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Some Classical Solutions of Matrix Model Equations

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Abstract: Based on work of P. Slodowy, P. Kronheimer, and a joint paper with C. Bachas and B. Pioline (hep-th/0007067), I will discuss the space of solutions of the matrix-equations $\dot{X}_a = \epsilon_{abc}X_bX_c - X_a$ for 3 antihermitean traceless $N \times N$ matrices $X_a(t)$, $t \in (-\infty, +\infty)$, interpolating between different representations of $su(2)$. I will also discuss solutions of $\ddot{X}_i = \sum_{j=1}^{d} [[X_i,\, X_j], X_j]$.

Consider 3 traceless, antihermitean $N \times N$ matrices $X_a(t)$, $t \in (-\infty, +\infty)$, developping in time according to the equations

$$\dot{X}_a = \epsilon_{abc}X_bX_c - mX_a\,. \tag{1}$$

The stationary points of this flow are representations of $su(2)$, i.e. $X_a = mJ_a$,

$$[J_a,\, J_b] = \epsilon_{abc}J_c\,. \tag{2}$$

The question is: given 2 such representations, ρ_+ and ρ_-, under which circumstances do there exist solutions $X_a(t)$ of (1) approaching the representation ρ_+ as $t \to +\infty$ and (being conjugate to) ρ_- as $t \to -\infty$?

Denoting the space of such solutions by $\mathcal{M}(\rho-,\, \rho+)$, Kronheimer [1], in parts building on work of Slodowy [2][3], proved that

$$\mathcal{M}(\rho_-,\, \rho_+) = \mathcal{N}(\rho_-) \cap S(\rho_+)\,, \tag{3}$$

where the r.h.s. is well known from singularity theory related to Lie algebras [2]. In the main part of my talk, based on joint work with C. Bachas and B. Pioline (see [4]; in particular concerning the physical relevance of (1), (3)) I will discuss (3):

Take

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{4}$$

as generators of $s\ell(2, \mathbb{C})$, the complexification of $su(2)$; denote by $H_\pm := \rho_\pm(h)$, $X_\pm := \rho_\pm(x)$, $Y_\pm := \rho_\pm(y)$, the corresponding $N \times N$ matrices in the representation $\rho_\pm$, i.e. satisfying the same commutation relations as those following from (4),

$$[x, y] = h,\ [h, x] = 2x,\ [h, y] = -2y\,. \tag{5}$$

$\mathcal{N}(\rho_\pm)$ is then defined as the orbit of $Y_\pm$ under the complexified gauge group, $SU(N)_{\mathbb{C}} = SL(N, \mathbb{C})$:

$$\mathcal{N}(\rho_{\substack{-\\(+)}}) := \{gY_{\substack{-\\(+)}}g^{-1} \mid g \in SL(N, \mathbb{C})\} \tag{6}$$

while

$$S(\rho_{\substack{+\\(-)}}) = Y_{\substack{+\\(-)}} + Z(X_{\substack{+\\(-)}}) \tag{7}$$

where

$$Z(X_{\substack{+\\(-)}}) := \{A \in s\ell(N, \mathbb{C}) \mid [A, X_{\substack{+\\(-)}}] = 0\} \tag{8}$$

is the centralizer of $X_{\substack{+\\(-)}}$.

Example ($N = 3$):

Let ρ_- be the irreducible 3-dimensional representation of $su(2)$, and $\rho_+ = 2 \oplus 1$ the direct sum of the irreducible 2-dimensional one, and the trivial 1-dimensional (putting all $J_a = 0$). Then one has

$$Y_- = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad Y_+ = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

$$X_- = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad X_+ = \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right)$$

$$H_- = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad H_+ = \left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & \\ \hline & & 0 \end{array}\right) . \quad (9)$$

In this example,

$$S(\rho_+) = \left\{ s = \begin{pmatrix} a & b & c \\ 1 & a & 0 \\ 0 & e & -2a \end{pmatrix} \mid a,b,e,c \in \mathbb{C} \right\} , \quad (10)$$

as can be found either by a simple explicit computation, or using the general fact that $s\ell(N,\mathbb{C})$ decomposes, under the adjoint action of $\rho_{\underset{(-)}{+}}$ into irreducible representation spaces of $s\ell(2,\mathbb{C})$, each of which contains exactly one 1-parameter family of elements of $Z(X_{\underset{(-)}{+}})$; in the above case, $s\ell(3,\mathbb{C})$ decomposes, under the action of $(Y_+ = E_{21},\ X_+ = E_{12},\ H_+ = E_{11} - E_{22})$ into one 3-dimensional representation space (ρ_+ itself, contributing $\mathbb{C}\cdot X_+$ to $Z(X_+)$), two 2-dimensional ones: spanned by E_{23} and $[E_{12}, E_{23}] = E_{13} \in Z(E_{12})$, resp. E_{31} and $[E_{12}, E_{31}] = -E_{32} \in Z(E_{12})$, and one 1-dimensional one ($\mathbb{C}\cdot(E_{11} + E_{22} - 2E_{33}) \in Z(E_{12})$). Instead of computing $\mathcal{N}(\rho_-)$ explicitly, $\mathcal{N}(\rho_-)\cap S(\rho_+)$ can, in the above example, be determined by simply demanding $s^3 = 0,\ s^2 \neq 0$ for the elements in (10); this gives $b = -3a^2,\ ec = 8a^3$, i.e.

