An $U(1) \times SO(9)$ invariant compactification of $D = 11$ supergravity to two dimensions

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Abstract: In this contribution we exhibit a compactification of $D = 11$ supergravity on $S^1 \times S^8$ to two space-time dimensions. The resulting theory is a maximal gauged supergravity in two dimensions with gauge group $U(1) \times SO(9)$ and some unusual features. In particular, the theory's ground state is not fully supersymmetric but only invariant under a "chiral" $N = 16$ supersymmetry.

1. Introduction

Numerous compactifications of $D = 11$ supergravity [1] are known and have been extensively studied over the past years (see e.g. [2] for a comprehensive survey and many references). However, for the most part these describe the compactification of $D = 11$ supergravity to dimensions four and higher, whereas relatively few results exist for low dimensional compactifications. In this contribution we study a solution corresponding to a compactification on $S^1 \times S^8$ to *two* space-time dimensions; for special values of the parameters, this solution has already appeared in the literature [3]. While that work is chiefly concerned with domain wall solutions and their interpretation in the context of D–branes, we are here motivated by the question whether there exists a maximal gauged supergravity in two dimensions. It is clear that such a gauged supergravity would be quite different from maximal gauged supergravity in four dimensions [4], already for the simple reason that in two dimensions there are *a priori* no vector fields that could be used to gauge the theory. Such vector fields exist in supergravities for $D \geq 4$, but are "dualized away" into scalar fields in three dimensions and below.

Our results provide further evidence for the existence of a maximal gauged $N = 16$ theory in two dimensions, albeit of an unusual type, with gauge group $U(1) \times SO(9)$. One of its unusual features is the fact that it does not appear to admit a ground state preserving the full $N = 16$ supersymmetry of the ungauged theory (for the latter, see [5] and references therein). Rather, this symmetry is broken down to a kind of "chiral" $(16, 0)$ supersymmetry. It remains to be worked out if this symmetry breaking is an intrinsic feature of the $D = 2$ gauged supergravity or merely a property of its ground states.

2. The metric and field equations

Let us first set up our notation and conventions. As already indicated we will study a special compactification of $D = 11$ supergravity to two dimensions on an internal manifold $S^1 \times S^8$, with the eleven coordinates x^M split as follows:

$$x^M \to (x^\mu, x^2, x^m) \qquad (2.1)$$

Here, the coordinates $x^0 = t$, $x^1 = r$ parametrize the two-dimensional space-time and x^2 is associated with the circle S^1; the remaining coordinates $\{x^3, \ldots, x^{10}\} \equiv \{y^1, \ldots, y^8\}$ describe the internal eight-sphere. We assume the metric to have signature $\{-++\ldots+\}$.

We start from the following ansatz for the vielbein $E_M{}^A$:

$$\begin{aligned} E_\mu{}^\alpha &= F(r)\,\overset{\circ}{e}_\mu{}^\alpha(x) \\ E_\mu{}^2 &= G(r)\,A_\mu(x) \\ E_2{}^2 &= G(r) \\ E_m{}^a &= H(r)\,\overset{\circ}{e}_m{}^a(y) \end{aligned} \tag{2.2}$$

with all other components set to zero. The zweibein part in (2.2) is proportional to the zweibein of AdS_2

$$\overset{\circ}{e}_\mu{}^\alpha = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \tag{2.3}$$

The associated AdS_2 has unit radius; different values of the radius may be absorbed into the pre-factor F. The spin connection and curvature scalar are easily computed

$$\overset{\circ}{\omega}_{001} = -1\,, \quad \overset{\circ}{\omega}_{101} = 0 \;\Rightarrow\; \overset{\circ}{R}_{\alpha\beta} = \eta_{\alpha\beta}. \tag{2.4}$$

The lower block $\overset{\circ}{e}_m{}^a(y)$ of (2.2) contains the metric on the sphere S^8 with unit radius.

The full $D = 11$ metric thus becomes

$$\begin{aligned} ds^2 &= F^2\left[-r^2dt^2 + dr^2/r^2\right] + H^2 d\Omega_8^2 \\ &\quad + G^2\left[dx^2 + A_\mu dx^\mu\right]^2 \end{aligned} \tag{2.5}$$

Apart from the S^8 line element $d\Omega_8^2$ (again normalized to unit radius for $H = 1$) the metric ansatz depends only on the coordinate r. An unusual feature is the "warp factor" multiplying the internal part of the metric. For previous compactified solutions of $D = 11$ supergravity with warp factor [6, 7, 8], the latter depended on the internal coordinates and multiplied the space-time part of the metric.

