# Statistical Physics Approach to M-theory Integrals*

**Werner Krauth and Matthias Staudacher**

*CNRS-Laboratoire de Physique Statistique, Ecole Normale Supérieure*
*24, rue Lhomond*
*F-75231 Paris Cedex 05*
*E-mail:* krauth@lps.ens.fr

*Albert-Einstein-Institut, Max-Planck-Institut für Gravitationsphysik*
*Am Mühlenberg 1*
*D-14476 Golm*
*E-mail:* matthias@aei-potsdam.mpg.de

ABSTRACT: We explain the concepts of computational statistical physics which have proven very helpful in the study of Yang-Mills integrals, an ubiquitous new class of matrix models. Issues treated are: Absolute convergence versus Monte Carlo computability of near-singular integrals, singularity detection by Markov-chain methods, applications to asymptotic eigenvalue distributions and to numerical evaluations of multiple bosonic and supersymmetric integrals. In many cases already, it has been possible to resolve controversies between conflicting analytical results using the methods presented here.

KEYWORDS: Monte Carlo Methods, M-theory, Matrix Models, Yang-Mills Theory.

## 1. Introduction

Recent work in field theory has revealed the existence of an important new class of gauge-invariant matrix models. At the difference of the classic Wigner-type models, interest now focusses on integrals of D non-linearly *coupled* matrices $X_\mu, \mu = 1, \dots D$. The $X_\mu$ are constructed from the generators $T^A$ of the fundamental representation of a given Lie algebra Lie($G$): $X_\mu = X_\mu^A T_A$, with $A = 1, \dots, \dim(G)$. The group $G$ may be $SU(N)$, but the orthogonal, symplectic and exceptional groups have also come under close scrutiny recently.

These ordinary, multiple Riemann integrals stem from a dimensional reduction of D-dimensional Euclidean continuum Yang-Mills theory to zero dimensions. They have important implications, as the integrals yield the bulk part of the Witten index of supersymmetric quantum mechanical gauge theories, and appear in multi-instanton calculations of large $N$ susy Yang-Mills theories. Furthermore, they appear in proposed formulations of string theory (the IKKT model) and M-theory. It remains to be elucidated whether they contain further non-perturbative information on gauge theories via the Eguchi-Kawai mechanism.

For the sake of brevity (cf. [1] for complete definitions), we write down (even for supersymmetric Yang-Mills theories) only the effective bosonic integral, which is obtained after integrating out the $\mathcal{N}$(= number of supersymmetries) Grassmann-valued fermionic matrices

$$\mathcal{Z}^{\mathcal{N}}_{D,G} = \int \prod_{A,\mu} \frac{dX_\mu^A}{\sqrt{2\pi}} \times e^{\frac{1}{4g^2}\mathrm{Tr}\,[X_\mu,X_\nu][X_\mu,X_\nu]} (\mathcal{P}\left\{X_\mu^A\right\}). \quad (1.1)$$

In this equation, $\mathcal{P}(\{X\})$ is the Pfaffian of a certain matrix $\mathcal{M}$, which can be constructed from the adjoint representation of the $X_\mu$.

During the last few years, intense effort has been brought to bear on these integrals, ranging from the rigorous exact solution for $SU(2)$ [2],[3]

*Talk presented by W. Krauth

to ultra-sophisticated analytical calculations [4] which lent support to earlier conjectures [5].

We have initiated a project with the aim to obtain direct non-perturbative information on these integrals by numerical Monte Carlo calculation. In several cases already, this approach has allowed to clarify analytic properties of the integrals, both in the supersymmetric and the purely bosonic case (where the Pfaffian in eq.(1.1) is simply omitted). We have also obtained very precise values (statistical estimates) of $\mathcal{Z}$ for several low-ranked groups, estimates which were sufficiently precise to decide between differing analytical conjectures. The basic strength of the numerical approach is however to allow the computation of a wide range of observables (Wilson loops, eigenvalue distributions), and much work remains to be done.

