 l_1 \ldots k_n l_n} . \tag{4.6}$$

For b_0 we can obtain a general formula. Indeed, if we write the action 4.1 in the form

$$\begin{aligned} 2\kappa_{d+1}^2 S &= \int d\rho\, d^d x \sqrt{G} f(R(G)) \\ &= \frac{l}{2} \int d\rho\, d^d x \rho^{-\frac{d}{2}-1} \sqrt{g} f(R(G)) \end{aligned} \tag{4.7}$$

then

$$b_0 = l f(R)|_{\rho=0} = l f(AdS) , \tag{4.8}$$

where in the last expression f is to be evaluated at the action of AdS_{d+1} space with

$$R_{\mu\nu\rho\sigma} = \frac{1}{l^2} (G_{\mu\rho} G_{\nu\sigma} - G_{\mu\sigma} G_{\nu\rho}) . \tag{4.9}$$

To see this one simply has to realize that $\sqrt{g} f(R)$ has an expansion in positive powers of ρ and that only the most singular (in ρ) contributions of $R_{\mu\nu\rho\sigma}$ contribute to b_0. It is straightforward to show that this contribution is as given in 4.9.

We observe that b_n appears multiplying a pole in $2n - d$: therefore b_n is a trace anomaly in $d = 2n$ dimensions, i.e. it could be expanded as a linear combination of a type A, the type B appropriate to the dimension and cohomologically trivial terms.

We will be interested in extracting universal information about trace anomalies from the effective action. The anomaly of type A is a clear candidate for this type of information. Indeed, as we remarked before, terms involving contractions of Weyl tensors (i.e. type B) appear with arbitrary coefficients in b_n; therefore we couldn't expect any simple, general expression for type B. In order to isolate the general expression we are looking for in type A, we use the following simple observation: an exactly Weyl invariant expression vanishes identically for a metric which is conformally flat, i.e. which has the form:

$$g_{(0)ij} = \exp(2\phi)\delta_{ij} . \tag{4.10}$$

Therefore the terms with arbitrary coefficients in the various recursion relations ("the homogeneous terms") will disappear. As a consequence, the unambiguous solution of the equations for a metric of the form 4.10 will give us the information about the surviving type A anomaly.

Indeed matching the ϕ dependence in the recursion formulae we obtain [1]:

$$b_n = \frac{l^{2n} b_0}{2^{2n}(n!)^2} E_{2n} + \text{cohomologically trivial terms} . \tag{4.11}$$

Using the expression 4.8 for b_0 evaluated for $d = 2n$ we have now a general formula for the type A trace anomaly in any even dimension corresponding to a given gravity action which admits an AdS solution. We have to think about the gravity action given abstractly as a polynomial in the curvature without specifying the dimension. The specific form of the action enters in determining the radius of the AdS solution and then in the evaluation of the action on the solution, to give b_0. All the factors depending on the dimension $d = 2n$ in the expression for the coefficient of the type A anomaly are then explicit.

5. The cohomological problem for the FG coefficients and their relation to the nonlocal anomalous action

As shown by FG [9], when the expansion 3.7 is inserted into the equations of motion following from the Einstein action with negative cosmological constant in $d + 1$ dimensions, one obtains a set of recursive equations for the $g_{(n)ij}(x)$. We summarize the features of the solutions as studied by FG:

1) In non-integer d dimensions all the $g_{(n)ij}$'s are local, diffeo-covariant expressions which are uniquely determined in terms of the boundary metric.

2) The functions $g_{(n)ij}$ have poles at $d = 2n$ but their covariant divergences $\nabla^i g_{(n)ij}$ and traces $g_{(0)}^{ij} g_{(n)ij}$ are finite.

3) When d is taken to $2n$, $n > 1$, the expansion breaks down since the equations become inconsistent. Starting with the ρ^n term, a $\log(\rho)$ dependence should be added but even after adding these terms the equations of motion determine only the covariant divergence and trace of $g_{(n)ij}$.

We will study the structure of the $g_{(n)ij}$, first directly in $d = 2n$. As first conditions abstracted from the study of FG we will require that $g_{(n)ij}$ has a local covariant divergence and trace which we will call $V_{(n)i}$ and $S_{(n)}$, respectively.

Next, we will use the transformation properties of $g_{(n)ij}$ under Weyl transformations. As discussed above the transformation has the general form:

$$\delta g_{(n)ij} = 2\sigma(1-n)g_{(n)ij} + A_{(n)ij}(\sigma)\,, \tag{5.1}$$

where $A_{(n)ij}(\sigma)$ is a local, finite, covariant functional of g_{ij} [3] and σ, linear in the infinitesimal σ.

The structure we want to study is contained in the three local functionals $V_{(n)i}$, $S_{(n)}$, $A_{(n)ij}$. Obviously, these functionals should fulfil some consistency conditions.