To compute the spin connection and Riemann tensor for the full metric we make use of the general formulas

$$\begin{aligned} \omega_{ABC} &= \tfrac{1}{2}\left(\Omega_{ABC} - \Omega_{BCA} + \Omega_{CAB}\right) \\ \Omega_{ABC} &= 2\,E_A{}^M E_B{}^N\,\partial_{[M}E_{N]C} \end{aligned} \tag{2.6}$$

In this way we derive from (2.2) the following nonvanishing components

$$\begin{aligned} \omega_{\alpha\beta\gamma} &= F^{-1}\,\overset{\circ}{\omega}_{\alpha\beta\gamma} + 2\,F^{-2}\eta_{\alpha[\beta}\,\overset{\circ}{\partial}_{\gamma]}F \\ \omega_{\alpha\beta 2} &= \tfrac{1}{2}\,GF^{-2}\,\overset{\circ}{A}_{\alpha\beta} \\ \omega_{2\alpha\beta} &= -\tfrac{1}{2}\,GF^{-2}\,\overset{\circ}{A}_{\alpha\beta} \\ \omega_{22\gamma} &= F^{-1}G^{-1}\,\overset{\circ}{\partial}_\gamma G \\ \omega_{ab\gamma} &= \delta_{ab}\,F^{-1}H^{-1}\,\overset{\circ}{\partial}_\gamma H \\ \omega_{abc} &= H^{-1}\,\overset{\circ}{\omega}_{abc} \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} \overset{\circ}{\partial}_\alpha &:= \overset{\circ}{e}_\alpha{}^\mu\,\partial_\mu \\ \overset{\circ}{A}_{\alpha\beta} &:= \overset{\circ}{e}_\alpha{}^\mu \overset{\circ}{e}_\beta{}^\nu A_{\mu\nu} \end{aligned} \tag{2.8}$$

denote the flat derivative and the Maxwell field strength $A_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$. From the Riemann tensor (with flat indices)

$$\begin{aligned} R_{AB}{}^{CD} &= 2\,\omega_{[A}{}^{CE}\,\omega_{B]E}{}^D + 2\,\partial_{[A}\,\omega_{B]}{}^{CD} \\ &\quad + 2\,\omega_{[AB]}{}^E\,\omega_E{}^{CD} \end{aligned} \tag{2.9}$$

we obtain the nonvanishing components of the Ricci tensor

$$\begin{aligned} R_{\alpha\beta} &= \tfrac{1}{2}\eta_{\alpha\beta}\,F^{-2}(\overset{\circ}{R} + 2\,\overset{\circ}{\Box}\,\ln F) \\ &\quad - \tfrac{1}{2}\,F^{-4}G^2\,\overset{\circ}{A}_{\alpha\gamma}\overset{\circ}{A}_\beta{}^\gamma \\ &\quad - F^{-2}G^{-1}\,(\overset{\circ}{\partial}_\alpha\overset{\circ}{\partial}_\beta + \overset{\circ}{\omega}_{\alpha\beta\gamma}\overset{\circ}{\partial}{}^\gamma)G \\ &\quad + F^3G^{-1}\,(\overset{\circ}{\partial}_\alpha F\overset{\circ}{\partial}_\beta G + \overset{\circ}{\partial}_\beta F\overset{\circ}{\partial}_\alpha G) \\ &\quad - \eta_{\alpha\beta}\,F^3G^{-1}\,\overset{\circ}{\partial}_\gamma F\overset{\circ}{\partial}{}^\gamma G \\ &\quad - 8F^{-2}H^{-1}\,(\overset{\circ}{\partial}_\alpha\overset{\circ}{\partial}_\beta + \overset{\circ}{\omega}_{\alpha\beta\gamma}\overset{\circ}{\partial}{}^\gamma)H \\ &\quad + 8F^3H^{-1}\,(\overset{\circ}{\partial}_\alpha F\overset{\circ}{\partial}_\beta H + \overset{\circ}{\partial}_\beta F\overset{\circ}{\partial}_\alpha H) \\ &\quad - 8\eta_{\alpha\beta}\,F^3H^{-1}\,\overset{\circ}{\partial}_\gamma F\overset{\circ}{\partial}{}^\gamma H \\[1ex] R_{\alpha 2} &= \tfrac{1}{2}\,F^{-1}G^{-2}H^{-8}\,\overset{\circ}{\partial}{}^\gamma(F^{-2}G^3H^8\,\overset{\circ}{A}_{\alpha\gamma}) \\[1ex] R_{22} &= -F^{-2}G^{-1}\,(\overset{\circ}{\partial}_\gamma\overset{\circ}{\partial}{}^\gamma + \overset{\circ}{\omega}_\alpha{}^\alpha{}_\gamma\,\overset{\circ}{\partial}{}^\gamma)G \\ &\quad - 8\,F^{-2}G^{-1}H^{-1}(\overset{\circ}{\partial}_\gamma G\overset{\circ}{\partial}{}^\gamma H) \\ &\quad + \tfrac{1}{4}F^{-4}G^2\,\overset{\circ}{A}_{\alpha\gamma}\overset{\circ}{A}{}^{\alpha\gamma} \\[1ex] R_{ab} &= H^{-2}\,\overset{\circ}{R}_{ab} \\ &\quad - 7\,\delta_{ab}\,F^{-2}H^{-2}(\overset{\circ}{\partial}_\gamma H\overset{\circ}{\partial}{}^\gamma H) \\ &\quad - \delta_{ab}\,F^{-2}H^{-1}\,(\overset{\circ}{\partial}_\gamma\overset{\circ}{\partial}{}^\gamma + \overset{\circ}{\omega}_\alpha{}^\alpha{}_\gamma\,\overset{\circ}{\partial}{}^\gamma)H \\ &\quad - \delta_{ab}\,F^{-2}G^{-1}H^{-1}\,\overset{\circ}{\partial}_\gamma G\,\overset{\circ}{\partial}{}^\gamma H \end{aligned} \tag{2.10}$$