The integrals in eq.(1.1) resemble partition functions in statistical physics. Our initial hope was to reduce eq.(1.1) to a standard form, most simply by the transformation

$$\mathcal{Z}^{\mathcal{N}}_{D,G} = \int \prod_{\mu,A} \frac{dX^A_\mu}{\sqrt{2\pi}} e^{-\frac{X^A_\mu{}^2}{2\sigma^2}} \left[ \frac{\mathcal{F}(\{X^A_\mu\})}{e^{-\sum \frac{X^A_\mu{}^2}{2\sigma^2}}} \right], \tag{1.2}$$

where $\mathcal{F}(\{X^A_\mu\})$ is the integrand in the second line of eq.(1.1). In this form, the integral can (in principle) be computed directly using Monte Carlo methods. To do so, it would suffice to generate Gaussian distributed random numbers $X^A_\mu$, and to average the term [ ] in eq.(1.2) over this distribution. The straightforward approach is thwarted by the fact that the integrals eq.(1.2) - and, equivalently, eq.(1.1) - only barely converge, if at all.

The reason for this bad convergence lies in the existence of "valleys" in the action in eq.(1.1). For example, any configuration of mutually commuting $X_\mu$ ($[X_\mu, X_\nu] = 0\ \forall \mu, \nu$) gives rise to a subspace of matrices with vanishing action, which stretches out to infinity, and leads to a large contribution to $\mathcal{Z}$. There is presently no mathematical proof that these singularities are integrable (cf. [9] [10] for perturbative results).

Very importantly, integrals may exist, without being computable by straightforward Monte Carlo methods. This distinction between existence and (Monte Carlo) computability is so crucial for Yang-Mills integrals that we present them in the next section in the simplified context of a $1-$dimensional integral.

## 2. Existence & Computability

Consider the integral

$$I(\alpha) = \int_0^1 dx \mu(x) x^{-\alpha} \tag{2.1}$$

with a constant weight function $\mu(x) = 1$, which we introduce for later convenience. In this toy problem, the singularity at $x = 0$ plays the role of a valley, as discussed before, in the more complex Yang-Mills integral.

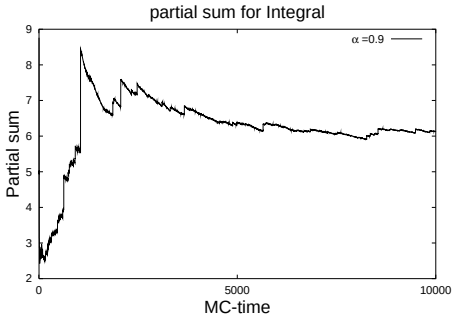


**Figure 1:** Partial sums $S_t$ (cf. eq.(2.2)) for $\alpha = 0.9$ vs Monte Carlo time $t$. The numerical estimate for this integral seems to converge to the wrong result.

We may compute the integral eq.(2.1) by the Monte Carlo method in the following way: as the weight function is constant ($\mu(x) = 1$), we pick $t$ uniformly distributed points $x_t$ with $0 \le x_t \le 1$ and compute

$$I \sim S_t = 1/t \sum_{i=1}^{t} x_i^{-\alpha}, \tag{2.2}$$

where $t = 1, 2 \ldots$ is the Monte Carlo time. A typical outcome for the partial sums $S_t$ during a Monte Carlo calculation for $\alpha = 0.9$ is shown in figure 1. The calculation is seemingly correct, as standard error analysis gives a result $I(\alpha = 0.9) = 6.13 \pm 0.46$, without emitting any warnings! Carrying on the simulation for much longer times, we would every so often generate an extremely small $x_t$, which in one step would hike up the partial sum, and change the error estimate. Repeatedly, we would get tricked into accepting "stabilized values" of the integral, which

would probably still not correspond to the true value $I(\alpha = 0.9) = 10$!

Clearly, there is a problem with the computability of the integral, which can be traced back to its infinite variance. Calling $\mathcal{O} = x^{-\alpha}$, the variance is given by

$$Var = \int dx \mu(x) \mathcal{O}(x)^2 - [\int dx \mu(x) \mathcal{O}(x)]^2. \tag{2.3}$$

The error in the Monte Carlo evaluation eq.(2.2) behaves like $\sqrt{Var/t}$, and, for $\alpha \geq 0.5$, is infinite. This situation is virtually impossible to diagnose from within the simulation itself.