$$\mathcal{N}(\rho_-) \cap S(\rho_+) = \quad (11)$$
$$= \left\{ \begin{pmatrix} a & -3a^2 & c \\ 1 & a & 0 \\ 0 & e & -2a \end{pmatrix} \middle|_{\substack{a,c,e\in\mathbb{C} \\ ec=8a^3}} \right\} .$$

According to (3), $\mathcal{M}(\rho_-, \rho_+)$ is therefore the 4-dimensional (singular) space (11). Let me now sketch (part of) the proof of (3) (cp [1]): One first 'gauges' (1) by introducing a 4-th traceless, antihermitean, $N\times N$ matrix, X_0, and going over to the equations

$$\dot{X}_a + [X_0, X_a] = \frac{1}{2}\epsilon_{abc}[X_b, X_c] - mX_a . \quad (12)$$

Due to their invariance under

$$\begin{aligned} X_a \to \tilde{X}_a &= U(t)X_aU^{-1}(t) , \\ X_0 \to \tilde{X}_0 &= UX_0U^{-1} - \dot{U}U^{-1} , \end{aligned} \quad (13)$$

a solution $\tilde{X}_a$ of (1) may be obtained from a solution X_a of (12) by closing U in (13) such that $\tilde{X}_0 = 0$. (12) is then split into one complex equation (from now on, $m = 2$)

$$\dot{\beta} + 2\beta + 2[\alpha,\beta] = 0 , \quad (14)$$

and one real equation,

$$\frac{d}{dt}(\alpha + \alpha^\dagger) + 2(\alpha + \alpha^\dagger) + 2[\alpha, \alpha^\dagger] + 2[\beta, \beta^\dagger] = 0 . \quad (15)$$

Due to $\alpha := \frac{1}{2}(X_0 - iX_3)$ and $\beta := -\frac{1}{2}(X_1 + iX_2)$ no longer having to obey any (anti)hermiticity conditions, the gauge-invariance of (14) is enhanced to complex (!) gauge transformations

$$\begin{aligned} \alpha &\to g\alpha g^{-1} - \tfrac{1}{2}\dot{g}g^{-1} \\ \beta &\to g\beta g^{-1} . \end{aligned} \qquad g \in SL(N,\mathbb{C}) \quad (16)$$

Kronheimer [1] then proved that any solution of (14) (with the required boundary conditions) is gauge equivalent to

$$\begin{aligned} \alpha_-(t) &= \frac{1}{2}H_-,\ \beta_-(t) = Y_- \quad \text{for } t \in (-\infty, 0] \\ \alpha_+(t) &= \frac{1}{2}H_+,\ \beta_+(t) = Y_+ + e^{-2t}e^{-t\,ad\,H_+}Z_+ \\ &\qquad \text{for } t \in [0, +\infty) , \end{aligned} \quad (17)$$

with $Z_+ \in Z(X_+)$. Stated the other way round (actually 0 may be replaced by any finite time, in (17)): for any given solution (α, β) of (14) there exist g_+ and g_- (approaching the identity, resp. a constant group element, at $t = +\infty$, resp. $t = -\infty$) such that, for any finite t,

$$\begin{aligned} \beta &= g_+^{-1}(Y_+ + e^{-(2+ad\,H_+)t}Z_+)g_+ \\ 2\alpha &= g_+^{-1}H_+g_+ + g_+^{-1}\dot{g}_+ \end{aligned} \quad (18)$$

AND

$$\begin{aligned} \beta &= g_-^{-1}Y_-g_- \\ 2\alpha &= g_-^{-1}H_-g_- + g_-^{-1}\dot{g}_- . \end{aligned}$$

This means that for any finite t,

$$Y_+ + e^{-(2+ad\,H_+)t}Z_+ , \quad (19)$$

which is $\in S(\rho_+)$, must be gauge-equivalent to Y_-, i.e. must be $\in \mathcal{N}(\rho-)$. Letting $t \to +\infty$, while noting that $(2 + ad\,H_+)$ is strictly positive*

*in the previous example one would have

$$[H_+ = E_{11} - E_{22}, \begin{matrix} X_+ \\ E_{13} \\ E_{32} \\ E_{11}+E_{22}-2E_{33} \end{matrix}] = \begin{matrix} 2X_+ \\ 1\cdot E_{13} \\ 1\cdot E_{32} \\ 0 \end{matrix}$$

on $Z(X_+)$, one finds that Y_+ (hence $\mathcal{N}(\rho_+)$!) must actually be contained in the closure of $\mathcal{N}(\rho_-)$ (for $\mathcal{M}(\rho_-,\rho_+)$ to be non-empty). If this condition is fulfilled, the dimension of $\mathcal{M}$, due to S_+ and $\mathcal{N}_-$ meeting transversely, can be computed as follows :

$$\begin{aligned} &\dim(S_+ \cap \mathcal{N}_-) \\ &= \dim S_+ + \dim \mathcal{N}_- - \dim(S_+ \cup \mathcal{N}_-) \\ &= \dim S_+ - \dim S_- \,. \end{aligned} \tag{20}$$