Unlike the well known Freund Rubin solutions [9], our ansatz has no source terms involving the three-form potential of $D = 11$ supergravity, and therefore the equations of motion simply read

$$R_{AB} = 0 \qquad (2.11)$$

The field equation for $(AB) = (0\alpha)$ and a particular combination of the other ones may be directly integrated and yield

$$\overset{0}{A}_{\alpha\beta} = C_0\,\epsilon_{\alpha\beta}\,F^2G^{-3}H^{-8} \qquad (2.12)$$

$$\partial_r(rGF) = C_1\,r^{-1}FH^{-8} \qquad (2.13)$$

with integration constants C_0 and C_1. Here we recognize that the Kaluza Klein vector A_μ plays a role analogous to the three-form potential in the standard Freund Rubin ansatz. There it is the vacuum expectation value of the four-index field strength which leads to preferential compactification to four dimensions, whereas here it is the vacuum expectation value of $A_{\mu\nu}$ which now leads to a two-dimensional vacuum.

The remaining field equations take the form

$$\partial_r(r^2H^8\partial_rG) = -\tfrac{1}{2}C_0^2F^2G^{-3}H^{-8} \qquad (2.14)$$

$$\partial_r(r^2GH^7\partial_rH) = 7\,F^2GH^6 \qquad (2.15)$$

$$H\partial_r(F^{-2}\partial_rG) = -8\,G\partial_r(F^{-2}\partial_rH) \qquad (2.16)$$

and will be further analyzed below.

3. Killing spinors

Before proceeding with the Einstein equations we discuss the Killing spinor equations resulting from our metric ansatz. The main advantage is, of course, that the first order Killing spinor equations are easier to analyze than the second order field equations, especially if one insists on maximal supersymmetry, just as for the $\mathrm{AdS}_4 \times S^7$ compactification of $D = 11$ supergravity [10]. It remains to verify afterwards that the solution in addition solves the Einstein equations. Since the three-form potential is assumed to vanish, the Killing spinor equations reduce to

$$\Big(\partial_A + \tfrac{1}{4}\,\omega_{ABC}\,\Gamma^{BC}\Big)\epsilon = 0\,. \qquad (3.1)$$

Here we represent the 11d Γ-matrices as

$$\Gamma^\alpha = \gamma^\alpha\otimes\Gamma^9 \qquad \Gamma^a = \mathbf{1}\otimes\Gamma^{a-2} \qquad (3.2)$$

for $\alpha = 0,1,2$ and $a = 3,...,10$, where the γ matrices on the r.h.s. are 2×2 and 16×16, respectively (for simplicity of notation we do not introduce new symbols for the $SO(9)$ Γ-matrices). For $A = 0$ (3.1) yields

$$\Big(F^{-2}\,\partial_r(rF)\,\gamma^2 + \tfrac{1}{2}\,C_0G^{-2}H^{-8}\,\gamma^0\Big)\epsilon = 0 \qquad (3.3)$$

with the elfbein (2.2) and connection coefficients (2.6). The condition (3.3) shows already that for nonvanishing C_0 we need to impose a nontrivial projection onto ϵ. Representing the 32-component spinor of $D = 11$ supergravity as direct product of a 2-component space-time spinor and an internal 16-component spinor (transforming as an $SO(9)$ spinor as expected), we employ the following ansatz:

$$\epsilon(r,y) \;=\; \begin{pmatrix}\alpha(r)\\ -\alpha(r)\end{pmatrix}\otimes\eta(y)\,. \qquad (3.4)$$