We have developed a highly efficient tool to numerically check for (absolute) convergence of integrals. The idea (translated to the case of the present toy problem) is to perform Markov chain random walk simulation with a stationary distribution $\mu'(x) = \mu(x)x^{-\alpha}$ and $\mu''(x) = \mu(x)x^{-2\alpha}$. to check for existence of the integral and finiteness of the variance, respectively. In an effort to be completely explicit, this means to choose a small displacement interval $\delta_t$, uniformly distributed between $+\epsilon$ and $-\epsilon$, and to go from $x_t$ to $x_{t+1}$ according to the following probability table

$$x_{t+1} = \begin{cases} x_t + \delta_t & \text{w/ probability} \\ & \min(1, \mu'(x_{t+1})/\mu(x_t)) \\ x_t & \text{else} \end{cases} . \tag{2.4}$$

(cf. [14]). During these simulations (which are neither used nor useful to compute the integral eq.(2.1) itself), we are exclusively interested in finding out whether the Markov chain eq.(2.4) gets stuck. If so, it has become attracted by a point $x_0$ with

$$\int_{x_0}^{x_0+\epsilon} dx \mu'(x) = \infty \tag{2.5}$$

i. e. a non-integrable singularity. In figures 2 and 3 , we show $x_t$ for Markov chains with stationary distributions $\mu'(x)$ and $\mu''(x)$, respectively. Figure 3, in particular, implies that the variance of the integral eq.(2.1) is infinite so that the result of fig. 1 cannot be trusted, while figure 2 assures us that the integral exists.

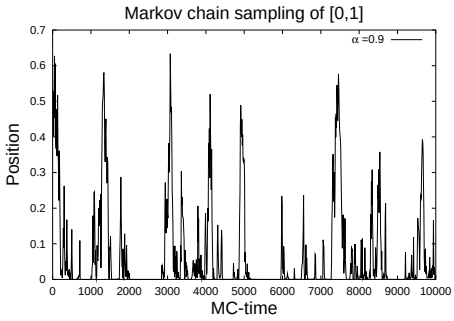


**Figure 2:** Position $x_t$ vs Monte Carlo time $t$ for the Markov chain with stationary distribution $\mu'(x) = x^{-0.9}$. The time evolution of $x_t$ does not get stuck, because the integral $\int_0^1 dx x^{-0.9}$ exists.

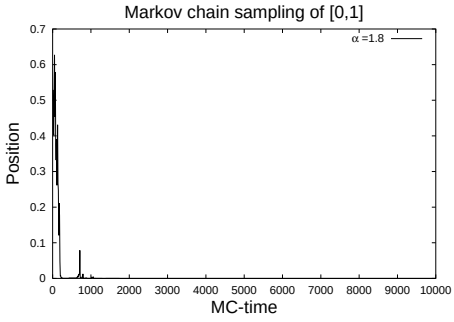


**Figure 3:** Same as figure 2, but for $\mu''(x) = x^{-1.8}$. $x_t$ gets stuck at $x \sim 0$ very quickly, signalling that the variance of the integral eq.(2.1) is infinite.

The method can be easily adapted to multidimensional integrals by monitoring an autocorrelation function rather than the position $x_t$,

In our applications, the method has been successful much beyond our initial expectations. Besides its "consulting" role within the Monte Carlo framework (as explained in the caption of figures 2 and 3), we have used it extensively to establish the existence conditions for bosonic and susy Yang-Mills integrals, which have not been obtained analytically beyond the 1-loop level. We have also adopted the method to obtain important information on the asymptotic behavior of integrals.

We conclude the discussion of our toy problem by showing how, after all, the integral eq.(2.1) can be computed by Monte Carlo methods. Con-

sider first

$$Q(\alpha_2,\alpha_1)=\frac{\int_0^1 x^{-\alpha_2}dx}{\int_0^1 x^{-\alpha_1}dx}=\frac{\int_0^1 \overbrace{x^{-\alpha_1}}^{measure}\ \overbrace{x^{\alpha_1-\alpha_2}}^{operator}dx}{\int_0^1 \underbrace{x^{-\alpha_1}}_{measure}\ dx} \tag{2.6}$$

According to the discussion in section 3, $Q(\alpha_2,\alpha_1)$ can be computed from random numbers distributed as $\mu(x)=x^{-\alpha_1}$ as long as

$$2\alpha_2-\alpha_1<1. \tag{2.7}$$

| $\alpha_1$ | $\alpha_2$ | $Q(\alpha_2,\alpha_1)$ | error (in %) |
|---|---|---|---|
| 0.0 | 0.15 | 1.179 | 0.2 % |
| 0.15 | 0.55 | 1.881 | 1 % |
| 0.55 | 0.75 | 1.799 | 1 % |
| 0.75 | 0.85 | 1.656 | 0.7 % |
| 0.85 | 0.9 | 1.508 | 0.6 % |

**Table 1:** Values of $\alpha_1,\alpha_2$, for which the quantity $Q(\alpha_2,\alpha_1)$ is computed according to eq.(2.6). The integral eq.(2.1) is well approximated by $\prod Q$.