In the second part of my talk (mostly based on [5],[6]) I would like to recall some solutions to the classical equations of motion,

$$\ddot{X}_i = -\sum_{j=1}^{d} [[X_i, X_j], X_j] \tag{21}$$

$$\sum_{j-1}^{d} [X_j, \dot{X}_j] = 0, \; X_i^\dagger = X_i \,,$$

of a regularized relativistic membrane in $d+2$-dimensional Minkowski-space (resp. a reduced $d+1$-dimensional $SU(N)$ Yang Mills theory in $A_0 = 0$ gauge, with the fields $A_i = X_i(t)$ being space-independent).

The Ansatz

$$X_i(t) = x(t) r_{ij}(t) M_j \tag{22}$$

with $(r_{ij}) = e^{A\varphi(t)} \in SO(d)$, $x^2\dot{\varphi} = L = \text{const}$, $\frac{1}{2}\dot{x}^2 + \frac{\lambda}{4}x^4 + \frac{L^2}{2x^2} = \text{const}$, reduces (22) to the equation

$$\sum_j [[M_i, M_j], M_j] = \lambda M_i \tag{23}$$

for a set of traceless hermitean $N \times N$ matrices M_i, $i = 1 \ldots \tilde{d}\,(\leq d$ if $A \equiv 0$, $\leq \frac{d}{2}$ if $A^2 \vec{M} = -\vec{M})$. Solutions of (23) include irreducible representations of semi-simple Lie-algebras, and

$$\vec{M} = \tag{24}$$

$$\frac{1}{\sqrt{2}} \left(\frac{U+U^{-1}}{2}, \frac{U-U^{-1}}{2i}, \frac{V+V^{-1}}{2}, \frac{V-V^{-1}}{2i}, 0 \ldots 0 \right)$$

with $VU = \omega UV$, $\omega = e^{\frac{4\pi i}{N}}$, N odd, $U^N = \mathbb{1} = V^N$. Due to the matrices $S_{\vec{m}} := \omega^{\frac{1}{2}m_1 m_2} U^{m_1} V^{m_2}$ satisfying commutation relations approaching those of $e^{i(m_1\varphi_1 + m_2\varphi_2)}$ (under $[f,g] \sim \in^{rs} \partial_r f \partial_s g$) (24) can be thought of as a discrete analogue of

$$\vec{m} = \frac{1}{\sqrt{2}}(\cos\varphi_1, \; \sin\varphi_1, \cos\varphi_2, \; \sin\varphi_2, \; 0, \ldots, 0) \tag{25}$$

defining a minimal Torus in the unit sphere S^3 (just as $\sum_i M_i^2 = \mathbb{1}$ for (24)).

It would be very interesting to find solutions of (23) that could be identified as discrete analogues of higher-genus minimal surfaces in S^3.

Another type of solutions of (21),

$$\begin{aligned} \vec{X} &= \textstyle\sum_\alpha r_\alpha(t)\, \vec{E}_\alpha \\ \vec{E}_1 &= \tfrac{1}{2}(S_{\vec{m}_1} + S_{-\vec{m}_1}, -i(S_{\vec{m}_1} - S_{-\vec{m}_1}), 0, \ldots, 0)\,, \\ \vec{E}_2 &= \tfrac{1}{2}(0, 0, S_{\vec{m}_2} + S_{-\vec{m}_2}, -i(S_{\vec{m}_2} - S_{-\vec{m}_2}), \\ &\qquad 0, \ldots, 0), \ldots \\ \ddot{r}_\alpha &= -4 r_\alpha \sum_\beta \sin^2(\frac{2\pi}{N}(\vec{m}_\alpha \times \vec{m}_\beta)) r_\beta^2\,, \end{aligned} \tag{26}$$

was given in [6] (including their $N = \infty$, $\vec{E}_a \to \vec{e}_a$ (φ^1, φ^2), continuum surface analogues). For recent stability analyses of a spherical analogue of such $(r_{\alpha=1\ldots6}, E_{a+3} := E_a(-\vec{m}_a)$ resp. $e_{a+3} := e_a(-\vec{m}_a)$, $\sphericalangle(\vec{m}_a, \vec{m}_b) = 2\pi/3, a, b = 1,2,3)$ solutions, resp. (22)/(23) with $[M_a, M_b] = i \in_{abc} M_c$ (cp. [5],[6]), see [7],[8].

Note added: I learned from E. Corrigan, that eq. (23) was studied by Wainwright, Wilson, and himself in a paper published in Comm.Math.Phys. 98 (1985) 259.

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