For flat space the two components of the first factor would be independent constants they here are related. The solution thus necessarily preserves only half of the maximal supersymmetries, but rather than an $N = 8$ supergravity, it yields a $(16,0)$ theory. However, this is not the usual chirality (defined w.r.t. to γ^2) but rather a "chirality" defined w.r.t. to γ^1 as is obvious from (3.3). Substituting this ansatz into (3.3) we arrive at

$$\partial_r(rF) = \tfrac{1}{2}\,C_0F^2G^{-2}H^{-8} \qquad (3.5)$$

For $A = 1$, equation (3.1) yields

$$\Big(rF^{-1}\partial_r + \tfrac{1}{4}\,C_0G^{-2}H^{-8}\,\gamma^1\Big)\epsilon = 0$$

which determines the prefactor α to be

$$\alpha(r) = \mathrm{const}\cdot\,(rF(r))^{1/2} \qquad (3.6)$$

For $A = 2$, equation (3.1) yields

$$\Big(-\tfrac{1}{2}\,C_0G^{-1}H^{-8}\,\gamma^2 + rF^{-1}\partial_rG\,\gamma^0\Big)\,\epsilon = 0$$

implying

$$rG^{-1}\partial_rG = -\tfrac{1}{2}\,C_0F^2G^{-2}H^{-8} \qquad (3.7)$$

whereas for $A = a$ we obtain

$$\Big(\overset{0}{D}_a + \tfrac{1}{2}\,rF^{-1}\partial_rH\,\gamma^1\,\Gamma_a\Gamma^9\Big)\,\epsilon = 0$$

whence

$$rF^{-1}\partial_r H = m \qquad (3.8)$$

$$\left(\overset{0}{D}_a + \tfrac{m}{2}\,\Gamma_a\Gamma^9\right)\eta(y) = 0 \qquad (3.9)$$

The second equation shows that $\eta(y)$ is a linear combination of Killing spinors on the sphere S^8; there are 16 independent such spinors whose explicit form will be worked out below. The parameter m here is proportional to the inverse S^8 radius and may be absorbed into H; we will set it to unity below.

In summary, the Killing spinor equations for the functions F, G, H are given by

$$\partial_r(rFG) = 0 \qquad (3.10)$$

$$r\partial_r H = mF \qquad (3.11)$$

$$\partial_r(rF) = \tfrac{1}{2}\,C_0 F^2 G^{-2} H^{-8} \qquad (3.12)$$

These equations may be shown to be compatible with the Einstein field equations, i.e. every solution of (3.10)–(3.12) automatically solves (2.12)–(2.16). In the following section, we derive the most general solution of (3.10)–(3.12).

4. Killing spinor equations

Equation (3.10) may be integrated to

$$F = C_2\, r^{-1} G^{-1} \qquad (4.1)$$

This leaves two equations for G and H:

$$r^2\partial_r H = mC_2\, G^{-1} \qquad (4.2)$$

$$r^2\partial_r G = -\tfrac{1}{2}\,C_0 C_2\, G^{-2} H^{-8} \qquad (4.3)$$

In order not to make the formulas too cumbersome, we will take $m = 1$ from now on and eliminate C_2 by rescaling $G \to C_2 G$, $C_0 \to C_2 C_0$.

Multiplying (4.2) and (4.3), we obtain (now with $m = 1$)

$$16\, H^{-1}\partial_r G = C_0\, G^{-1}\partial_r(H^{-8}) \qquad (4.4)$$

which may be integrated to

$$G^2 = \tfrac{C_0}{7}\, H^{-7} + C_3 \qquad (4.5)$$

At this point we are left with only one differential equation which is written out in (4.12) below.

4.1 Nontriviality of the solution

The most general solution of the Killing spinor equations (4.2), (4.3), has the nonvanishing components of the Riemann tensor

$$R_{0101} = R_{2121} = -\frac{4C_0}{G^2H^9} \qquad (4.6)$$

$$R_{1201} = \frac{4C_0}{G^2H^9}$$

$$R_{0a0b} = R_{0a2b} = R_{2a2b} = \frac{C_0\,\delta_{ab}}{2G^2H^9}$$

with the functions G and H related as above. It is now straightforward to verify that the associated Ricci tensor vanishes, and hence the metric satisfies the Einstein field equations (2.11), provided the equations derived above are also obeyed.

In particular, (4.6) shows that every solution with vanishing C_0 (i.e. with vanishing vector field A_μ) is flat space. In other words, every nontrivial solution of this type has a non-vanishing vacuum expectation value for $A_{\mu\nu}$, and it is charged with respect to the $11d \to 10d$ Kaluza-Klein vector field.

For the higher dimensional situation one usually analyzes the content of Killing spinor equations in terms of their compatibility conditions [10]. In our setting the compatibility conditions are very simple, viz.[1]

$$R_{ABCD}\,\Gamma^{CD}\,\epsilon = 0 \qquad (4.7)$$

For instance, the special case

$$\left(R_{0101}\gamma^{01} + R_{0112}\gamma^{12}\right)\epsilon = 0 \qquad (4.8)$$

illustrates once more why our solution has a "chiral" residual supersymmetry only: substituting the relevant values of the Riemann tensor, we see that this equation becomes a projection condition condition. This feature may be traced back to the separate S^1 factor, which is absent from the higher dimensional Freund Rubin-type solutions.