It is easy to see that all of the pairs $(\alpha_1,\alpha_2)$ in table 1 satisfy the bound of eq.(2.7), and the Monte Carlo data for $Q(\alpha_2,\alpha_1)$ can thus be trusted, just as the final result

$$\int_0^1 dx\ \ x^{-0.9}=\prod Q=9.98\pm 0.16. \tag{2.8}$$

## 3. "Measurement" = "Comparison"

After these preliminary steps, we finally confront the Monte Carlo measurement of the Yang-Mills integrals. In this context, we recall from our basic physics training the heading of this section. Translated to the context of a Monte Carlo calculation, the measure/compare equivalence means that the integral eq.(1.1) has always to be written as

$$\mathcal{Z}^{\mathcal{N}}_{D,G}=\int\prod_{A,\mu}\frac{dX^A_\mu}{\sqrt{2\pi}}\mu(\{X^A_\mu\})\left\{\frac{\mathcal{F}(\{X^A_\mu\})}{\mu(\{X^A_\mu\})}\right\}. \tag{3.1}$$

In the $\{\ \ \}$ in eq.(3.1), we *compare* $\mathcal{F}$ to the measure, which we are free to choose (but which we have to be able to integrate analytically). As mentioned before, the Gaussians of eq.(1.2) are too different from $\mathcal{F}$ to work. A straightforward generalization of the approach eq.(2.6) was found to be wanting: the mismatch between $\mathcal{F}$ and $\mu$ could only be smoothed with a very large number of steps $(\alpha_1,\alpha_2)$ in eq.(2.6).

A much better approach has come from the observation that $\mathcal{F}$ can be *compactified* onto the surface of a hypersphere, because both the action and the Pfaffian are homogeneous functions of the radius $R=\sqrt{\sum X^A_\mu}$, which can therefore be integrated out. Introducing polar coordinates $R,\Omega$ and noting $\tilde{\mathcal{F}}=\int_0^\infty dRR^{d-1}\mathcal{F}(\Omega,R)$, we arrive at the ultimate formulation of the integral

$$\mathcal{Z}^{\mathcal{N}}_{D,G}=\int d\Omega\tilde{\mathcal{F}}(\Omega). \tag{3.2}$$

This means that the integrand $\tilde{\mathcal{F}}$ is compared to the constant function on the surface of a hypersphere in dimension $d=dim(G)\mathcal{N}$ .

The integral eq.(3.2) can still not be evaluated directly, so that the strategy of eq.(2.6) has to be used. Here, we simply compute a few ratios of the integrals $\int d\Omega\left[\tilde{\mathcal{F}}(\Omega)\right]^\alpha$ for different values $0<\alpha<1$. In this case, of course, pairs $(\alpha_2,\alpha_1)$ are tested by the qualitative Monte Carlo algorithm, as analytical convergence conditions in the spirit of eq.(2.7) are lacking. After having expended an extraordinary amount of rigor on these very difficult integrals, we nevertheless obtain well-controlled predictions, to be surveyed below.

## 4. Synopsis

The methods presented in the previous sections were used to compute a number of results which are fully discussed in [1], [6], [7], [8]. For complementary Monte Carlo studies, using somewhat different techniques, see [9], [11]. To give an indication of the scope and the quality of the data, we present here our recent calculations for gauge groups other than $SU(N)$, as well as an intriguing qualitative result concerning the asymptotics of the eigenvalue distributions.