We start with $A_{(n)ij}$: if we perform a second Weyl variation on 5.1 the result should not depend on the order since the Weyl group is abelian. This gives a Wess-Zumino type condition for $A_{(n)ij}$:

$$\begin{aligned} &2(1-n)\sigma_1 A_{(n)ij}(\sigma_2) + \delta_{\sigma_2}A_{(n)ij}(\sigma_1) \\ &= 2(1-n)\sigma_2 A_{(n)ij}(\sigma_1) + \delta_{\sigma_1}A_{(n)ij}(\sigma_2) \end{aligned} \tag{5.2}$$

Following from the definitions of $V_{(n)i}$ and $S_{(n)}$ as covariant divergence and trace, respectively, of $g_{(n)ij}$, their Weyl variation is given in terms of $A_{(n)ij}(\sigma)$. Using 5.1 and calculating explicitly the variations in $d=2n$ we obtain:

$$\delta S_{(n)} = -2n\sigma S_{(n)} + g^{ij}A_{(n)ij(\sigma)}\,, \tag{5.3}$$

$$\delta V_{(n)i} = -2n\sigma V_{(n)i} + \nabla^j A_{(n)ij}(\sigma) - S_{(n)}\nabla_i\sigma\,. \tag{5.4}$$

Now we can formulate our "cohomological" problem: We search for triples of *local* functionals $V_{(n)i}$, $S_{(n)}$, $A_{(n)ij}(\sigma)$ which satisfy the consistency conditions 5.2,5.3,5.4; clearly, any local functional $g_{(n)ij}$, by taking its covariant divergence, trace and Weyl variation 5.1, generates such a triple. When a consistent triple does not have a local generator, the nonlocal $g_{(n)ij}$ producing the triple constitutes a nontrivial solution. Of course, like in all problems of this type, the nontrivial solutions are really equivalence classes under the addition of local generators. In the above treatment we will consider all the quantities to be covariant under d-dimensional diffeomorphisms.

[3] When there is no danger of confusion, we will simply denote by g_{ij} the boundary metric $g_{(0)ij}$.

The cohomologically nontrivial solutions are given by the following construction [2]:

If a diffeo-invariant, dimensionless functional of g_{ij}, $W(g)$, has a Weyl variation for an infinitesimal Weyl parameter σ, which can be expressed as an integral over a *local* density $D(g)$

$$\delta_\sigma W(g) = \int d^{2n}x\sqrt{g}D(g)\sigma \tag{5.5}$$

then $A_{(n)ij}$, defined as

$$A_{(n)ij} = \frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{ij}}\delta_\sigma W(g)\,, \tag{5.6}$$

fulfils 5.2. The proof is simple. Consider the commutator between the operators:

$$\left[\frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{ij}(x)}, \delta_\sigma\right] = 2(n-1)\sigma(x)\frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{ij}(x)}\,. \tag{5.7}$$

Applying 5.7 to $W(g)$ we obtain 5.1 for a $g_{(n)ij}$ defined as:

$$g_{(n)ij} = \frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{ij}}W(g)\,. \tag{5.8}$$

Therefore, $A_{(n)}$ will fulfil 5.2 automatically and it is local by assumption. Since, obviously, a local $W(g)$ will give a local $g_{(n)ij}$ 5.8, the interesting solutions will reflect nonlocal $W(g)$'s, *i.e.* the actions which generate the Weyl anomalies. Then $A_{(n)ij}$ is expressed in 5.6 directly in terms of the anomaly $\sqrt{g}D(g)\sigma$.

Now $D(g)$ can be either E_{2n}, the Euler characteristic ("type A" anomaly), or an expression which together with $\sqrt{g}$ is locally Weyl invariant ("type B", whose number increases with the dimension). The "type A" gives by itself a nontrivial solution . On the other hand the "type B" does not fulfil 5.3,5.4. In order to arrive at a consistent solution the FG expansion should be modified including logarithmic terms:

$$g_{ij}(x,\rho) = \sum_{m=0}^{\infty} g_{(m)ij}(x)\rho^m + h_{(n)ij}(x)\rho^n\log(\rho) + \ldots \tag{5.9}$$

in $d=2n$ dimensions, where we made explicit just the first new term. The PBH transformations

$$\begin{aligned} \rho &= \rho' e^{-2\sigma(x')} \simeq \rho'(1-2\sigma(x'))\,, \\ x^i &= x'^i + a^i(x',\rho')\,, \end{aligned} \tag{5.10}$$

induce now a modified transformation of $g_{(n)ij}(x)$:

$$\bar{\delta} g_{(n)ij} = \delta g_{(n)ij} - 2\sigma h_{(n)ij}\,, \qquad (5.11)$$

where $\delta g_{(n)ij}$ is given in 5.1 and give for the transformation of $h_{(n)ij}$:

$$\bar{\delta} h_{(n)ij} = 2(1-n)\sigma h_{(n)ij}\,. \qquad (5.12)$$

From 5.12 it follows that $h_{(n)ij}$ should be an expression which transforms homogeneously. As a consequence now there is a consistent nontrivial solution corresponding to each "type B" anomaly. The nonlocal $g_{(n)ij}$ which is the solution of the cohomology problem is given by 5.8 with $W(g)$ being one of the nonlocal actions responsible for the anomalies.

We have now a general mechanism for generating the nonlocal pieces in the effective action responsible for the anomalies : the action has boundary terms [4] whose functional derivatives with respect to the boundary metric are combinations containing $g_{(n)ij}$ [12, 13].

As discussed above $g_{(n)ij}$ contains cohomologically nontrivial terms given by 5.8. Integrating this equation we'll get the anomalous nonlocal terms in the effective action . The general proof that the normalization of these terms coincides with the normalization of the pole terms which gave our universal results is not yet completed.

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