[1] As already mentioned this equation by itself does not imply $R_{AB} = 0$, but only $R_{AB}\xi^B = 0$, where ξ^A is the associated Killing vector which can be expressed as a fermionic bilinear of Killing spinors.

4.2 Two extremal cases

Let us first discuss two special "extremal" solutions. For $C_0 = 0$ we have

$$F = \frac{1}{\sqrt{C_3}\,r}\,, \qquad G = \sqrt{C_3}\,, \qquad H = H_\infty - \frac{1}{\sqrt{C_3}\,r} \tag{4.9}$$

which is just the flat space solution as already mentioned (to recover the flat Minkowski metric, we have to redefine $r \to 1/r$).

For $C_3 = 0$ we obtain (after rescaling $C_0 \to 7C_0$)

$$\begin{aligned} F &= \left(\tfrac{5}{2}C_0^{-1/7}\,r^{-2/7} + H_\infty^{-5/2}\,C_0^{5/14}\,r^{5/7}\right)^{-7/5} \\ G &= \left(\tfrac{5}{2\,r}\,C_0^{-1/7} + C_0^{5/14}\,H_\infty^{-5/2}\right)^{7/5} \\ H &= \left(\tfrac{5}{2\,r}C_0^{-1/2} + H_\infty^{-5/2}\right)^{-2/5} \end{aligned} \tag{4.10}$$

which has no smooth limit to flat space since we cannot scale out the factor C_0 in such a way that the limit $C_0 \to 0$ exists. Its curvature diverges at zero and tends to a constant at infinity. At $H_\infty = \infty$, this solution simplifies to

$$\begin{aligned} F &= \left(\tfrac{5}{2}\right)^{-7/5}\,C_0^{1/5}\,r^{2/5} \\ G &= \left(\tfrac{5}{2}\right)^{7/5}\,C_0^{-1/5}\,r^{-7/5} \\ H &= \left(\tfrac{5}{2}\right)^{-2/5}\,C_0^{1/5}\,r^{2/5} \end{aligned} \tag{4.11}$$

This is the metric derived in [3] (in the $10d$ dual frame). Its curvature vanishes at infinity but diverges at zero.

4.3 The general case

Starting from (4.5) and (4.2), the remaining differential equation for H is

$$r^2\partial_r H\,\sqrt{\tfrac{C_0}{7}\,H^{-7} + C_3} \;=\; 1 \tag{4.12}$$

Upon rescaling the function $H \to |C_3|^{-1/2}\,H$, and the constant $C_0 \to 7\,|C_3|^{5/2}\,C_0$, we get

$$r^2\partial_r H\,\sqrt{C_0\,H^{-7} + \epsilon_1} \;=\; 1 \tag{4.13}$$

with $\epsilon_1 = \pm 1 = \operatorname{sgn} C_3$. The negative branch of the square root corresponds to the solution $H \to -H$, $C_0 \to -C_0$.

It follows directly that the asymptotics near $r \sim 0$ must be of one of the following forms:

$$H(r) = \begin{cases} -r^{-1}\,(1 + \mathcal{O}(r)) \\ \left(\tfrac{2}{5}\right)^{2/5}\,C_0^{1/5}\,r^{2/5}\,(1 + \mathcal{O}(r)) \end{cases} \tag{4.14}$$

where the first case is possible only for $\epsilon_1 = +1$. Moreover, $r = 0$ is the only value of r where H can diverge or take the value 0 (since in either case one of the term under the square root dominates such that (4.13) may be integrated and leads to (4.14)). In particular, in the limit $r \to \infty$, H remains regular:

$$H(r) \overset{r\sim\infty}{=} H_\infty - \frac{1}{r}\,(C_0 H_\infty^{-7} + \epsilon_1)^{-1/2} + \mathcal{O}(r^{-2}) \tag{4.15}$$

The differential equation (4.13) further requires

$$C_0\,H^{-7} \;\geq\; -\,\epsilon_1 \tag{4.16}$$

Since $H(r) \neq 0$ for $r \neq 0$ there are hence two different types of solutions:

$H_\infty > 0$: In this case, (4.16) requires $C_0 > 0$. Since $\partial_r H$ is positive, (4.16) is satisfied for all r iff it is valid for H_∞. For $\epsilon_1 = 1$ this is always true, for $\epsilon_1 = -1$ this condition gives the upper bound $H_\infty < C_0^{1/7}$. All these solutions have regular behavior (4.15) at infinity and $r^{2/5}$ behavior (4.14) at zero. The computation of the Riemann tensor (4.6) shows that they have constant curvature at infinity and a singularity at the origin. In the limit $C_0 \to 0$ they tend to

$$H(r) \to \begin{cases} 0 & \text{for } \; r < H_\infty^{-1} \\ H_\infty - 1/r & \text{for } \; r > H_\infty^{-1} \end{cases} \tag{4.17}$$

and thus have no smooth limit to the flat space.

