# Non-linear Realizations and Bosonic Branes

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Abstract: In this talk we presented the results of hep-th/9912226, 0001216 and 0005270. The reader may consult these references. In this very short note, following hep-th/0001216, we express the well known bosonic brane as a non-linear realisation. References to related work can be found in this paper.

## 1. The Theory of Non-linear Realizations

We first briefly review the theory of non-linear realizations set out by Coleman, Wess and Zumino and extended by Volkov. Rather than consider a general group we restrict ourselves to the (super) group $\underline{G}$ whose generators can be divided into the two sets $\underline{K}$ and $\underline{H}$. The sets $\underline{K}$ and $\underline{H}$ are both supgroups of $\underline{G}$ and $\underline{H}$ is the automorphism group, of $\underline{K}$. The generators in each of the above two classes are further divided as $\underline{K} = \{K, K'\}$ and $\underline{R} = \{R, R'\}$. The generators $K$ and $R$ both form subalgebras of the Lie (super) algebra $G$ whose associated groups we denote by $K$ and $H$ respectively. The generators of $H$ are an automorphisms of $K$. The division of the generators of $\underline{G}$ into these four classes corresponds to:

K, unbroken generators associated with positions in (super) space-time,

K' spontaneously broken generators associated with positions in (super) space-time,

R, unbroken automorphism generators,

R', spontaneously broken auotmorphism generators.

The automorphism generators include those of the Lorentz group or its covering group the corresponding Spin group, in some cases internal group generators, but also other generators which are not usually considered.

¿From now on we will drop the prefix (super), but it is to be understood to be present. As will be clear, the objects associated with the group $\underline{G}$ carry an underlined indices while those associated with the subgroup $G$ carry no underline. The decomposition of the former into the latter and the remainder is achieved using unprimed and primed indices respectively.

We wish to consider the coset $\underline{G}/H$. For simplicity we will use an exponential description of group elements and we may then choose the coset representatives to be

$$g = e^{X\cdot K} e^{X'\cdot K'} e^{\phi'\cdot R'} \tag{1.1}$$

In this equation $\cdot$ denotes the relavent summation over the indices. Under any rigid group transformation $g_0$ we find that

$$g \rightarrow \ g_0 g = \hat{g} = e^{\hat{X}\cdot K} e^{\hat{X}'\cdot K'} e^{\hat{\phi}'\cdot R'} h_o \tag{1.2}$$

where $h_0$ is an element of $H$. The Cartan forms are given by

$$g^{-1}dg = F\cdot K + F'\cdot K' + \omega'\cdot R' + \omega\cdot R \tag{1.3}$$

Under equation (1.2) the Cartan forms transform as

$$\hat{g}d\hat{g} = h_0(g^{-1}dg)h_0^{-1} + h_0 dh_0^{-1} \tag{1.4}$$

It can happen that the Cartan forms carry a reducible representation of $H$, in which case, certain of the forms can be set to zero. This is the so called inverse Higgs effect. It has the effect of eliminating some of the Goldstone fields in terms of the others. The action or equations of motion are to be constructed from the Cartan forms in such a way that they are invariant under the transformations of equation (1.2).

The conventional interpretation of the above equations is to regard the $X$ as the coordinates of (super) space-time and to take the fields $X'$ and $\phi'$ to depend on them. This leads to a field theory on the coset space $G/H$. This approach has been almost universally adopted. However, when considering branes it is instructive to consider a more general possibility. The brane is moving through the coset space $\underline{G}/\underline{H}$ with tangent group $\underline{H}$ and sweeps out a submanifold that has the dimensions of the coset $G/H$ and a tangent space group $H$. We therefore consider the representatives of the coset $\underline{G}/H$ of equation (1.2) to depend on the variables $\xi$ which parametrises the embedded submanifold. Since the Cartan forms involve the exterior derivative $d$ they are independent of the coordinate system used. The Cartan forms associated with $K$ are given by $F\cdot K = d\xi\cdot F\cdot K$ and similarly for the other Cartan forms. We can think of the $F$ in this equation as the vielbein on the embedded submanifold. The covariant derivatives of the Goldstone fields associated with $K'$ are defined by

$$F'\cdot K' \equiv F\cdot \Delta X'\cdot K' = d\xi\cdot F^{-1}\cdot F'\cdot K'. \quad (1.5)$$