The first example concerns the evaluation of the integrals for the gauge groups $SO(N)$, $Sp(2N)$ and $G_2$. These calculations can be connected to other theoretical work essentially by dividing $\mathcal{Z}$ by the volume $\mathcal{F}_G$ of the group $G$. In this way,

we arrive at a numerical value for the bulk contribution to the quantum-mechanical Witten index, which is given by

$$\mathrm{ind}_0^D(G) = \frac{1}{\mathcal{F}_G}\mathcal{Z}^{\mathcal{N}}_{D,G}. \qquad (4.1)$$

In table 3 we list our Monte Carlo results for this bulk index, obtained by the methods explained above, for groups up to rank three. We furthermore compare these data to analytic predictions from the generalization of the deformation method of Moore et al. [4] to these groups [8]. Note the excellent precision (2% statistical error) for groups up to $SO(7)$, where the integral eq.(1.1) lies in 84 dimensions. Intriguingly, both our numerical and analytical results are at variance with a previous conjecture [12] for non-unitary groups. In the special case of $SO(7)$, e.g., ref. [12] obtains the fraction 15/128 which is incompatible with our data.

| Group $G$ | Monte Carlo $\mathrm{ind}_0^{D=4}(G)$ | Exact |
|---|---|---|
| SO(3) | 0.2503 ± 0.0006 | 1/4 |
| SO(4) | 0.0627 ± 0.0013 | 1/16 |
| SO(5) | 0.1406 ± 0.001 | 9/64 |
| SO(6) | 0.0620 ± 0.001 | 1/16 |
| SO(7) | 0.0966 ± 0.0017 | 25/256 |
| Sp(2) | 0.2500 ± 0.0002 | 1/4 |
| Sp(4) | 0.139 ± 0.0015 | 9/64 |
| Sp(6) | 0.0973 ± 0.003 | 51/512 |
| $G_2$ | 0.173 ± 0.003 | 151/864 |

**Table 2:** Monte Carlo results versus proposed (BRST deformation method) exact values for the $D = 4$ bulk index.

Let us mention that at present the calculations for $D = 4$ and $D = 6$ are considerably simpler than the case $D = 10$, because the Pfaffian can be reduced to a determinant for $D = 4, 6$ [1]. In $D = 10$, this possibility does not exist generically (for an exception for $SU(3)$ cf. [1]). We have now developed new methods to compute Pfaffians which should allow computations for $D = 10$ in the near future. It is possible if tedious to work out the predictions of the BRST deformation technique for $\mathrm{ind}_0^{D=10}(G)$ *cf* [13], which again differ from the conjectures of [12]. It would be interesting to check the results of [13] by our Monte Carlo methods.

A further strength of the Monte Carlo approach is to allow the calculation of quantities other than just the integral $\mathcal{Z}$. We briefly review as a second illustration of the here advocated approach the study of the correlation functions $< \mathrm{Tr}\, X_\mu^k\} >$, where $X_\mu$ is an arbitrary single matrix. This correlation function allows to infer the eigenvalue distribution of the matrices. Indeed, denoting the normalized eigenvalue density of individual matrices by $\rho(\lambda)$, one has

$$< \mathrm{Tr}\, X_\mu^k\} >= \int_{-\infty}^{\infty} d\lambda\ \rho(\lambda)\ \lambda^k. \qquad (4.2)$$

Here the calculation was immediately feasible also for $D = 10$ case, since we only needed to test for absolute convergence, i.e. it suffices to consider a simplified measure obtained from the absolute value of the original measure:

$$\mu'(\{X_\mu^A\}) = \mathrm{Tr}\, X_\mu^k \left|\mathcal{F}(\{X_\nu^k\})\right|. \qquad (4.3)$$

This is algorithmically far more efficient, because now the problem may again be reduced to the computation of the square of a Pfaffian, which is readily available, avoiding the calculation of the Pfaffian itself.

The final result for the asymptotic eigenvalue densities as $\lambda \to \infty$ supersymmetric systems in $D = 4, 6, 10$, for the supersymmetric system with gauge groups SU($N$), is

$$\rho_{\mathbf{D}}^{\mathrm{SUSY}}(\lambda) \sim \begin{cases} \lambda^{-3} & D = 4 \\ \lambda^{-7} & D = 6 \\ \lambda^{-15} & D = 10 \end{cases}. \qquad (4.4)$$

These laws were obtained by applying the above Markov chain random walk tool to the measure eq.(4.3), i.e. we established divergence of eq.(4.2) iff $k \geq 2, 6, 14$ (respectively for $D = 4, 6, 10$), leading to eq.(4.4). Note that these power laws are independent of $N$. They demonstrate that the present matrix models are very different from the classic Wigner-type models. It would be interesting to obtain the generalization of eq.(4.4) to other gauge groups.

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