$H_\infty < 0$: In this case, (4.16) requires $\epsilon_1 = 1$. Then, (4.16) is satisfied automatically for $C_0 < 0$, whereas for $C_0 > 0$ it further requires the upper bound $H_\infty < -C_0^{1/7}$ on the asymptotic value. All these solutions have regular behavior (4.15) at infinity and r^{-1} behavior (4.14) at zero. The computation of the Riemann tensor (4.6) shows that they have constant curvature at infinity and vanishing curvature at the origin. In the limit $C_0 \to 0$ they smoothly tend to the flat space.

4.4 Killing spinors on S^8

We here summarize some properties of S^8 Killing spinors. For this purpose we need explicit expressions for the S^8 spin connection. With a convenient (and standard, see [11]) choice of coordinates the achtbein on S^8 is given by

$$\overset{0}{e}_m{}^a = \delta_m^a - \frac{y^2\omega(y)}{1+y^2\omega(y)}\frac{y_m y^a}{y^2} . \quad (4.18)$$

In terms of these coordinates the S^8 spin connection and Ricci tensor become

$$\begin{aligned} \overset{0}{\omega}_{abc} &= 2\omega(y)\delta_{a[b}\, y_{c]} , \quad (4.19) \\ \overset{0}{R}_{ab} &= 7\,\delta_{ab} \end{aligned}$$

where

$$\omega(y) \equiv -\frac{1}{y^2}(1-\sqrt{1-y^2})$$

The Killing spinor equation now reads

$$\left(\partial_a + \tfrac{1}{2}\,\kappa\,\omega(y)\,y^b\,\Gamma_{ab} + \tfrac{m}{2}\,\Gamma_a\right)\,\eta(y) = 0 \quad (4.20)$$

with

$$\partial_a = \frac{\partial}{\partial y^a} + y^2\omega(y)\,n_a n^b \frac{\partial}{\partial y^b} \quad (4.21)$$

where $n^a = \frac{y^a}{|y|}$ and the $SO(9)$ Γ matrices obey the standard Dirac algebra

$$\Gamma_a\Gamma_b = \delta_{ab} + \Gamma_{ab}$$

such that

$$\Gamma_{a(b}\Gamma_{c)} = \delta_{bc}\Gamma_a - \delta_{a(b}\Gamma_{c)}$$

The solution can be written in the form

$$\eta(y) = \Big(A(|y|) + B(|y|)n_a\Gamma^a\Big)\eta_0 \quad (4.22)$$

with

$$\begin{aligned} A(|y|) &= \frac{m}{\sqrt{2\kappa\omega(y)}} , \\ B(|y|) &= -\sqrt{\frac{\kappa\omega(y)}{2}}\,|y| . \end{aligned}$$

To see how these Killing spinors are related to the Killing vectors on S^8, we consider the bilinear expressions

$$\begin{aligned} K_m^{IJ}(y) &= \eta^{[I}(y)\,\Gamma_{m9}\,\eta^{J]}(y) \\ &= \eta_0^{[I}\,\Gamma_{m9}\,\eta_0^{J]} - \omega y_m y^a \eta_0^{[I}\,\Gamma_{a9}\,\eta_0^{J]} \\ &\quad - y^a\eta_0^{[I}\,\Gamma_{am}\,\eta_0^{J]} \quad (4.23) \end{aligned}$$

or

$$K_{IJ}^m(y) = \Gamma_{IJ}^{m9} - \omega\, y^m y_a\, \Gamma_{IJ}^{a9} - y_a\,\Gamma_{IJ}^{am} \quad (4.24)$$

Thus there are as many Killing vectors as there are independent matrices Γ^{ab}, i.e. 36 Killing vectors as expected. The presence of the extra factor Γ_9 in the above equation is the reason that there are not 120 independent bilinear combinations as naive counting would have suggested; there is no such extra factor in the corresponding expression for the 28 S^7 Killing vectors in terms of bilinears of the eight S^7 Killing spinors [10].

4.5 Properties of the solution

Among the noteworthy features of the compactification described here is its breaking of supersymmetry. As discussed above, this implies that either the induced gauged theory is not fully supersymmetric but only invariant under a "chiral" $N = 16$ supersymmetry or at least its ground state breaks half of the supersymmetry.

We further note, that there is no nontrivial solution of the Killing spinor equations with constant functions F or H. The two-dimensional geometry hence is not AdS_2 but only conformally equivalent (of course, all metrics are conformally equivalent in two dimensions).

The fact that the solution requires a nontrivial "warp factor" in the internal part of the metric seems to indicate that the potential of the gauged theory in fact does not admit stationary points for any constant values of the scalar fields. Rather, it suggests that this potential can be minimized only with r-dependent scalar fields, a feature which has no analogue in the known higher dimensional gauged theories.