The $\Delta X'$ are independent of reparamenterisations in the paprameters $\xi$. A similar construction can be made for the covariant derivatives of $\phi'$. When identifying the fields that can be set to zero, i.e. use the inverse Higgs mechanism, we must not only maintain the $\underline{G}$ invariance of equation (1.2) and also reparameterisation invariance. In effect, this means setting only those covariant derivatives of the Goldstone fields in equation (1.5) that transform in a covariant manner under $h_0$ to zero. The equations of motion, or action, are to be constructed from the vielbein and the covariant derivatives of the Goldstone fields that remain. In this way one can find a formulation of brane dynamics that is reparameterisation invariant and also invariant under the rigid $\underline{G}$ transformations of equation (1.2). From this approach we can recover the more conventional approach by using the reparameterisation invariance to choose static gauge, i.e. $X = \xi$ for those coordinates that lie in the brane directions.

## 2. Bosonic Branes

In this section we will show that the dynamics of bosonic p-branes in a flat background in $D$ dimensional space-time arises as a non-linear realization in the sense of the previous section. We take $\underline{G} = ISO(D-1,1)$ with $\underline{K} = \{P_{\underline{n}}\}$ and $\underline{H} = \{J_{\underline{nm}}\}$ where $\underline{n},\underline{m} = 0,1\ldots,D-1$ and $G = ISO(p,1)$ with $K = \{P_n\}$ and $H = \{J_{nm}, J_{n'm'}\}$ where $n,m = 0,1,\ldots,p+1$ and $n',m' = p+1,\ldots,D-1$. This is to be expected as the presence of the p-brane clearly breaks the background space-time group $ISO(D-1,1)$ to $ISO(p,1)\times SO(D-p-1)$. The Lie algebra of $ISO(D-1,1)$ is given by

$$[J_{\underline{nm}}, J_{\underline{pq}}] = -\eta_{\underline{np}}J_{\underline{mq}} - \eta_{\underline{mq}}J_{\underline{np}} + \eta_{\underline{nq}}J_{\underline{mp}} + \eta_{\underline{mp}}J_{\underline{nq}} \quad (2.1)$$

$$[P_{\underline{n}}, J_{\underline{pq}}] = +\eta_{\underline{np}}P_{\underline{q}} - \eta_{\underline{nq}}P_{\underline{p}} \quad (2.2)$$

We can write the coset representatives in the form

$$\begin{aligned} g(X,\phi) &= exp(X^nP_n + X^{n'}P_{n'})exp(\phi^{nm'}J_{nm'}) \\ &\equiv exp(X^{\underline{n}}P_{\underline{n}})exp(\phi\cdot J) \end{aligned} \quad (2.3)$$

We distinguish $X$ from $X'$ by the indices they carry, in other words the prime on the $X$ is understood to be present and we just write $X^{n'}$. We also drop the prime on $\phi$ from now on. The Cartan forms are given by

$$\begin{aligned} g^{-1}dg \equiv\ & e^nP_n + f^{n'}P_{n'} + \Omega^{nm'}J_{nm'} \\ & + w^{n'm'}J_{n'm'} + w^{nm}J_{nm} \end{aligned} \quad (2.4)$$

which we may express as

$$\begin{aligned} g^{-1}dg = &\ exp(-\phi\cdot J)(dX^{\underline{n}}P_{\underline{n}})exp(\phi\cdot J) \\ & + exp(-\phi\cdot J)dexp(\phi\cdot J) \end{aligned} \quad (2.5)$$