5. Remarks on the $D = 2$ theory

Finally, we sketch the complementary approach to the construction of the two-dimensional gauged theory, which is by deformation of the ungauged i.e. toroidally compactified theory. The latter is described by a field theory coupled to gravity in conformal gauge

$$e_\mu{}^\alpha = \lambda\,\delta_\mu{}^\alpha\,. \tag{5.1}$$

The scalar sector is given by a dilaton field ρ and an $E_{8(8)}$ valued matrix $\mathcal{V}$ which defines the currents

$$\mathcal{V}^{-1}\partial_\mu\mathcal{V} \;=\; P_\mu^A Y^A + \tfrac{1}{2}Q_\mu^{IJ}X^{IJ}\,, \tag{5.2}$$

where Y^A and X^{IJ} denote the 128 noncompact and 120 compact generators of $E_{8(8)}$, respectively. The theory thus has a manifest global $E_{8(8)}$ symmetry

$$\mathcal{V} \rightarrow \Lambda_a t^a \mathcal{V}\,, \tag{5.3}$$

if by t^a we denote all 248 generators of the algebra $\mathfrak{e}_8$: $t^a = \{Y^A, X^{IJ}\}$. On the fermionic side, the theory comprises the $2d$ gravitino ψ_μ^I and the dilatino ψ_2^I transforming in the **16** of $SO(16)$, as well as the fermionic matter denoted by $\chi^{\dot A}$ and transforming in the $\overline{\mathbf{128}}$. The Lagrangian of the ungauged theory is given by [5]

$$\begin{aligned}\mathcal{L}^{(0)} \;=\; & -\tfrac{1}{4}\rho E^{(2)}R^{(2)} + \tfrac{1}{4}\rho E^{(2)}P^{\mu A}P_\mu^A \\ & - \rho E^{(2)}\epsilon^{\mu\nu}\overline{\psi}{}_2^I D_\mu \psi_\nu^I \\ & - \tfrac{\mathrm{i}}{2}\,\rho E^{(2)}\overline{\chi}{}^{\dot A}\gamma^\mu D_\mu \chi^{\dot A} \\ & - \tfrac{1}{2}\,\rho E^{(2)}\overline{\chi}{}^{\dot A}\gamma^\nu\gamma^\mu\psi_\nu^I\Gamma^I_{A\dot A}P_\mu^A \\ & - \tfrac{\mathrm{i}}{2}\,\rho E^{(2)}\overline{\chi}{}^{\dot A}\gamma^3\gamma^\mu\psi_2^I\Gamma^I_{A\dot A}P_\mu^A\,. \end{aligned}\tag{5.4}$$

Gauging this theory corresponds to promoting a subgroup of the global symmetry (5.3) in a local symmetry. As discussed above, one of the peculiarities of the two-dimensional theory with respect to its higher dimensional relatives is the a priori absence of vector fields which are dualized away in the process of compactification. More precisely, the vector fields which are present in the toroidal compactification of the eleven-dimensional theory have been dualized into the scalar sector in three dimensions. The corresponding $D = 3$ dualization equation takes the form

$$\epsilon^{\mu\nu\rho}\,\partial_\mu B_\nu{}^a \;=\; E^{(3)}\,\mathcal{V}^a{}_A\,P^{A\,\rho} \tag{5.5}$$

where $\mu, \nu \ldots$ now run over 0, 1, 2. It explicitly exhibits the duality between the scalar fields contained in the $E_{8(8)}$ matrix $\mathcal{V}$ and a set of 248 vector fields combined into an $\mathfrak{e}_8$ valued matrix $B_\mu{}^a\,t_a$. There are hence different equivalent formulations of the $D = 3$ theory, depending on the choice of particular complementary subsets of scalars and vector fields, cf. [12]. For our purpose we consider the reduction of (5.5) back to two dimensions with the following ansatz

$$e^{(3)}{}_\mu{}^\alpha \;=\; \begin{pmatrix} \lambda\,\delta_\mu{}^\alpha & \rho A_\mu \\ 0 & \rho \end{pmatrix} \tag{5.6}$$

$$B_\mu^{(3)} \;=\; (B_\mu - A_\mu B_2, B_2) \tag{5.7}$$

for vielbein and vector fields. Equation (5.5) then splits into two equations

$$\partial_\mu B_2{}^m \;=\; \rho\epsilon_{\mu\nu}\mathcal{V}^m{}_A P^{A\,\nu} \tag{5.8}$$

$$B_{\mu\nu}{}^m \;=\; -B_2{}^m\,A_{\mu\nu} \tag{5.9}$$

i.e. into a two-dimensional dualization equation between scalars $\mathcal{V}$ and B_2^a and a relation between the $2d$ field strength $B_{\mu\nu}$ and the field strength of the Kaluza-Klein vector field A_μ. We emphasize here, that the construction of the gauged theory will require explicit appearance of both, the original scalar fields $\mathcal{V}$ as well as their duals $B_2{}^m$. In toroidal compactification, one uses the equations of motion for the Kaluza-Klein vector field A_μ to show that

$$A_{\mu\nu} = C_0\,\rho^{-3}\,\epsilon_{\mu\nu} \tag{5.10}$$

with constant C_0. Asymptotically flat solutions require $C_0 = 0$ and hence, via (5.9), vanishing field strength of the two dimensional vector fields. In other words, under the reduction, the three-dimensional vector fields split into dual scalars B_2 and nonpropagating vector fields which are usually dropped from the theory. However, as it turns out, once the two-dimensional theory is gauged with these vector fields, the relations (5.9) also get modifications in order of the coupling constant.

We can now follow the standard recipe [4] by first covariantizing the derivatives (5.2) w.r.t. a subgroup of $E_{8(8)}$

$$\begin{aligned}\mathcal{V}^{-1}\mathcal{D}_\mu\mathcal{V} &\equiv \mathcal{V}^{-1}\partial_\mu\mathcal{V} + g\,B_\mu^a\Theta_{ab}\mathcal{V}^{-1}t^b\mathcal{V} \\ &= \mathcal{P}_\mu^A Y^A + \tfrac{1}{2}\mathcal{Q}_\mu^{IJ}X^{IJ}\,. \end{aligned}\tag{5.11}$$

The constant tensor Θ_{ab} here encodes the embedding of the gauged subgroup into $E_{8(8)}$. Continuing the standard procedure, the fermionic supersymmetry transformations rules get modified

$$\begin{aligned}\delta\psi^I_\mu &= \mathcal{D}_\mu\epsilon^I + \mathrm{i}gA_{1\,IJ}\,\gamma_\mu\,\epsilon^J \\ \delta\psi^I_2 &= \rho^{-1}\partial_\mu\rho\,\gamma^\mu\,\epsilon^I \\ &\quad + \mathrm{i}g\left(A_{2\,[IJ]} + A_{2\,(IJ)}\gamma^3\right)\epsilon^J \\ \delta\chi^{\dot{A}} &= \tfrac{\mathrm{i}}{2}\,\gamma^\mu\epsilon^I\,\Gamma^I_{A\dot{A}}\,\mathcal{P}^A_\mu \\ &\quad + \mathrm{i}g\left(A_{3\,I\dot{A}} + A_{4\,I\dot{A}}\gamma^3\right)\epsilon^I \qquad (5.12)\end{aligned}$$

with tensors $A_1, \ldots, A_4$ to be determined as functions of the scalar fields. The original Lagrangian (5.4) is changed by adding a general fermionic bilinear term in order g of the coupling constant and a potential term in order g^2. These terms are determined by requiring maximal supersymmetry of the gauged theory. Moreover, supersymmetry imposes strong consistency conditions which eventually select the possible gauged subgroups.

A detailed discussion of the construction for the two-dimensional case shall be reported elsewhere [13]. Here, we just state the result, which is that the consistency conditions for the existence of a gauged deformation of toroidally compactified $D = 2$ supergravity may be given in closed form as a set of linear algebraic equations for the embedding matrix Θ_{ab}. Group theoretical arguments then allow to find explicit nontrivial solutions Θ_{ab} to these equations and thus to construct a class of two-dimensional gauged supergravities.

Some of their common features may already be observed from (5.12). Since the tensor $A_{2\,(IJ)}$ turns out to have a nonvanishing constant trace part, the variation of the dilatino ψ^I_2 in (5.12) e.g. already shows that no solution of the gauged theory can preserve the full supersymmetry. Rather, we find the same type of chiral symmetry breaking as was exhibited in the compactification discussed above. Moreover, since also $A_{4\,I\dot{A}}$ turns out to be nonvanishing, the variation of the matter fermions $\chi^{\dot{A}}$ shows the necessity to invoke nonconstant scalar fields in a supersymmetric solution as was likewise predicted by our compactification scenario.

Finally, we note, that the equation for the Kaluza-Klein field strength in the gauged theory is modified by a contribution quadratic in the dual scalar fields ${B_2}^a$

$$\begin{aligned}A_{\mu\nu} &= C_0\,\rho^{-3}\,\epsilon_{\mu\nu} \\ &\quad - \tfrac{1}{2}\,g\,\rho^{-3}\,\epsilon_{\mu\nu}\left({B_2}^a\Theta_{ab}{B_2}^b\right) \qquad (5.13)\end{aligned}$$

This shows that unlike in the toroidal compactification, the Kaluza-Klein vector field A_μ and hence also the vector fields B_μ can no longer be dropped. Rather, the solutions appear nontrivially charged under the $U(1)$, as was also explicitly found in our compactification.

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