A straightforward calculation shows that

$$\begin{aligned} e^n &= dX^{\underline{m}}\Phi_{\underline{m}}{}^n = -2\phi^{nm'}dX_{m'} + dX^n + \ldots \\ f^{n'} &= dX^{\underline{m}}\Phi_{\underline{m}}{}^{n'} = -2\phi^{n'm}dX_m + dX^{n'} \\ \Omega^{nm'} &= d\phi^{nm'} \\ w^{n'm'} &= (\phi^{pn'}d\phi_p{}^{m'} - (n'\leftrightarrow m'), \\ w^{nm} &= (\phi^{p'n}d\phi_{p'}{}^m - (n\leftrightarrow m)) \end{aligned} \quad (2.6)$$

where $\Phi_{\underline{n}}{}^{\underline{m}}$ is defined by $exp(-\phi\cdot J)P_{\underline{n}}exp(\phi\cdot J)\equiv\Phi_{\underline{n}}{}^{\underline{m}}P_{\underline{m}}=P_{\underline{n}}+2\phi_{\underline{n}}{}^{\underline{m}}P_{\underline{m}}+\dots$ and $+\dots$ means higher order terms in $\phi^{\underline{nm}}$.

Under a group transformation, $g_0 g(X,\phi)=g(\hat{X},\hat{\phi})h_0$, the Cartan forms transform in accord with equation (1.4). The effect of taking $P_{\underline{n}}$ transformations in $g_0$ is to simply to shift the $X^{\underline{n}}$ while the Cartan forms are left invariant. Writing $h_0=1+r^{nm}J_{nm}$, we find that

$$\begin{aligned}\hat{e}^n &= e^n-2e^p r_p{}^n, \quad \hat{f}^{n'}=f^{n'},\\ \hat{\Omega}^{nm'} &= \Omega^{nm'}+2r^{pn}\Omega_p{}^{m'}-2r^{pm'}\Omega_p{}^n\end{aligned} \tag{2.7}$$

$$\hat{w}^{nm}=w^{nm}-2(r^{np}w_p{}^m-(n\leftrightarrow m))+dr^{nm} \tag{2.8}$$

to lowest order in $r^{nm}$. Similar results hold for $J_{n'm'}$. The fields $e^n$ and $f^{n'}$ transform as expected under the Lorentz group $SO(p,1)\times SO(D-p-1)$.

Clearly, we can set $f^{n'}=0$ and preserve SO(1,D-1) and reparameterisation symmetry. At the linearized level, examining equation (2.7) we find it implies that $dX^{n'}=2\phi^{n'}_m dX^m$ or

$$2\phi^{n'}_m=\frac{\partial\xi^p}{\partial X^m}\frac{\partial X^{n'}}{\partial\xi^p} \tag{2.9}$$

If we choose static gauge this equation becomes,

$$2\phi^{n'}_m=\frac{\partial X^{n'}}{\partial X^m}, \tag{2.10}$$

While solving for $f^{n'}=0$ to all orders may be complicated it is clear that its content is to solve for $\phi^{n'}_m$ in terms of $\partial_m X^{n'}$.

What is really of interest to us is the non-linear form of $e^n$ once we have solved this constraint $f^{n'}=0$. Examining equation (2.7) we find that the vector $f^{\underline{n}}\equiv(e^n,f^{n'})$ is related by a Lorentz transformation to the vector $(dX^n,dX^{n'})$ $=dX^{\underline{n}}$. As such,

$$e_p{}^n\eta_{nm}e_q{}^m=\partial_p X^{\underline{n}}\partial_q X^{\underline{m}}\eta_{\underline{nm}} \tag{2.11}$$

since $f^{n'}=0$. The above expression is invariant under $g_0$ transformations. A worldvolume reparameterisation and group invariant action is therefore given by

$$\int d^p\xi\ dete=\int d^p\xi\sqrt{-det(\partial_p X^{\underline{n}}\partial_q X^{\underline{m}}\eta_{\underline{nm}})} \tag{2.12}$$

In other words, the well known generalisation of the Nambu action for the string to a p-brane follows in a straightforward consequence of taking the non-linear realization of $ISO(D-1,1)$ with subgroup $SO(p,1)\times SO(D-p-1)$.

Clearly, had we not included the Lorentz group in our coset and introduced the corresponding Goldstone bosons the veilbein on the brane would have been trivial and we would not have found the above action.

Although in this note we have only realised the bosonic branes as non-linear realisations, the branes of M- theory follow a similar pattern. For these branes the Lorentz group is replaced by the automorphism group of the corresponding supersymmetry algebra and the field strengths of the world-volume gauge fields arise as the Goldstone bosons for some of the genreators in this automorphism group.